Name ________________________ Student ID# ____________

Final Exam. Tuesday, 12-10-96 4:00-6:00. Math 100. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 150 points out of a quarter total of 450.

Definition 2.1 The logical symbols NOT, AND, OR, and $\Rightarrow$ are defined by the following rules for combining the labels **T** and **F** for sentences A and B.

A	B	NOT A	A AND B	A OR B	$A \Rightarrow B$
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**F**	**F**	**T**

Definition 2.2 $\mathbb{R}^2$ is the set $\{(x, y) : x \in \mathbb{R} \text{ AND } y \in \mathbb{R}\}$, of "ordered pairs" of real numbers.

Definition 2.3 The x-axis is a subset of $\mathbb{R}^2$, defined by

$$\{(x, y) \in \mathbb{R}^2 : y = 0\}.$$

The y-axis is the set

$$\{(x, y) \in \mathbb{R}^2 : x = 0\}.$$

Definition 2.4 A subset $G \subseteq \mathbb{R}^2$ is "the graph of a function" if the following implication is true for any points (x_1, y_1) and (x_2, y_2) in the set G:

$$x_1 = x_2 \implies y_1 = y_2.$$

Definition 2.5 A "function" is the rule that assigns to the number x the number y such that (x, y) is on the graph.

There can't be two numbers y that get assigned to x by the definition of "graph of a function." However, there may be some $x \in \mathbb{R}$ that are not assigned a number y.

Definition 2.6 The "domain" of a function with graph G is the set $\{x \in \mathbb{R} : \text{there is a point } (x, y) \in G\}$.

Questions #1–31 2 points each unless otherwise noted.

1. State the definition of "x is an odd integer."

2. State the definition of "x is a rational number," in terms of the set $\mathbb{Z}$ of integer numbers.

3. State the definition of "r is a root of the polynomial $ax^4 + bx^3 + cx^2 + dx + e$."

4. State the definition of "rational expression" in terms of polynomial expressions.

5. If z is the complex number $z = x + iy$, calculate and simplify z^2.

6. State the "quadratic formula" for the roots of the quadratic polynomial $ax^2 + bx + c$.

7. State the definition of the open interval $(a, b) \subseteq \mathbb{R}$.

8. State the Theorem on Transitivity of Subsets.

9. State the formula for the average of eight numbers $a_1, a_2, a_3, \ldots, a_8$.

10. State the definition of $|x|$. Also graph the function $y = |x|$.

11. State the Triangle Inequality for elements $x, y \in \mathbb{R}$.

12. For two points $x, y \in \mathbb{R}$, state the definition of the "distance" between x and y.

13. Consider the sentence "If p is odd, then p^2 is odd." State the following implication sentences:

- Converse: "If ____________________, then ____________________."
- Contrapositive: "If ____________________, then ____________________."

14. Fill in this table with **T** or **F**. (4 pts)

A	B	NOT(A AND B)	(NOTA) AND (NOTB)	NOT($A \Rightarrow B$)	(A OR B) $\Rightarrow B$
T	**T**				
T	**F**				
F	**T**				
F	**F**				

15. State the definition of the sum and composite functions:

- $(f + g)(x) =$
- $(f \circ g)(x) =$

Questions #16–20 refer to points $P = (a, b)$ and $Q = (c, d)$ in $\mathbb{R}^2$ such that $a \neq c$. Let $\overline{PQ}$ be the line segment connecting them. State the formulas:

16. The length of $\overline{PQ} =$

17. The slope of $\overline{PQ} =$

18. The circle C with center P and radius $r > 0$ has equation:

19. The line with slope m that contains P has equation:

20. Several parabolas go through point P. Give an example of one of them.

21. Assume $B \neq 0$. State the formula for the slope of the line

$$\{(x, y) \in \mathbb{R}^2 : Ax + By + C = 0\},$$

and calculate the intersection of this line with the y-axis.

22. State the definition of "the function $f(x)$ is decreasing on the set I."

23. State, in terms of absolute values, an inequality whose solution set is the set of numbers x whose distance from c is less than δ.

24. What is the domain of the function $f(x) = \frac{1}{\sqrt{1-x^2}}$?

25. Give an example of a pair of functions $f(x)$ and $g(x)$ so that $f \circ g \neq g \circ f$:

- $f(x) =$
- $g(x) =$
- $f \circ g(x) =$
- $g \circ f(x) =$

26. Adam goes to a restaurant and spends \$114.91. Suggest a generous tip. (Hint: calculating 15% to the nearest thousandth of a cent would be cheap.)

27. Why is the implication $a < b \implies a^2 < b^2$ false? (give an example of numbers a, b, where $a < b$ and $a^2 \geq b^2$)

28. Why is $a < b \implies \frac{1}{b} < \frac{1}{a}$ false, even for $a,\ b \neq 0$?

29. For $q \neq 0$, $s \neq 0$, why is "$\frac{p}{q} = \frac{r}{s} \implies p = r$ and $q = s$" false?

30. For $a < b \in \mathbb{R}$, why is $[a, b] \subseteq (a, b)$ false?

31. Give an example of a polynomial with domain $\mathbb{R}$ and range contained in, but not equal to $\mathbb{R}$.

32. Solve the following equalities, that is, find **all** numbers $x \in \mathbb{R}$ that satisfy the given equation. You don't have to explain your reasoning or unnecessarily simplify your answer. The solutions x might involve both numbers and real constants such as m and b. (Hints: There will always be at least one real solution. Check your answers!) (Give an answer for each question; I will count your nine best answers out of twelve for one point each.)

- ____________ $\frac{1}{2}x - \frac{3}{4} = 0.$
- ____________ $mx + b = \frac{1}{2}x + 3,\ m \neq \frac{1}{2}.$
- ____________ $7(2x+5) - 6(x+8) = 7.$
- ____________ $\frac{5+x}{6} - \frac{10-x}{3} = 1.$
- ____________ $\sqrt{x} = x^3.$
- ____________ $\sqrt{|x|} = m,\ m > 0.$
- ____________ $\frac{m}{x} = \frac{x}{b}.\ (m > 0,\ b > 0)$
- ____________ $2x^2 = 5x.$
- ____________ $x^2 - 11x + 30 = 0.$
- ____________ $x = \frac{2x^2+8x-10}{x-1}.$
- ____________ $|x-2| = \frac{1}{10}.$
- ____________ $|x| = x + 1.$

33. If $m \neq 1$ then the lines $y = mx + b$ and $y = x - 5$ intersect at one point. Calculate the coordinates of the point. (1 pt)

34. **Use the quadratic formula** to show that the two roots of the polynomial $x^2 - c$ are $\sqrt{c}$ and $-\sqrt{c}$. Assume $c > 0$, and show all work. (2 pts)

Graph the following functions. Each graph should be larger than a quarter. (#35–#40 4 pts each)

For #35 and #36, indicate where the graph intersects the x-axis and the y-axis. (plug in $x = 0$ and $y = 0$ and plot these ordered pairs on your graph)

35. $y = \frac{x}{2} + 3$.

36. $y = -3x + 6$.

37. $y = (x + 2)^2 - 1$.

38. $y = \begin{cases} 3x + 5 & \text{if } x \geq 0 \\ \frac{1}{x} & \text{if } x < 0 \end{cases}$

39. Graph the circle $x^2 + (y-5)^2 = 16$. Calculate the intersection of the circle with the line $y = 4$.

40. Let S be the half-plane $\{(x, y) \in \mathbb{R}^2 : x > 1\}$, and let T be the half-plane $\{(x, y) \in \mathbb{R}^2 : y \geq 1\}$. (Note that S does not contain its boundary line, but T does.) Sketch, on separate coordinate axes, the sets S, T, $S \cup T$, $S \cap T$, and the set $\{(x, y) \in \mathbb{R}^2 : (x, y) \in S \cup T \text{ AND } (x, y) \notin S \cap T\}$.

41. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, 4, \ldots\}$. The capital letters A, B, C refer to points in $\mathbb{R}^2$. You need not explain your answer.$^{(1\ each)}$

__________ If $a \in \mathbb{R}$, $b \in \mathbb{N}$, $c \in \mathbb{N}$, then $a^b \cdot a^c = a^{b+c}$.

__________ If $a \in \mathbb{R}$, $b \in \mathbb{N}$, $c \in \mathbb{N}$, then $(a^b)^c = a^{(b^c)}$.

__________ If $a \in \mathbb{R}$, $b \in \mathbb{N}$, $c \in \mathbb{N}$, then $(a^b)^c = (a^c)^b$.

__________ If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{N}$, then $a^c \cdot b^c = (a \cdot b)^c$.

__________ If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{N}$, then $a^c + b^c = (a + b)^c$.

__________ $(c < 0 \text{ AND } |x - c| < \left|\frac{c}{2}\right|) \Rightarrow x > 0$.

__________ $(4, 5] \cap [4\frac{1}{2}, 7)$ is a closed interval.

__________ $(4, 5] \cup [4\frac{1}{2}, 7)$ is a closed interval.

__________ If $A \neq B$ and the lines L and M both contain A and B, then $L \subseteq M$.

__________ If the distance from A to C is not equal to the distance from A to B plus the distance from B to C, then A, B, and C do not lie on the same line.

__________ A subset containing only two points, $\{B, C\} \subseteq \mathbb{R}^2$, is always the graph of a function.

__________ If line L is the set $\{(x, y) : 2x + 3y + 4 = 0\}$, and line M is the set $\{(x, y) : 6x + 9y + 12 = 0\}$, then $L \neq M$.

42. Simplify the following expressions until they are integers or rational numbers in lowest terms. $^{(1\ each)}$

- ____________ $\frac{7}{8} + \frac{5}{6}$.
- ____________ $(\frac{2}{5} - 1)^2$.
- ____________ $(-\frac{1}{27})^{1/3}$.
- ____________ $(\frac{1}{27})^{-1/3}$.
- ____________ i^6, $(i \in \mathbb{C})$.

43. Solve the following inequalities, that is, find **all** numbers $x \in \mathbb{R}$ that satisfy the given statement. You don't have to explain your reasoning or unnecessarily simplify your answer, but you must phrase your answer in **interval notation**, e.g., $[4, 5)$. The solutions x might involve both numbers and real constants such as m, b, c, δ. The solutions $\emptyset$ and $\mathbb{R}$ might also occur, and the solutions might be disconnected unions of intervals, e.g., $[4, 5) \cup (10, \infty)$. Check your answers! (Give an answer for each question; I will count your eight best answers out of twelve for 1.5 points each.)

- ____________ $2x - \frac{4}{3} > 0$.
- ____________ $2x - \frac{4}{3} \neq 0$.
- ____________ $6 < -2x < 8$.
- ____________ $mx + b > 0$, $m < 0$.
- ____________ $-11 \leq 2x - 3 < 7$.
- ____________ $|x - 3| \leq 5$.
- ____________ $|2x + 1| \geq 1$.
- ____________ $x \in (-1, 3)$ OR ($x \in (2, 4]$ AND $x \neq 2.5$).
- ____________ ($x \in (-1, 3)$ OR $x \in (2, 4]$) AND $x \neq 2.5$.
- ____________ $0 < |x - c| < \delta$
- ____________ $x^2 \leq 4$.
- ____________ $x^2 - 2x < 0$ AND $x \notin \mathbb{Z}$.

44. Here is a list of statements which could be used to justify steps in a proof. Some of them might get used several times in the proof below, or not at all. Before each line, write the reason why that line follows from the previous line. I have already written one of the answers. (*16 pts total*)

A. Associative law for $\cdot$

B. Associative law for $+$

C. Commutative law for $\cdot$

D. Commutative law for $+$

E. Theorem: $a \leq b + c \implies -c + a \leq b$

F. Multiplicative identity: $a \cdot 1 = a$

G. Additive Identity: $a + 0 = a$

H. Distributive Law: $a \cdot (b + c) = a \cdot b + a \cdot c$

I. Theorem: Squares are nonnegative

J. a^2 is defined to be $a \cdot a$

K. $1 + 1 = 2$

Theorem. For x, y real numbers, $-2xy \leq x^2 + y^2$.
PROOF:

______________ $0 \leq (x + y)^2$
______________ $0 \leq (x + y) \cdot (x + y)$
______________ $0 \leq (x + y) \cdot x + (x + y) \cdot y$
______________ $0 \leq x \cdot (x + y) + y \cdot (x + y)$
______________ $0 \leq (x \cdot x + x \cdot y) + (y \cdot x + y \cdot y)$
______________ $0 \leq (x^2 + x \cdot y) + (y \cdot x + y^2)$
______________ $0 \leq x^2 + (x \cdot y + (y \cdot x + y^2))$
______________ $0 \leq x^2 + ((x \cdot y + y \cdot x) + y^2)$
______________ $0 \leq x^2 + (y^2 + (x \cdot y + y \cdot x))$
______________ $0 \leq x^2 + (y^2 + (x \cdot y + x \cdot y))$
______________ $0 \leq x^2 + (y^2 + ((x \cdot y) \cdot 1 + (x \cdot y) \cdot 1))$
______________ $0 \leq x^2 + (y^2 + (x \cdot y) \cdot (1 + 1))$
______________ $0 \leq x^2 + (y^2 + (x \cdot y) \cdot 2)$
______________ $0 \leq x^2 + (y^2 + 2 \cdot (x \cdot y))$
______________ $0 \leq (x^2 + y^2) + 2 \cdot (x \cdot y)$
______________ $-(2 \cdot (x \cdot y)) + 0 \leq x^2 + y^2$
______________ $-(2 \cdot (x \cdot y)) \leq x^2 + y^2$. $\blacksquare$

45. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. The volume of a cube with base, width, and height s is s^3. Suppose the volumes of a sphere S and a cube Q are equal and positive. (5 pts total):

a. Calculate $\frac{s}{d}$, where $d = 2r$ is the diameter of S. (Hint: get a formula for the volume of S in terms of d, and set this equal to the volume of Q)

b. Which is larger, d or s? (Hint: π is about 3.1.)

c. The surface area of S is $4\pi r^2$, and of Q is $6s^2$. Calculate the surface area of S in terms of d.

d. Calculate the ratio of the surface area of S to the surface area of Q. (Hint: only constants should appear in the answer.)

e. Which area is larger?