Midterm 3
Wednesday, 5-24-96 10:30-11:30
Math 153
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Name ____________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Theorem 2.1 (Quotient Rule for Derivatives)
$g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Theorem 2.2 If Σa_n is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.3 (Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then Σa_n is a divergent series. (This is just the contrapositive of the previous theorem)

Theorem 2.4 (Basic Comparison Test) Let Σa_k be a series with nonnegative terms.

- Σa_k converges if there exists a convergent series Σc_k with nonnegative terms such that $a_k \leq c_k$ for all k sufficiently large.
- Σa_k diverges if there exists a divergent series Σd_k with nonnegative terms such that $d_k \leq a_k$ for all k sufficiently large.

Theorem 2.5 (Root Test) Let Σa_k be a series with nonnegative terms, and suppose that $\lim_{k\to\infty} (a_k)^{1/k} = \rho$.

- If $\rho < 1$, then Σa_k converges.
- If $\rho > 1$, then Σa_k diverges.
- If $\rho = 1$, this limit test is inconclusive

(first six problems 3 pts each. Give only one answer.)

1. Given a function $f(x)$ with $n+1$ continuous derivatives on an open interval containing 0, define $P_n(x)$, its Taylor polynomial of degree n.

2. Taylor's theorem states that, for f as in the previous question, $f(x) = P_n(x) + R_{n+1}(x)$. State a formula or estimate for $R_{n+1}(x)$.

3. State the definition of the Taylor series of $f(x)$. Also state conditions on $f(x)$ and $R_{n+1}(x)$ that imply the convergence of this series to $f(x)$.

4. For a continuous function $g(x)$ on $[a, b]$, state the definition of the "average value of $g(x)$ on $[a, b]$."

5. State the Theorem on the Differentiability of Power Series.

6. Use Taylor's theorem for power series in $(x-a)$ to express the polynomial $f(x) = 9x^3 - x + 2$ in the form $\Sigma a_k(x-5)^k$. (Hint: you don't have to simplify your answer)

7. a. Consider $f(x) = 4\sin(2x) + \frac{\cos(9x)}{1-x}$. Use Taylor's theorem to calculate the Taylor polynomials, centered at 0, of $f(x)$ of degree 0, 1, 2, and 3. Hints:

$$f''(x) = -16\sin(2x) - \frac{81\cos(9x)}{(1-x)} - \frac{18\sin(9x)}{(1-x)^2} + \frac{2\cos(9x)}{(1-x)^3},$$

$$f'''(x) = -32\cos(2x) + \frac{729\sin(9x)}{(1-x)} - \frac{243\cos(9x)}{(1-x)^2} - \frac{54\sin(9x)}{(1-x)^3} + \frac{6\cos(9x)}{(1-x)^4}.$$

b. Here's a graph of the function $f(x) = 4\sin(2x) + \frac{\cos(9x)}{1-x}$. Sketch, on this graph or a copy of it, the graphs of the Taylor polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$, which you just found in part a. (8 pts total)

8. Calculate the exact sum of each of the following series, as a constant or in terms of x. (3 each, best 5 of 6) The following two examples may be useful:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } x \in (-1,1),$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x, \text{ for } x \in \mathbb{R}.$$

- For $x \in \mathbb{R}$, $\displaystyle\sum_{k=3}^{\infty} \frac{x^k}{k!} =$

- For $x \in (-1,1)$, $\displaystyle\sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} =$

- For $x \in \mathbb{R}$, $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} =$

- $\displaystyle\sum_{k=0}^{\infty} \frac{(-\pi^2)^k}{(2k)! \cdot 4^k} =$

- For $x \in (0,1)$, $\displaystyle\sum_{n=0}^{\infty} x^{n/2} =$

- For $x \in \mathbb{R}$, $\displaystyle\sum_{k=0}^{\infty} \frac{x^{k+3}}{k!} =$

12. True or False. Write **T** or **F** next to each sentence. The numbers i, k, and n refer to natural numbers in the domain of a sequence. $^{(1\ each)}$

_______ If $\Sigma a_k x^k$ converges on $(-c, c)$, and $0 < d < c$, then $\Sigma a_k x^k$ converges on $(-d, d)$.

_______ If $\sum a_i x^i$ converges, then so does $\sum (-1)^i a_i x^i$.

_______ $\sum (-1)^i$ diverges.

_______ $\sum ((-1)^i + (-1)^{i+1})$ diverges.

_______ The operation "average value of $f(x)$ on $[a, b]$," with domain $\{f(x)$ continuous on $[a, b]\}$ and target $\mathbb{R}$, is a linear operator.

13. Give an example of a sequence a_n such that the power series $\Sigma a_k x^k$ converges only for $x = 0$, and prove that your example diverges for $x \neq 0$. (2 pts)

14. a. (2 pts) At time $t = 0$, a light switch is turned on. At time $t = 10$ seconds, it is switched off, and then on again 5 seconds later (at $t = 15$), off in 2.5 seconds, on in 1.25 more seconds, etc., so that the light is switched between on and off at intervals of time decreasing in steps of 50%. At what time t will this pattern terminate? (Express the elapsed time after n steps as a finite sum, and then state the limit of the sum as $n \to \infty$.)

b. (extra credit) After this experiment, is the light switch on or off?