Name ________________________ Student ID# ____________

Final Exam. Thursday, 3-20-97 4:00-6:00. Math 101. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 154 points out of a quarter total of 460.

There are no questions on this page

Definition 1.1 $\mathbb{R}^2$ is the set $\{(x,y) : x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$, of "ordered pairs" of real numbers.

Definition 1.2 The x-axis is a subset of $\mathbb{R}^2$, defined by

$$\{(x,y) \in \mathbb{R}^2 : y = 0\}.$$

The y-axis is the set

$$\{(x,y) \in \mathbb{R}^2 : x = 0\}.$$

Definition 1.3 A subset $G \subseteq \mathbb{R}^2$ is "the graph of a function" if the following implication is true for any points (x_1, y_1) and (x_2, y_2) in the set G:

$$x_1 = x_2 \implies y_1 = y_2.$$

Definition 1.4 A "function" is the rule that assigns to the number x the number y such that (x,y) is on the graph.

There can't be two numbers y that get assigned to x by the definition of "graph of a function." However, there may be some $x \in \mathbb{R}$ that are not assigned a number y.

Definition 1.5 The "domain" of a function with graph G is the set $\{x \in \mathbb{R} : \text{there is a point } (x,y) \in G\}$.

Recall the positive integers, or natural numbers, are elements of the set $\{1, 2, 3, 4, 5, \dots\}$.

The "unit circle" is the circle with center $(0,0)$ and radius 1.

Recall π is the constant $3.14159265\dots$.

The circumference of a circle with radius r is $2\pi r$.

The letter "theta" is written as θ and I'll use it to denote some angles.

(#1-11 1 pt each)

1. State the equation of the vertical line through the point (c, d).

2. For the parabola $y = a(x-h)^2 + k$, $a \neq 0$, state the equation of the axis line, and the coordinates of the vertex.

3. For the parabola $y = ax^2 + bx + c$, state the equation of the axis line.

4. State the definition of "r is a root of the polynomial $x^3 + ax^2 + bx + c$."

5. State the formula for the x-intercepts of the quadratic $y = ax^2 + bx + c$.

6. Under what condition on the coefficients a, b, c can the polynomial $ax^2 + bx + c$ be factored as $a(x-h)^2$?

7. State the definition of "$R(x)$ is a rational function."

8. In terms of a center (p, q) and radius $r > 0$, state the geometric definition of a circle by completing the sentence,
"A circle is the set of points (x, y) such that ... "

9. In terms of a focus $F = (q,p) \in \mathbb{R}^2$ and a directrix line $D = \{y = d\} \subseteq \mathbb{R}^2$, state the geometric definition of a parabola $P \subseteq \mathbb{R}^2$ by completing the sentence,

"A parabola is the set of points (x, y) such that … "

10. If b is a positive integer, state the definition of x^b.

11. State the definition of "$\log_b x = y$."

12. (2 pts) State the definition of the function "$\ln(x) = y$." Also state its domain.

13. (1 pt) State the definition of "angle A is acute" in terms of radians.

14. (3 pts) On the same pair of axes, draw the three parabolas $y = x^2$, $y = -\frac{1}{2}x^2$, and $y = (x-4)^2$.

15. (3 pts) On the same pair of axes, draw the two hyperbolas $y = \frac{1}{x}$, $y = \frac{1}{x} + 1$. Calculate the x-intercept of the second hyperbola and indicate it on the graph.

16. (3 pts) Sketch a graph of the cubic $y = \frac{1}{2}(x+1)(x-1)(x-3)$, and calculate its y-intercept.

17. (2 pts) Factor $3x^2 - 42x + 120$ as $a(x - r_1)(x - r_2)$.

18. (2 pts) By completing the square, write $x^2 - 2x + 5$ in the $a(x-h)^2 + k$ form.

19. (3 pts) By completing the square, write $2x^2 - 20x + 53$ in the $a(x-h)^2 + k$ form.

20. (2 pts) State the domain of the function $y = \frac{2x}{x^2+8x-9}$.

21. Calculate the following numbers: (assume $b > 1$) (1 pt each)

- $\log_b b =$
- $\log_b 1 =$
- $\log_2 16 =$
- $\log_3 \frac{1}{3} =$
- $\log_5 \sqrt{5} =$
- $e^{2\ln 9} =$
- $e^{(\frac{1}{2}\ln 4 + 2\ln 5)} =$
- $\log_{10}(4) + \log_{10}(250) =$
- $\log_{10}(2) - \log_{10}(200) =$

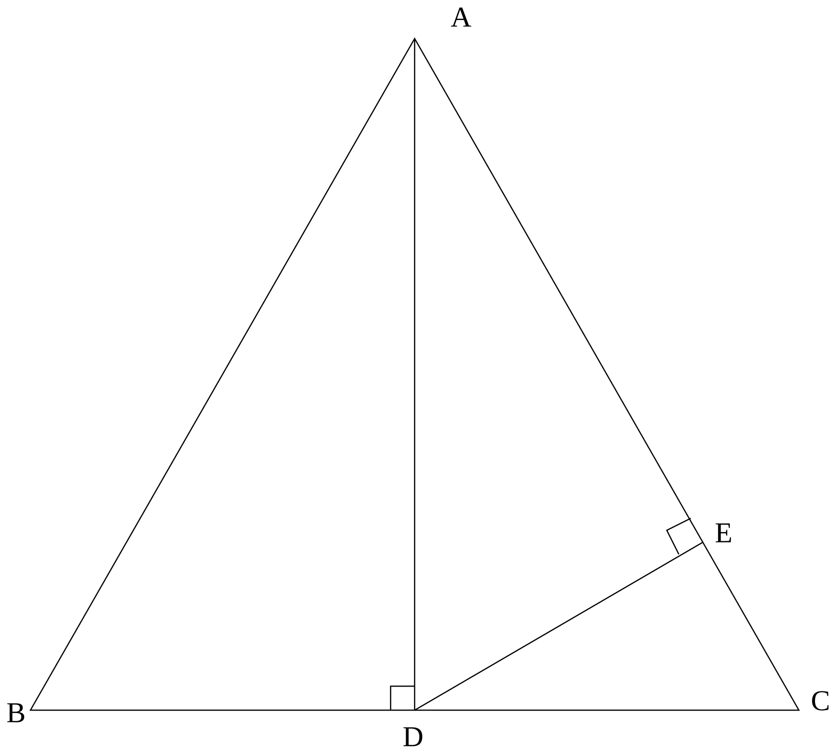
22. Manipulate the following expressions, assuming x and y are positive: (1 pt each)

- simplify: $e^{(5\ln(x)-3\ln(y))} =$
- expand as a sum or difference: $\ln\left(\sqrt{\frac{4}{4+x}}\right) =$
- simplify: $e^{(\frac{1}{2}\ln(x^2-2x+1))} =$
- expand: $(e^x - e^{-x})^2 =$
- Solve for x: $e^{5\ln(x)} - 32 = 0$:
- Find the two solutions for x: $e^{2x} - 5e^x + 6 = 0$:

23. (each line 1 pt) Fill in the blanks with numbers that measure an angle:

360 degrees	=	________ radians	=	________ revolutions,
________ degrees	=	$\pi/4$ radians	=	________ revolutions,
________ degrees	=	________ radians	=	1/6 revolutions,
30 degrees	=	________ radians	=	________ revolutions,
________ degrees	=	1 radian	=	________ revolutions,
________ degrees	=	________ radians	=	3/4 revolutions.
________ degrees	=	________ radians	=	10 revolutions.

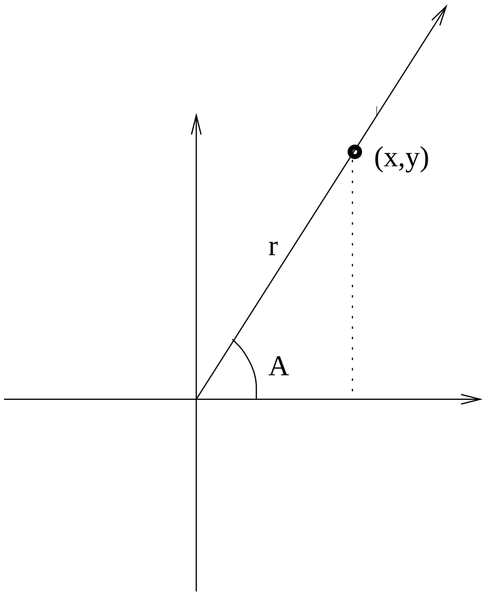
24. (4 pts) Here's an equilateral triangle $\triangle ABC$, with each side length **4** units. Let D be the midpoint of BC so that AD and BD form a right angle. Calculate the distance between A and D. Also, assuming DE and AE form a right angle, calculate the lengths of segments DE, AE, and CE.



25. (1 pt) Show that $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$ satisfies the equation $x^2 + y^2 = 1$.

26. (1 pt) Show that $y = \frac{1-t^2}{1+t^2} = -1$ has no solutions $t \in \mathbb{R}$.

27. For the angle A in standard position through the point (x, y), state a formula for the following quantities in terms of x and y. (1 pt each)



- r = distance from $(0, 0)$ to $(x, y) =$
- $\sin(A) =$
- $\cos(A) =$
- $\tan(A) =$
- $\sec(A) =$
- $\csc(A) =$
- $\cot(A) =$
- $\tan(-A) =$

28. (2 pt) In terms of the quantities r, $\sin(A)$, and $\cos(A)$, state formulas for x and y:

29. (3 pt) Using your definition of $\sin(A)$ and $\cos(A)$, show that

$$(\sin(A))^2 + (\cos(A))^2 = 1.$$

30. (3 pts) Prove the formula $\ln(AB) = \ln(A) + \ln(B)$ for $A,\ B > 0$. Hint: use both sides as exponents.

31. (2 pts) Graph the functions $\sin(x)$ and $\cos(x)$, where x is in radians, on the same pair of axes on the domain $[0, 4\pi]$.

32. (3 pts) Graph the functions e^x and $\ln(x)$ on the same pair of coordinate axes.

33. (3 pts) Graph the functions $\tan(x)$ and $\sec(x)$ on the domain $(-\pi/2, \pi/2)$.

34. (2 pts) Calculate the x-intercept of the line $y = mx + b$, assuming $m \neq 0$. The line $y = mx + b$ and the x-axis meet at this point— what is the tangent of the angle formed by these two lines?

34. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. P_1 and P_2 refer to parabolas

$$y = a_1x^2 + b_1x + c_1,\ \ a_1 \neq 0,$$

and

$$y = a_2x^2 + b_2x + c_2,\ \ a_2 \neq 0.$$

You need not explain your answer.$^{(1\ each)}$

____________ It is possible for P_1 and P_2 to intersect in exactly one point.

____________ If P_1 and P_2 both "open upward," then P_1 and P_2 intersect in at least one point.

____________ P_1 never intersects the parabola

$$y = a_1(x-1)^2 + b_1(x-1) + c_1.$$

____________ If $a_1 > 0$, then the y-coordinate of the vertex of P_1 is a minimum value.

____________ P_1 can intersect a cubic $\{y = ax^3 + bx^2 + cx + d,\ a \neq 0\}$ in exactly three points.

____________ P_1 can intersect a cubic $\{y = ax^3 + bx^2 + cx + d,\ a \neq 0\}$ in exactly four points.

____________ P_1 can intersect a circle $\{(x-a)^2 + (y-b)^2 = r^2\}$, in exactly four points.

____________ (Recall i is the complex number such that $i^2 = -1$)

$$(x-i)^3 = x^3 - 3ix^2 - 3x + i.$$

____________ If $f(x)$ is a polynomial with exactly one real root, and $g(x)$ is a polynomial with exactly one real root, then the product $f(x) \cdot g(x)$ is a polynomial with exactly one real root.

35. More True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. Assume a, c, M, and N are all positive, and b is an integer greater than 1. (1 pt each)

____________ $(b^x)^y = b^{(x^y)}$.

____________ $(b^x)^y = b^{(x \cdot y)}$.

____________ $(a+c)^b = a^b + c^b$.

____________ $(a+c)^b = a^b + bac + c^b$.

____________ $(ac)^b = a(c^b)$.

____________ $a^{(b+1)} = a^b + a$.

____________ $b^{\log_b c} = c$.

____________ $\log_b(b^y) = y$

____________ $\log_b(M \cdot N) = \log_b M + \log_b N$.

____________ $\log_b(M \cdot N) = (\log_b M) \cdot (\log_b N)$.

____________ $\log_b(M \cdot N) = (\log_b M)^{(\log_b N)}$.

____________ $\log_b(M + N) = \log_b M + \log_b N$.

____________ $\log_b(M + N) = (\log_b M) \cdot (\log_b N)$.

____________ $\log_b(M + N) = (\log_b M)^{(\log_b N)}$.

____________ If $\theta > 2\pi$ then $\sin(\theta) > 1$.

____________ $\ln(\tan(\theta)) = \ln(\sin(\theta)) - \ln(\cos(\theta))$.

____________ $\csc(\theta) = \frac{1}{\cos(\theta)}$.

____________ $\sin(-\theta) = -\sin(\theta)$.

____________ $\cos(-\theta) = -\cos(\theta)$.

____________ For a circle with radius r, an angle of one radian defines a larger arc than the arc defined by one degree.

____________ If $p(x)$ and $q(x)$ are polynomials , and r is a root of the polynomial $p(x)$, then r is a root of the polynomial $p(x)q(x)$.

____________ If $p(x)$ and $q(x)$ are polynomials , and r is a root of the polynomial $p(x)q(x)$, then r is a root of the polynomial $p(x)$.

____________ If a non-vertical line is tangent to a parabola, then the line and the parabola intersect at exactly one point.

____________ If a non-vertical line is tangent to the curve $y = ax^3 + bx^2 + cx + d$, then the line and the curve intersect at exactly one point.

36. Here are ten functions, and on the next three pages are ten Maple printouts of their graphs, labeled $A(x)$, $B(x)$, ... , $J(x)$. The variable x represents **radians** when trigonometric functions are involved. Match the function to the graph by writing the corresponding letter next to each function. (There are no repetitions, and don't leave any blank.) (1.5 pts each)

- ____________ $y = (x+5)^3 - (x+5)$
- ____________ $y = \frac{1}{2}(x+6.5)$
- ____________ $y = 3\sqrt{x}$
- ____________ $y = \sin(x)$
- ____________ $y = \cos(x)$
- ____________ $y = e^x$
- ____________ $y = \sec(x)$
- ____________ $y = \ln(x)$
- ____________ $y = (\frac{1}{2})^x$
- ____________ $y = (x+5)^2 + 1$

37. Here are ten functions, and below the list is a set of "applications" which can be mathematically modeled by one of the listed functions. The variable t always stands for elapsed time, and k, c, and C are positive constants. Match the function to the application by writing the corresponding letter next to each function. (There are no repetitions, and don't leave any blank. Also, the functions are unrelated to each other.) (2 pts each)

- ____________ $y = -kt^2 + ct + C.$
- ____________ $y = C + 100kt.$
- ____________ $y = C - 100kt.$
- ____________ $y = C(1+k)^t.$
- ____________ $y = C(1-k)^t.$
- ____________ $y = Ce^{kt}$
- ____________ $y = Ce^{-kt}$
- ____________ $y = c + (C-c)e^{-kt}$
- ____________ $y = C\sin(t).$
- ____________ $y = -\ln(kt + C).$

A. Tuition increases by $100k$ dollars every year.
B. Tuition increases by $100k\%$ every year.
C. Financial aid decreases by $100k$ dollars every year.
D. Financial aid decreases by $100k\%$ every year.
E. Interest accumulates continuously at $100k\%$ nominal annual rate.
F. The height of a falling object.
G. The height of a rising drawbridge.
H. The pH of a solution as an acid is mixed in.
I. Radioactivity diminishes continuously with decay constant k.
J. A hot object cools off.