Final Exam
Wednesday, 3-15-95 8:00-10:00 AM
Math 132
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Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 120 points out of a quarter total of 369.

Here are some formulas you might need:

$$[\![t]\!] = \text{greatest integer less than } t \tag{1}$$

$$n \neq 0 \Rightarrow \frac{d}{dx}x^n = nx^{n-1} \tag{2}$$

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x) \tag{3}$$

$$g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \tag{4}$$

$$\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x) \tag{5}$$

$$\int_a^b k \cdot f(x)dx = k\int_a^b f(x)dx \tag{6}$$

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \tag{7}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{8}$$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \tag{9}$$

Write your name on each page.

1. (5 pts each)
State the definition of "$f(c)$ is the maximum value of f on the set S."

State the definition of "$f(c)$ is a local maximum value of $f(x)$."

State the definition of "$f(x)$ is integrable on $[a, b]$."$^{(2)}$

State the definition of "$f(x)$ is increasing on the set S."

State the definition of "$f(x)$ is concave down on the interval (a, b)."

2. Use the Max/Min Existence theorem and the Critical Point theorem to prove the following special case of Rolle's Theorem: (Hint: a sketch might suggest a proof.) (5 pts)

Theorem 4.1 Suppose $f(x)$ is continuous on the closed interval $[a, b]$, and is differentiable on (a, b). Suppose also that $f(x) \geq 0$ on $[a, b]$, $f(x)$ is not constant, and $f(a) = f(b) = 0$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

3.State the following four theorems. (Use the reverse side of one of these pages.)(4 pts each)

- Max/Min Existence Theorem
- Critical Point Theorem
- Fundamental Theorem of Calculus
- Mean Value Theorem

4. Prove one of the above theorems. (6 pts)

5. State and Prove the Monotonicity Theorem. (Use either MVT or FTC) (6 pts)

6. (3 pts) The function $h(t)$ is defined on the domain $[0, \infty)$ as follows:

$h(t) = 1$ on the intervals $[1, 2)$, $[3, 4)$, $[5, 6)$, $[7, 8)$, etc., where the left endpoint is a positive odd integer.

$h(t) = 0$ on the remaining intervals $[0, 1)$, $[2, 3)$, $[4, 5)$, $\ldots$, where the left endpoints are even.

Sketch a graph of $h(t)$.

Using the previous $h(t)$, define, on the domain $[0, \infty)$, the function $A(x)$ by

$$A(x) = \int_0^x h(t)dt.$$

Sketch a graph of $A(x)$. Hint: **Theorem 4.2** If $h(t)$ is continuous at c, then $A(x)$ is differentiable at c, with derivative $A'(c) = h(c)$.

Another hint: $A(3.5) = 1.5$. Plot this point.

7. Consider the function $f(x) = 2x + 1$ on the interval $[3, 5]$. Choose a partition P dividing $[3, 5]$ into four subintervals, and choose four sample points x_i^* from these subintervals. Then calculate the Riemann Sum of $f(x)$ on $[3, 5]$ using your partition. Hint; By the FTC, $\int_3^5 f(x)dx = 18$, so your answer should be a number close to this. (6 pts)

8. True or False. Write **T** or **F** next to each sentence. You need not explain your answer.$^{(2\ each)}$

In the following questions, A, B, and $A \cap B$ ("A intersect B") refer to subsets of the coordinate plane, each of which is in the domain of the Area function. k is a constant real number, n is a positive integer.

______ $f(x)$ differentiable on $\mathbb{R} \implies f(x)$ integrable on $[a, b]$.

______ $f(x)$ integrable on $[a, b] \implies f(x)$ differentiable on (a, b).

______ $A \neq B$ AND $A \subseteq B \implies \text{Area}(A) < \text{Area}(B)$.

______ $f(c)$ is the maximum value of $f(x) \implies -f(c)$ is the minimum value of the function $-f(x)$.

______ $f(c)$ is the maximum value of $f(x) \implies kf(c)$ is the maximum value of the function $kf(x)$.

______ $\text{Area}(A \cap B) > 0 \implies \text{Area}(A \cup B) < \text{Area}(A) + \text{Area}(B)$.

______ $\left(\sum_{i=1}^{5} a_i\right) + \left(\sum_{i=3}^{7} b_i\right) = \sum_{i=1}^{5}(a_i + b_{i+2})$

______ $\sum_{i=1}^{n} a_{(i^2)} = \sum_{i=1}^{(n^2)} a_i$

______ $\lim_{x\to\infty} f(x) = \infty$ AND $\lim_{x\to\infty} g(x) = -\infty \implies \lim_{x\to\infty} (f+g)(x) = 0$.

9. Sketch (on another page) a graph of the following function:

$$f(x) = \frac{x^2 - 1}{x^3}.$$

Indicate local max/min, and horizontal and vertical asymptotes. Hint: Finding $f'(x)$ is easier if you write $f(x)$ as a sum of fractions. Another hint: Before graphing, figure out when $f(x)$ is positive, negative, and zero. (15 pts)

10. One of the hypotheses in FTC is that the antiderivative is continuous on a closed interval $[a, b]$. Explain why (theoretically or by example) requiring only that the antiderivative be differentiable on the open interval (a, b) is insufficient to guarantee a correct calculation of a definite integral. (3 pts)

11. Consider $f(x) = \begin{cases} x^{-2/3} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$. This function is integrable, but not bounded! Use FTC to calculate $\int_0^8 f(t)dt$. (Hint: the function $F(x) = 3x^{1/3}$ satisfies the hypotheses of FTC. This shows that $F(x)$ does not have to have a derivative at the endpoint 0, even though it is right-continuous there.) (2 pts)

12. Do the following calculations any way you want, using a different page for scratchwork. Clearly indicate your answer on this page. (3 pts each)

- $\displaystyle\sum_{i=1}^{100} i$
- $\displaystyle\left(\sum_{i=0}^{100} i(i-1)(i-100)\right) - \left(\sum_{i=2}^{99} i(i-1)(i-100)\right)$
- $\displaystyle\int \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} dx$
- $\displaystyle\int_{-2}^{2} \llbracket x \rrbracket \, dx$
- $\displaystyle\int \frac{-1}{\sqrt{-x}} dx$