Final Examination
Monday, 12-12-94 8-10 AM
Math 131
Adam Coffman

Name ______________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 130 points out of a quarter total of 400.

Hypotheses for theorems 1–7:
k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

1. (4)Suppose $f(x)$ is differentiable on an open interval containing c. State the "$h \to 0$" and "$x \to c$" definitions of $f'(c)$.

2. (4)Use one of these definitions and the sum of limits theorem to prove derivative rule #4.

3. (4)Use (p. 1) rules #3, #4 to prove $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$.

4. (4)Use the $h \to 0$ definition of the derivative (not theorems $\#1 - \#7$) to show that, if $f(x) = 3x^2 + x + 1$, then $f'(4) = 25$.

5. (4)Assume that f, g, h are functions that are differentiable on $\mathbb{R}$ and that their derivatives are also differentiable. What is the derivative of $f(g(h(x)))$, and also what is its second derivative?

8. Compute the following derivatives any way you want. You don't have to explain your reasoning. Write either a function value after the "=," or "**DNE**" if the derivative Does Not Exist. Some of the questions refer to the following functions. $^{(2\ each)}$

$$r(x) = \frac{1}{x} \qquad\qquad s(x) = x^2 + 1$$

.

$t(x)$ is an unknown but differentiable function with domain $\mathbb{R}$

.

- $(r \cdot s)'(2) =$
- $(r \circ s)'(2) =$
- $(s \circ r)'(2) =$
- $(t \circ r)'(2) =$
- (the second derivative:) $(r + s)''(2) =$

9. Compute the following limits any way you want. You don't have to explain your reasoning. Write either a number after the "=," or "**DNE**" if the limit Does Not Exist. $^{(2\ each)}$

- $\lim_{x \to 5} mx + b =$
- $\lim_{x \to 0} |x| =$
- $\lim_{x \to 0} \frac{|x|}{x} =$
- $\lim_{x \to 0^+} \frac{|x|}{x} =$
- $\lim_{x \to -1} \frac{x^2 - 1}{x + 1} =$

10. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

____________ $\lim_{x \to c} f(x) = L \Rightarrow \lim_{x \to c}(1 + f(x)) = 1 + L$.

____________ $\lim_{x \to c} f(x) = L \Rightarrow f(c) = L$.

____________ $f(c) = L \Rightarrow \lim_{x \to c} f(x) = L$.

____________ $f(x)$ continuous at $c \Rightarrow f(x)$ differentiable at c.

____________ $f(x)$ differentiable at $c \Rightarrow f(x)$ continuous at c.

11. $^{(3)}$State the ϵ/δ defintion of $\lim_{x \to c} f(x) = L$.

12. $^{(1)}$State the ϵ/δ defintion of $\lim_{x \to c^+} f(x) = L$.

13. $^{(3)}$State the $\lim_{x \to c}$ definition of $f(x)$ is continuous at c.

14. $^{(10)}$Use the ϵ/δ definition of limit to show that $f(x) = 3x + 1$ is continuous at $c = 4$.

Scratchwork:

Formal Proof:

15. Recall the definition of absolute value:
For $x \in \mathbb{R}$,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

(2)Explain why $x \leq |x|$.

16. (3)Show, by taking a $h \to 0^+$ and a $h \to 0^-$ limit, that $|x|$ is not differentiable at $c = 0$.

17. (2)State the Triangle Inequality for real numbers.

18. (1 *each*)Graphing two functions on the same x-y axes might help solve the following equalities. (find all solutions x.)

a. $|x| = x$.

b. $|x| = 1 + \frac{x}{2}$.

c. $|x| - 2 = 1 - |x|$.

19. (2)What is the distance between the ordered pairs $(3, 5)$ and $(4, -1)$?

20. Solve the following inequalities, that is, find the subset of $\mathbb{R}$ consisting of elements x with the required property or properties. You don't have to explain your reasoning, but do **write your answer in interval notation**. (*2 each; I will count your best 8*)

- ____________ $x + 4 < 9$.

- ____________ $6 < -3x \leq 9$.

- ____________ $x^2 \leq 16$. Hint: $x = -5$ doesn't have this property.

- ____________ $x^4 > -1$.

- ____________ $0 < |x - c| < \delta$.

- ____________ $|f(x) - L| < \epsilon$. Solve for $f(x)$.

- ____________ $x < 3$ AND $x \in (2, 4]$.

- ____________ $x \in (-1, 3)$ OR ($x \in (2, 4]$ AND $x \neq 2.5$).

- ____________ ($x \in (-1, 3)$ OR $x \in (2, 4]$) AND $x \neq 2.5$.

21. This question gives a careful proof of the alternate triangle inequality, $|a| - |b| \leq |a - b|$.

Here is a list of arguments used to justify statements. Some of them might get used twice in the proof below, or not at all; write the letter from the list next to the step that it justifies.(1 each)

A. Definition of subtraction: $x - y = x + (-y)$.

B. Distributive Law.

C. Associative Law for $+$.

D. Commutative Law for $+$.

E. Additive identity: $x + 0 = x$.

F. Triangle Inequality.

G. Theorem: $x \leq y \Rightarrow x - z \leq y - z$.

H. Additive inverse: $x + (-x) = 0$.

I. Definition of Absolute Value.

PROOF:

$\underline{\qquad\qquad\qquad}$ $|a| = |a + 0|$

$\underline{\qquad\qquad\qquad}$ $|a| = |a + (b + (-b))|$.

$\underline{\qquad\qquad\qquad}$ $|a| = |a + ((-b) + b)|$.

$\underline{\qquad\qquad\qquad}$ $|a| = |(a + (-b)) + b|$.

$\underline{\qquad\qquad\qquad}$ $|a| = |(a - b) + b|$.

$\underline{\qquad\qquad\qquad}$ $|a| \leq |a - b| + |b|$.

$\underline{\qquad\qquad\qquad}$ $|a| - |b| \leq |a - b|$.

$\blacksquare$

22. Pierre is my cat. Evidence suggests that the mass in grams of Pierre's fur coat is given interms of the celsius temperature H by the function $M(H) = 50 - H^2/30$.

$^{(2)}$Graph this function with domain $0 \leq H \leq 30$ on the horizontal axis.

$^{(2)}$Define Pierre's *shedding rate* by the formula $-\frac{d}{dt}M$. Suppose the temperature H is a function of time t, so that $M(H(t)) = 50 - (H(t))^2/30$. Find Pierre's shedding rate using the Chain Rule.

$^{(3)}$Suppose t is measured in months, and that the temperature in the Midwest is given by $H(t) = 15(1 - \cos(\frac{\pi \cdot t}{6}))$. Use the fact that $\frac{d}{du}\cos(u) = -\sin(u)$ and the Chain Rule to calculate $\frac{d}{dt}H(t)$ and the shedding rate.

(circle one) $^{(1)}$ Would Pierre shed more in Chicago, or in a place like Hawaii where the temperature is constant?

23. $^{(2)}$The slope of the hypoteneuse of the following right triangle is m. The length of the horizontal segment is 3. Find the the length of the other two sides in terms of m.