Final Examination
Monday, 12-11-95 10:30-12:30
Math 151
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Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 151 points out of a quarter total of 450.

Hypotheses for theorems 1–7:
k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\dfrac{f(x)}{g(x)}) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Answers to the following questions should be exactly one sentence. Do not write two different definitions. (2 pts each)

1. State the associative law for multiplication.

2. State the equation of a line with slope m, through the point (x_0, y_0).

3. State the definition of $|x|$, the absolute value of x.

4. State the Triangle Inequality for real numbers.

5. State the definition of $f \circ g$.

6. State the definition of the closed interval $[a, b]$.

7. Recall $\mathbb{Z}$ is the set of integers. State the definition of the set of rational numbers, $\mathbb{Q}$.

8a. Recall that $\mathbb{R}^2$ is the set of ordered pairs of real numbers. If $S \subset \mathbb{R}^2$, what property must S have to be the graph of a function?

8b. If $S \subset \mathbb{R}^2$, what additional property must S have to be the graph of an increasing function?

9. State the δ, ϵ definition of

$$\lim_{x \to c} f(x) = L.$$

10. State the δ, ϵ definition of

$$\lim_{x \to c^+} f(x) = L.$$

11. State the definition of "$f(x)$ is continuous at c."

12. State the definition of "$f(x)$ is continuous on $[a, b]$."

13. State the definition of the derivative of $f(x)$ at c. (Hint: it's a number.)

14. State the definition of the derivative of $f(x)$ on the open interval (a, b). (Hint: it's a function.)

15. State the definition of "$f(x)$ has a minimum value on the set I."

16. State the definition of "B is an upper bound of the set S."

17. State the definition of "β is the least upper bound of the set S."

18. State the δ, ϵ definition of "$f'(c)$ is the derivative of $f(x)$ at c."

19. State the definition of "$f(x)$ is weakly increasing."

20. State the definition of "$f(x)$ is a rational function."

21. Suppose $\mathbb{F}$ is a set of objects with the following properties:

A. The associative axiom for $+:$ $(x+y)+z = x+(y+z)$.

B. Right additive identity: $\exists 0 \in \mathbb{F}$ such that $x = x + 0$.

C. Left additive identity: $x = 0 + x$.

D. Right additive inverse: $\forall x \in \mathbb{F},\ \exists -x \in \mathbb{F}$ such that $x + (-x) = 0$.

E. Left additive inverse: $(-x) + x = 0$.

F. Multiplicative identity: $\exists 1 \in \mathbb{F}$ such that $x = 1 \cdot x$.

G. Commutative axiom for $\cdot:$ $x \cdot y = y \cdot x$.

H. The left distributive law: $x \cdot (y+z) = x \cdot y + x \cdot z$.

I. The right distributive law: $(y+z) \cdot x = y \cdot x + z \cdot x$.

a. (2 pts) Prove, justifying each equality, that rule "I" is a consequence of rules "G" and "H".

b. (1 pt each) Any system of numbers with the above properties must also satisfy the commutative law for addition, $x + y = y + x$. The following proof is careful with left and right, avoiding the use of commutativity, since that's what we're trying to prove. Fill in the blanks, with one letter (A–I) listing the justification for each step. (I've filled in two already)

____________ $x + y = 0 + (x+y)$

____________ $= (0 + (x+y)) + 0$

____________ $= (((-x)+x) + (x+y)) + 0$

____________ $= (((-x)+x) + (x+y)) + (y + (-y))$

____________ $= ((-x) + ((x+x) + (y+y))) + (-y)$

____________ $= ((-x) + ((1 \cdot x + 1 \cdot x) + (1 \cdot y + 1 \cdot y))) + (-y)$

____________ $= ((-x) + ((1+1) \cdot x + (1+1) \cdot y)) + (-y)$

____________ $= ((-x) + (1+1) \cdot (x+y)) + (-y)$

____________ $= ((-x) + (1 \cdot (x+y) + 1 \cdot (x+y))) + (-y)$

____________ $= ((-x) + ((x+y) + (x+y))) + (-y)$

____________ $= (((-x)+x) + (y+x)) + (y + (-y))$

____________ $= (((-x)+x) + (y+x)) + 0$

____________ $= (0 + (y+x)) + 0$

____________ $= 0 + (y+x)$

____________ $= y + x.$

22. Solve the following inequalities, that is, find the subset of $\mathbb{R}$ consisting of all elements x with the required property or properties. You don't have to explain your reasoning, but do **write your answer in interval notation**. $(1 each)$

- ____________ $|x| \neq -1$.
- ____________ $x \geq |x|$.
- ____________ $|x - c| < \delta$.
- ____________ $0 < x - c < \delta$.
- ____________ $|f(x) - f(c)| < \epsilon$. Solve for $f(x)$.
- ____________ $2 - |x| < |x| + 2$.
- ____________ $3 - |x| < |x| + 2$.

23. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

____________ $f(x) = f'(x) \Rightarrow f(x)$ is increasing.

____________ $f(x)$ is bounded on $[a, b] \Rightarrow f(x)$ has a maximum value on $[a, b]$.

____________ $(A \subseteq B$ and $B \subseteq C) \Rightarrow A \subseteq C$.

____________ $(A \subseteq B$ and $C \subseteq B) \Rightarrow A \subseteq C$.

____________ $f(x)$ not continuous at c, k is a constant $\Rightarrow k \cdot f(x)$ is not continuous at c.

____________ If $0 < |x - c| < \delta \Rightarrow |k(x) - M| < 4$, then $0 < |x - c| < \delta/2 \Rightarrow |k(x) - M| < 4$.

____________ If $0 < |x - c| < \delta \Rightarrow |k(x) - M| < 4$, then $0 < |x - c| < \delta/2 \Rightarrow |k(x) - M| < 2$.

24. Prove that if $f(x)$ is increasing on an open interval (a, b) and differentiable at $c \in (a, b)$, then $f'(c) \geq 0$. (5 pts)

25. Give an example of a function $f(x)$ which is increasing on the open interval $(0, 1)$ and $f'(\frac{1}{2}) = 0$. (5 pts)

26. Give an example of a function which has positive derivative at every point in its domain but is not increasing. (5 pts)

27. (5 pts) Give an example of a function defined on $[0, 2]$ such that

- $f(0) < f(2)$
- and, there is no interval $(a, b) \subseteq [0, 2]$ such that $f(x)$ is increasing on (a, b).

28. (5 pts) Give an example of a function $g(x)$ such that

- $g(x)$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$
- and, $\lim_{x \to 0^+} g'(x)$ does not exist.

29a. (11 pts) Prove, using the δ, ϵ definition,

$$\lim_{x \to 4} -3x + 2 = -10.$$

scratchwork:

formal proof:

29b. (11 pts) Prove, using the definition,

$$\lim_{x \to 5} x^2 - 3x - 1 = 9.$$

scratchwork:

formal proof:

30. Use the "$h \to 0$" definition of derivative to show that $\frac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$. (4 pts)

31. Use the "$h \to 0$" definition of the derivative to prove Theorem #3 on the cover page. (4 pts)

32. Use the derivative theorems on the first page one at a time to calculate the following derivatives. Label each step with the corresponding number. Assume f is a differentiable function, with derivative $f'(x) = \frac{df}{dx}(x)$. (4 pts each)

a. $\frac{d}{dx}(x^2 + 1) =$

b. $\frac{d}{dx} f(2x) =$

c. $\frac{d}{dx}((f(x))^2 + x) =$

d. $\frac{d^2}{dx^2}\left(\frac{x+1}{x}\right) =$

33. Use the theorem on products of limits,

$$\lim_{x \to c}(f(x) \cdot g(x)) = (\lim_{x \to c} f(x)) \cdot (\lim_{x \to c} g(x)),$$

and the fact that differentiable functions are continuous, to prove the product rule for derivatives. (#5 on the cover sheet) Hints: Use the definition of derivative. At a certain point, add and subtract $f(x+h) \cdot g(x)$ in the numerator. (9 pts)