Midterm 3
Monday, 3-4-96 10:30-11:20
Math 152
Adam Coffman

Name ____________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–7:
k is a constant.
$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.
The notation $\frac{d}{dx}$ is also used to mean "the derivative of".
Conclusions:

1. $\frac{d}{dx}k = 0.$
2. $\frac{d}{dx}x = 1.$
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x).$
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x).$
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x).$

Theorem 2.1 If $f(x)$ is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), with derivative $F'(x) = f(x)$.

Theorem 2.2 The function $\log(x) = \int_1^x \frac{1}{t}dt$ is differentiable on $(0, \infty)$. $\frac{d}{dx}\log|x| = 1/x$ on $(-\infty, 0) \cup (0, \infty)$.

Theorem 2.3 (The Inverse Function Theorem) If $f(x)$ is differentiable and one-to-one on (a, b), and $f'(x) \neq 0$, then f^{-1} is differentiable, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

In the following questions, give only one definition, version of the required statement, or example. In the proofs, you may cite any theorem from the cover page.

1. State the definitions I gave in class of $\exp(x) = e^x$, a^b, $\sinh(x)$, and $\sin^{-1}(x)$. $^{(2\ pts\ each)}$

2. Use the Inverse Function Theorem to calculate $\frac{d}{dx}e^x$ and $\frac{d}{dx}\sin(x)$.$^{(2\ pts\ each)}$

3. Calculate and simplify $\frac{d}{dx}\left(\sin^{-1}(x) + x\sqrt{1-x^2}\right)$.$^{(2pts)}$

4. Use your answer to #3, the Fundamental Theorem of Calculus, the Triangle Area formula $\frac{1}{2}$base·height, and the identities $\sin^{-1}(1) = \frac{\pi}{2}$, $\cos(\theta) = \sin(\frac{\pi}{2} - \theta)$ to show that the area of this "sector" of the unit circle is $\theta/2$ for $\theta \in [0, \pi/2)$. $^{(3pts)}$

5. Integrate by parts twice to calculate $\int \cos(\log(x))dx$.$^{(3pts)}$

12. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

______ $e^{(x^2)} = (e^x)^2$.

______ $\forall x \in (-1, \infty),\ \ln(\frac{1}{1+x}) = -\ln(1+x)$

______ $\forall x \in (0, \infty),\ \ln(\frac{1}{1+x}) = -\ln(1+x)$

______ $\forall x \in \mathbb{R},\ e^{-x} \leq e^x$.

______ If $h(x)$ and $f(x)$ are two solutions of the differential equation $y'' = -y$, then so is $(h+f)(x)$.

______ If $h(x)$ is a solution of $y'' = -y$, then so is $k \cdot h(x)$.

______ If $h(x)$ is a solution of $(y')^2 = y$, then so is $k \cdot h(x)$.

11. Give any one solution, y, to each of the following differential equations: $y = y(x)$ is a function of x, and derivatives are in x, using the abbreviation $y' = \frac{d}{dx}y(x)$. Some of the following questions require that y satisfy "initial conditions." g, m, and k are non-zero constants. It will help a lot if you clearly indicate your answer on this page and check it on another.$^{(2\ pts\ each;best\ 7)}$

- $y' = 2y,\ y(0) = k$
- $y'' = g,\ y(0) = 100,\ y'(0) = 1$
- $y'' = 2y$
- $y'' = -2y$
- $y' = x(\ln(x) - 1)$
- $y' = \tan(x)\sec^2(x)$
- $y' = x^2y^4,\ y(1) = 1$
- $y' = \frac{g}{m} + ky$
- $x^2y'' = y$

14. The following questions refer to the attached excerpt from:

Brian Dennis, Patricia L. Munholland, and J. Michael Scott, *Estimation of growth and extinction parameters for endangered species*, Ecological Monographs, Vol. 61 #2 (June 1991) pp. 115–143.

a. The function $G(t; x_d, \mu, \sigma^2)$ is defined on line (16), as a function of t, with domain $t > 0$. The quantities x_d, μ, and $\sigma > 0$ are constants. The function Φ is not specified, but it has the property

$$\frac{d}{dz}\Phi(z) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}z^2).$$

It is claimed that the function g defined on line (17) is the derivative of G with respect to t. Verify this by showing all steps of the calculation.

(Hints: When I checked this using the chain rule, both the following expressions appeared: $\frac{d}{dt}\frac{-x_d+|\mu|t}{\sigma\sqrt{t}} = \frac{x_d+|\mu|t}{2\sigma t^{3/2}}$, $\frac{d}{dt}\frac{-x_d-|\mu|t}{\sigma\sqrt{t}} = \frac{x_d-|\mu|t}{2\sigma t^{3/2}}$.)

(7 pts)

b. The graph of $g(t)$ appears in the article's Figure 1. Sketch a graph of $G(t)$ on the domain $(0, 600) \subseteq \mathbb{R}$ and indicate its critical points and inflection points, if any. (Hints: your sketch does not have to be precise, but should be larger than a quarter. Many functions G have derivative g, but only sketch the graph of one of them. 2 pts)