Name ________________________ Student ID# ____________

Final Exam. Thursday, 6-12-97 4-6PM. Math 102. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. This test is worth 150 points out of a quarter total of 450.

There are no questions on this page

θ radians	θ degrees	$\sin(\theta)$
0	0	0
$\pi/6$	30	$1/2$
$\pi/4$	45	$\sqrt{2}/2$
$\pi/3$	60	$\sqrt{3}/2$
$\pi/2$	90	1
$7\pi/12$	105	$\frac{1}{2}\sqrt{2+\sqrt{3}}$
π	180	0

Definition 1.1 $\sin^2(\theta)$ is an abbreviation for $(\sin(\theta))^2$, and there are similar abbreviations for the other trig functions: cos, tan, cot, sec, csc.

Theorem 1.2 $\sin^2(\theta) + \cos^2(\theta) = 1$.

Theorem 1.3 $\cos(\theta) = \sin(90^\circ - \theta)$. ($\theta$ degrees)

Theorem 1.4 $\sin(\theta) = \cos(90^\circ - \theta)$. ($\theta$ degrees)

Theorem 1.5 $\sin(-\theta) = -\sin(\theta)$.

Theorem 1.6 $\cos(-\theta) = \cos(\theta)$.

Theorem 1.7 $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$.

Theorem 1.8 $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$.

$\mathbb{N}$ refers to the set of natural numbers $\{1, 2, 3, 4, 5, 6, 7, \ldots\}$.

Theorem 1.9 For sequences a_k and b_k,

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Theorem 1.10

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Theorem 1.11 For $r \neq 0$ and $r \neq 1$,

$$\sum_{k=1}^{n} r^{(k-1)} = \frac{1 - r^n}{1 - r}.$$

Questions 1, 2, 3 are derivations of trigonometric identities. Show all work, using the page 1 theorems *one at a time*. (3 pts each) Here's an example:

Example 1.12 Show $\cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$.

$$\begin{aligned} \cos^2(\theta) - \sin^2(\theta) &= (1 - \sin^2(\theta)) - \sin^2(\theta) \quad (Thm.\ 1.2) \\ &= 1 - 2\sin^2(\theta) \end{aligned}$$

1. Use Theorem 1.8 to derive the "double-angle" formula for $\sin(2\theta)$.

2. Derive the "cosine of a difference" formula for $\cos(\alpha - \beta)$.

3. Show $\sin(180° - \theta) = \sin(\theta)$ for θ degrees. Hint: start with

$$\sin(180° - \theta°) \quad = \quad \sin(90° - (\theta - 90°))$$

Questions 4–8 refer to triangle $\triangle ABC$ with (positive) side lengths a, b, c. (2 pts each)

4. State the definition of the "semiperimeter" s.

5. State Heron's formula for the area of $\triangle ABC$ in terms of s, a, b, c.

6. If γ is the angle $\angle ACB$, state the "Law of Cosines" formula for c^2 in terms of a, b, and γ.

$c^2 =$

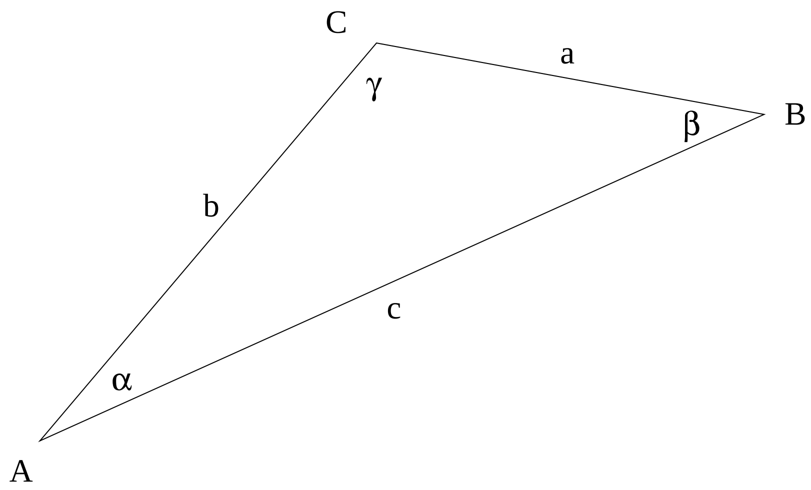
7. Solve the "Law of Cosines" formula for γ in terms of a, b, and c.

$\gamma =$

8. $\triangle XYZ$ is called "similar to $\triangle ABC$" if its side lengths x, y, z are constant multiples (k) of the side lengths of $\triangle ABC$, that is: $x = ka$, $y = kb$, and $z = kc$.

Calculate and simplify:

- $\frac{\text{perimeter } XYZ}{\text{perimeter } ABC} =$
- $\frac{\text{area } XYZ}{\text{area } ABC} =$

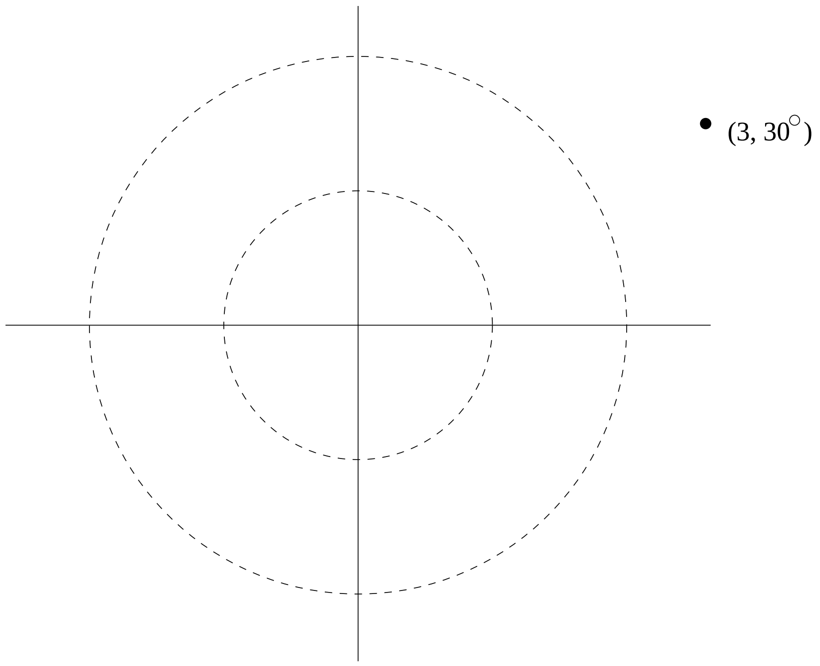


9. (3 pts) In this triangle, γ is an obtuse angle. Use the definition of the sine function to prove the Law of Sines:

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b}.$$

10. (1 pt each) Suppose, in the above triangle, $\beta = 45°$ and $\gamma = 105°$ and $a = 10$. Calculate $\angle\alpha$ and side lengths b and c exactly (no decimals, and not necessarily simplifying fractions):

11. (2 pts) Suppose, in the above triangle, $\gamma = 135°$, $a = 8$, and $b = 6$. Use the Law of Cosines to calculate side length c exactly (not necessarily simplifying fractions or radicals. Hint: for obtuse γ, $\cos\gamma < 0$).



12. On the above picture are the circles of radius 1 and 2 centered at the origin of the polar coordinate system. I've plotted the point $(r, \theta) = (3, 30°)$. Plot and label the following points. Also, below, convert to rectangular coordinates. (1.5 pts each)

- $(r, \theta) = (2, -60°)$. $(x, y) =$
- $(r, \theta) = (2.5, 0°)$. $(x, y) =$
- $(r, \theta) = (1, 210°)$. $(x, y) =$
- $(r, \theta) = (\sqrt{2}, 5\pi/4 \text{ radians})$. $(x, y) =$
- $(r, \theta) = (1.5, 5\pi \text{ radians})$. $(x, y) =$
- $(r, \theta) = (0, \pi/7 \text{ radians})$. $(x, y) =$

13. (1 pt) If the rectangular coordinates (x, y) are given, state the formula for conversion to polar coordinates:

$r =$

$\theta =$

Questions 14–16 refer to the number $s \in \mathbb{R}$ and two vectors $\vec{v}$, $\vec{w} \in \mathbb{R}^2$ with rectangular coordinates $\vec{v} = (x, y)$ and $\vec{w} = (a, b)$.

14. (1 pt) If the angle between $\vec{v}$ and $\vec{w}$ is θ, state the definition of the dot product $\vec{v} \cdot \vec{w}$.

15. (1 pt) State the definition of "$\vec{v}$ and $\vec{w}$ are orthogonal."

16. (1 pt each) In terms of the numbers x, y, a, b, s, state formulas for:

- additive inverse: $-\vec{v} =$
- vector sum: $\vec{v} + \vec{w} =$
- scalar multiplication: $s\vec{v} =$
- dot product: $\vec{v} \cdot \vec{w} =$
- complex product: $(x + iy)(a + ib) =$

17. Calculate the following numbers exactly, with angles in radians (unmarked θ) or degrees ($\theta°$). You do not have to simplify fractions or radicals in your answer. (1 pt each, best 8 of 10)

$$
\begin{aligned}
\sin(120°) &= \underline{\hspace{3cm}} \\
\sin(\frac{3\pi}{4}) &= \underline{\hspace{3cm}} \\
\sin^{-1}(\frac{1}{2}\sqrt{2+\sqrt{3}}) &= \underline{\hspace{3cm}} \ \textit{radians} \\
\sin(210°) &= \underline{\hspace{3cm}} \\
\cos(210°) &= \underline{\hspace{3cm}} \\
\cos(15°) &= \underline{\hspace{3cm}} \\
\sin(52.5°) &= \underline{\hspace{3cm}} \\
\sin(41°)\sin(4°) - \cos(41°)\cos(4°) &= \underline{\hspace{3cm}} \\
\sin(\pi/3) + \sin(\pi/6) &= \underline{\hspace{3cm}} \\
2\sin(\pi/12)\cos(\pi/12) &= \underline{\hspace{3cm}}
\end{aligned}
$$

18. (1 pt) State the definition of $\sin^{-1}(x) = y$.

19. (2 pts) Sketch a graph of $y = \sin^{-1}(x)$, indicating its domain and range.

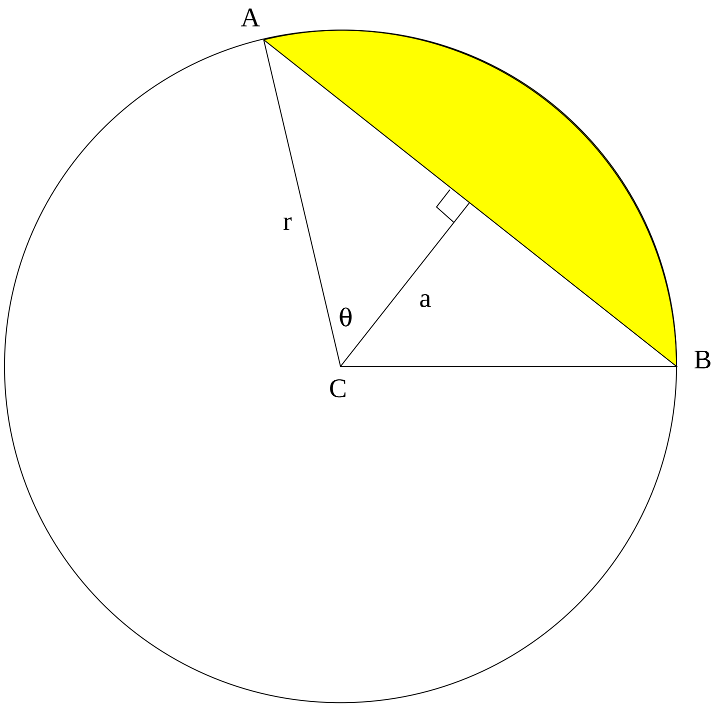
20. (1 pt) State the definition of $\cos^{-1}(x) = y$.

21. (2 pts) Sketch a graph of $y = \cos^{-1}(x)$, indicating its domain and range.

22. (1 pt) State the definition of $\tan^{-1}(x) = y$.

23. (2 pts) Sketch a graph of $y = \tan^{-1}(x)$, indicating its domain and range.

24. (2 pts each) In the following circle with center C and radius $r > 0$, assume θ is an acute angle.



a. Calculate length AB using the Pythagorean theorem (twice).

b. Using an inverse trig function, express θ in terms of only a and r.

c. Recall that the area of the shaded region is $r^2(\cos^{-1}(\frac{a}{r}) - \frac{a}{r}\sqrt{1-(\frac{a}{r})^2})$. Calculate the area when $a = 0$.

25. True or False. Write **T** or **F** next to each sentence. You need not explain your answer.$^{(1\ each)}$

The first five questions are about polar coordinates with radius r and polar angle θ.

_________ $(r, \theta°) = (-r, -\theta°)$.

_________ $(r, \theta°) = (-r, \theta° + 180°)$.

_________ The scalar multiplication formula is $s(r, \theta) = (sr, s\theta)$.

_________ The scalar multiplication formula is $s(r, \theta) = (sr, \theta)$.

_________ The scalar multiplication formula is $s(r, \theta) = (sr, s + \theta)$.

_________ Every complex number has two different square roots which are also complex numbers.

_________ Every nonzero complex number has two different square roots which are also complex numbers.

_________ (scalars s, t and vectors $\vec{v}$, $\vec{w}$): $s\vec{v} + t\vec{w} = (s + t)(\vec{v} + \vec{w})$.

The variables α, β, and θ are always angles in radians.

___________ $\cos^2(\alpha/2) = (1 + \cos(\alpha))/2$.

___________ $\cos(\alpha/2) = \left|\sqrt{(1 + \cos(\alpha))/2}\right|$.

___________ $|\cos(\alpha/2)| = \sqrt{(1 + \cos(\alpha))/2}$.

___________ $\tan(-\theta) = \tan(\theta)$ for $\theta \in$ domain of tan.

___________ $\sin(\alpha)\sin(\beta) = \sin(\frac{\alpha+\beta}{2})$.

___________ $\sin^4(\theta) - 2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta) = \cos^2(2\theta)$.

___________ $\sin^4(\theta) + 2\sin^2(\theta)\cos^2(\theta) + \cos^4(\theta) = \cos^2(2\theta)$.

____________ A triangle with side lengths 3, 4, 5 must be a right triangle.

____________ A triangle with one side twice as long as another must be a 30°, 60°, 90° triangle.

In an isosceles triangle $\triangle ABC$, lengths AC and BC are equal, and side AB is divided into three equal segments AD, DE, EB. True or false, continued:

____________ Area $\triangle ACD =$ Area $\triangle DCE$.

____________ Perimeter $\triangle ACD =$ Perimeter $\triangle DCE$.

____________ $\angle CDB = \angle CEA$.

____________ $\tan \angle CDB = 3\sqrt{3}$.

m and n are natural numbers, and a_k and b_k are sequences (also assume a_0 is defined).

__________ $\displaystyle\sum_{k=1}^{n}(a_k)^2 = \left(\sum_{k=1}^{n} a_k\right)^2.$

__________ $\displaystyle\sum_{k=1}^{n} a_{(k-1)} = \sum_{k=0}^{n-1} a_k.$

__________ $\displaystyle\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} a_{(n-k)}.$

__________ $\displaystyle\sum_{k=1}^{n}(a_k - a_{(k-1)}) = a_n - a_0.$

__________ $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n.$

__________ $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = n^2.$

26. By giving an example of two sequences a_k and b_k show that the following statements about sums are False. Calculate both and get different answers:

First, pick two sequences (specify at least the first five entries):

- $\{a_k\} =$
- $\{b_k\} =$

2 pts each:

a. $\displaystyle\sum_{k=1}^{4}(a_k \cdot b_k) = \left(\sum_{k=1}^{4} a_k\right) \cdot \left(\sum_{k=1}^{4} b_k\right).$

b. $\displaystyle\sum_{k=2}^{5} a_{(k-1)} = \left(\sum_{k=2}^{5} a_k\right) - 4.$

c. $\displaystyle\sum_{k=1}^{3} a_k + \sum_{k=1}^{2} b_k = \sum_{k=1}^{5}(a_k + b_k).$

27 By drawing and labeling a right triangle, show that $\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1+x^2}}$ for $x \geq 0$. (2 pts)

(#28–38 2 pts each)

28. a. For $n \in \mathbb{N}$, state the definition of the factorial

$$n! =$$

b. Also, the factorial of zero is defined to be: $0! =$

29. For $n \in \mathbb{N}$ and $j \in \mathbb{N}$ with $j < n$, state the definition of the binomial coefficient

$$\binom{n}{j} =$$

30. The "Principle of Mathematical Induction" is used to prove statements S_1, S_2, S_3, … , about natural numbers. State the two steps in such a proof:

Step 1:

Step 2:

31. State the definition of a sequence.

32. By writing out the first several terms, give an example of an arithmetic sequence with common difference 2:

33. By writing out the first several terms, give an example of a geometric sequence with common ratio 3:

34. If a_k is a sequence, state the definition of the series

$$\sum_{k=1}^{n} a_k =$$

35. Let c be a constant. Use the definition of the Sigma notation to show that

$$\sum_{k=1}^{n} (c \cdot a_k) = c \cdot \left(\sum_{k=1}^{n} a_k \right).$$

36. Here are the first few terms of the "Fibonacci sequence" F_n. Write down the next five:

$$1, 1, 2, 3, 5,$$

37. Here are the first three rows of "Pascal's triangle." Write down the next five rows.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

38. If a_1, a_2, ... , a_{10} are ten numbers, state the formula for their average using Sigma notation.

39. Here are some "recursive formulas" for sequences. Write down the first five ($k = 1, 2, 3, 4, 5$) terms of the sequence. (I've already written the first one.) Some of the expressions may be complicated fractions or exponents, which you need not simplify. (1.5 pts each)

- $\begin{cases} a_1 = 10 \\ a_k = \frac{1}{5} a_{(k-1)} \end{cases}$
- $\begin{cases} a_1 = 2 \\ a_k = a_{(k-1)} + k \end{cases}$
- $\begin{cases} a_1 = 1 \\ a_k = (a_{(k-1)}) + (a_{(k-1)})^2 \end{cases}$
- $\begin{cases} a_1 = 3 \\ a_k = -(a_{(k-1)}) \end{cases}$
- $\begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_k = a_{(k-2)} \cdot a_{(k-1)} \end{cases}$

40. Use the definition of the Sigma notation (not the theorems from page 1) to calculate the following numbers. Simplify your answer. (1 pt each)

- $\displaystyle\sum_{k=1}^{4} 2k - 1 =$
- $\displaystyle\sum_{i=2}^{5} 2^i =$
- $\displaystyle\sum_{b=1}^{5} (-1)^b \cdot b^2 =$
- $\displaystyle\sum_{k=1}^{30} \sin(k\pi) =$
- $\displaystyle\sum_{k=1}^{30} \cos(k\pi) =$

- $\sum_{c=10}^{16}(13-c) =$

41. (1 pt) Expand $(x+y)^6 =$

42. (2 pts) Prove the following statement about $n \in \mathbb{N}$ by mathematical induction:

$$n^2 - n \geq 0.$$

43. (1.5 pts each) Use the Theorems from page 1 to calculate the following sums. You do not have to show your work.

- $\sum_{k=2}^{100} k =$
- $\sum_{k=1}^{19}(2k+1) =$
- $\sum_{k=1}^{10} 2^{k-1} =$
- $\sum_{k=1}^{10}(-1)^k =$
- $\sum_{i=1}^{91} 10^{(1-i)} =$

44. A frog is at the bottom of a thirty meter well. Each day, it can climb up three meters, but each night it slides back down two meters. How many days will it take the frog to escape the well?