Final examination
Friday, 6-9-95, 8-10 AM
Math 133
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Name ______________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 144 points.

Here are some facts you might need:

Theorem 4.1 (Quotient Rule for Derivatives)
$g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Theorem 4.2 (the Inverse Function Theorem) If f is one-to-one, and differentiable at c with $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$, with derivative

$$(\frac{d}{dx}f^{-1})(f(c)) = \frac{1}{f'(c)}.$$

This is also written, when $x = f(c)$,

$$(\frac{d}{dx}f^{-1})(x) = \frac{1}{f'(f^{-1}(x))}.$$

Definition 4.3 A sequence a_n converges to L if $\forall \epsilon > 0$, $\exists N$ such that $n > N \implies |a_n - L| < \epsilon$.

Theorem 4.4 (Ordinary Comparison Test) Suppose $\exists N$ s.t. $n \geq N \Rightarrow 0 \leq a_n \leq b_n$. Then $\sum_{i=1}^{\infty} b_i$ converges $\Rightarrow \sum_{i=1}^{\infty} a_i$ converges.

1. Graph (on separate sets of coordinate axes) the following functions. Explicitly state their domain and range, and calculate all stationary points. (recall that this means finding points x such that $f'(x) = 0$.) (6 pts each)

- $f(x) = \ln(x)$.
- $g(x) = e^{-x^2}$.
- $h(x) = \frac{1}{\ln(x)}$.
- $j(x) = \tan(x)$ (on the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$).

2. Using the definitions of $\ln(x)$ and e^x, explain why e^x is a differentiable function. (2 pts)

3. Show, by computing a lower sum and an upper sum with four rectangles, that $0.9 \leq \ln(3) \leq 1.3$. (3 pts)

4. State the formula for Integration by Parts and prove it. (3 pts)

5. State the formal definition of $\sum_{i=1}^{\infty} c_i = S$. (hint: use the $\epsilon \ldots$ definition on the cover sheet. (2 pts)

5. State the formal definition of $\lim_{n\to\infty} a_n + b_n = L + M$. (2 pts)

6. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, prove $\lim_{n\to\infty} a_n + b_n = L + M$.
(hints: start with the definitions in the hypotheses, using $\frac{\epsilon}{2} > 0$ and the triangle inequality. and as usual, try to end up with the definition of the conclusion.) (5 pts)

7. State the definition of the following sentences a.–e. (3 pts each):

a. a_n is an injective sequence.

b. $x = f^{-1}(y)$. (Hint: 6 symbols)

c. The sequence a_n diverges to positive infinity. (the formal definition of the limit statement)

d. $L(\vec{x})$ is a linear operator.

e. $y = \cosh(x)$. (state the definition of this function in terms of e^x.)

8. Do the following calculations any way you want. (4 pts each)

a. $\displaystyle\int \frac{\sin(x)}{\cos(x)} dx =$

b. $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{x} =$

c. $f(x) = \dfrac{1}{\sqrt{x-3}} \implies f^{-1}(x) =$
(also state the domain and range of f^{-1}.)

d. $e^{\ln(x+y)-3\ln(y)} =$

e. $\log_{10}(x+4) + \log_{10} x = 1 \implies x =$

9. Use a trig identity to show $x = \sin(\theta) \implies \cot(\theta) = \frac{\sqrt{1-x^2}}{x}$. (1 pt)

10. Calculate the indefinite integral $\displaystyle\int \frac{\sqrt{1-x^2}}{x} dx$.

Hints: Use the substitution $x = \sin(\theta)$. One of the terms that should come up is $\int \csc(\theta) d\theta$, which equals $-\ln(\csc(\theta) + \cot(\theta))$. (7 pts)

11. Check your answer to #10 (by differentiating it). (2 pts)

12. (10 pts) Use the identity $\sec^2(\theta) - 1 = \tan^2(\theta)$ to calculate the indefinite integral

$$\int \frac{\sqrt{x^2-1}}{x} dx =$$

Fill in the blanks to complete the following theorems. (2 pts each for #13–15)

13. **Theorem 4.5** (L'Hôpital's Rule for forms of type ∞/∞). Suppose that $\lim_{x\to a}|f(x)| =$ ____________ $=$ ____________. If $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists,

then:________________________.

14. **Theorem 4.6** (Limit Comparison Test) Suppose $a_n \geq 0$, $b_n > 0$, and $\lim_{n\to\infty}\dfrac{a_n}{b_n} = L$.

If __________________, then ($\sum_{i=1}^{\infty} a_i$ converges $\iff$ $\sum_{i=1}^{\infty} b_i$ converges).

If __________________, then ($\sum_{i=1}^{\infty} b_i$ converges $\Rightarrow$ $\sum_{i=1}^{\infty} a_i$ converges).

15. **Theorem 4.7** (The Integral Test) Let $f(x)$ be a continuous, positive, weakly decreasing function on $[1, \infty)$, and suppose $a_k = f(k)$ for $k \in \mathbb{N}$. Then

____.

(4 pts each for #16–18) 16. Show that $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2 - 0.1}$ diverges, by comparison with the harmonic series.

17. Use the Limit Comparison Test to show that $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^3 - n - 1}$ converges.

18. Use the Ratio Test to determine whether $\displaystyle\sum_{k=1}^{\infty}\frac{3^k}{(k!)^2}$ converges or diverges.

19. True or False. Write **T** or **F** next to each sentence. You need not explain your answer.$^{(1\ each)}$

The following questions assume that a_n and b_n are sequences, satisfying the hypothesis of the Ordinary Comparison Test (Thm. 4.4 on the cover page): $0 \le a_n \le b_n$ for large enough n. "$\sum$" will be used to abbrevate $\sum_{i=1}^{\infty}$. k is a constant.

______ If $\sum a_i$ converges, then so does $\sum(-1)^i a_i$.

______ If $\sum(-1)^i a_i$ converges, then so does $\sum a_i$.

______ If $\sum a_i$ converges, then so does $\sum(ka_i)^2$.

______ If $\sum a_i$ converges, then so does $\sum a_{(i^2)}$.

______ $\sum(-1)^i$ converges.

______ $\sum\left((-1)^i + (-1)^{i+1}\right)$ converges.

______ If $\lim_{n\to\infty} a_n = 0$, then $\sum a_i^2$ converges.

______ If $\sum a_i$ diverges, then (a_n) is an increasing sequence.

______ If $h(x)$ and $f(x)$ are two solutions of the differential equation $y'' = -y$, then so is $(h+f)(x)$.

______ If $h(x)$ is a solution of the differential equation $y'' = -y$, then so is $k \cdot h(x)$.

______ If $h(x)$ is a solution of the differential equation $(y')^2 = y$, then so is $k \cdot h(x)$.

______ $\tan(x)$ is increasing on the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$.

______ $\tan(x)$ is increasing on the domain $(-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$.

______ $\int_0^1 x^{-2}$ converges.

______ $\int_1^\infty x^{-2}$ converges.

______ $\int_0^1 x^{-1/2}$ converges.

______ $\int_1^\infty x^{-1/2}$ converges.

______ $\int_1^\infty f(x)$ converges $\implies \lim_{x\to\infty} f(x) = 0$.

______ $\lim_{x\to 0^+} \sqrt{x}\ln(x) = 0$.

(2 pts): Calculate the following indefinite integral using integration by parts: $\int e^x \cos(x)dx =$.

(The following True/False are optional and non-graded. Do not waste time giving any thought to your answers.)

______ Math was my most difficult subject this year.

______ Math was my most interesting subject this year.

______ The answer to this question is **F**.

______ This course would be better if computers were involved.

______ Tutorials were useful.

______ Having class at 9:30 is too early in the day.

______ Having class at 9:30 is too late in the day.

20. Here are a list of axioms and theorems about vectors. a and b will denote scalars (real numbers) and $\vec{x}$ and $\vec{y}$ will denote vectors. Scalar multiplication is written as $a\vec{x}$.

A. Distributive law #1: $(a+b)\vec{x} = a\vec{x} + b\vec{x}$.

B. Distributive law #2: $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y}$.

C. Scalar Identity: $1\vec{x} = \vec{x}$.

D. Associative law for Vector Addition: $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$.

E. **Theorem 4.8** $\vec{x} + \vec{y} = \vec{x} + \vec{z} \implies \vec{y} = \vec{z}$.

F. **Theorem 4.9** $\vec{y} + \vec{x} = \vec{z} + \vec{x} \implies \vec{y} = \vec{z}$.

The following proof demonstrates that the "commutative law" for vector addition is actually a theorem, following as a consequence of the other axioms for vector algebra. The idea is to use the two Distributive laws to expand a product in two different ways. Each step is numbered; after the number, write the letter of the axiom or theorem that justifies the step. I've filled one in already. (1 pt each)

$$\begin{aligned}
(1+1)(\vec{x}+\vec{y}) &= (1+1)\vec{x} + (1+1)\vec{y} && (1)\\
&= (1\vec{x} + 1\vec{x}) + (1\vec{y} + 1\vec{y}) && (2)\\
&= (\vec{x} + \vec{x}) + (\vec{y} + \vec{y}). && (3)
\end{aligned}$$

Also, (expanding the other way)

$$\begin{aligned}
(1+1)(\vec{x}+\vec{y}) &= 1(\vec{x}+\vec{y}) + 1(\vec{x}+\vec{y}) && (4)\\
&= (1\vec{x} + 1\vec{y}) + (1\vec{x} + 1\vec{y}) && (5)\\
&= (\vec{x} + \vec{y}) + (\vec{x} + \vec{y}) && (6)
\end{aligned}$$

Since we started with the same thing both times, the end results are equal:

$$\begin{aligned}
(\vec{x}+\vec{x}) + (\vec{y}+\vec{y}) &= (\vec{x}+\vec{y}) + (\vec{x}+\vec{y}) \\
\vec{x} + (\vec{x} + (\vec{y}+\vec{y})) &= \vec{x} + (\vec{y} + (\vec{x}+\vec{y})) && (7)\\
\vec{x} + (\vec{y}+\vec{y}) &= \vec{y} + (\vec{x}+\vec{y}) && (8)\\
(\vec{x}+\vec{y}) + \vec{y} &= (\vec{y}+\vec{x}) + \vec{y} && (9)\\
\vec{x}+\vec{y} &= \vec{y}+\vec{x}, && (10)
\end{aligned}$$

which is the commutative law we wanted to show.