Midterm 3
Monday, 12-4-95 10:30-11:20
Math 151
Adam Coffman

Name _____ Student ID# _____

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–7:

k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Theorem 0.1 The function $\sqrt{x}$ is continuous on $[0, \infty)$, and differentiable on $(0, \infty)$. The derivative function $\frac{d}{dx}\sqrt{x}$ is equal to $\frac{1}{2\sqrt{x}}$.

1. State the definition of

$$f(x) \text{ is differentiable at } c.$$

(Hint, this is a sentence, not a number) (2 *points*)

2. State the definition of

$$f(x) \text{ is differentiable on } (a, b).$$

(2 *points*)

3. State the definition of "$f(x)$ is bounded on $[a, b]$."
(2 *points*)

4. State the definition of "$f(x)$ has a maximum value on the set I."
(2 *points*)

5. If f is a differentiable function on (a, b), and $c \in (a, b)$, state the equation of the line tangent to the graph of f at the point $(c, f(c))$.
(2 *points*)

6. Let $f(x) = |x|$, and let $g(x) = x^3$. Compute the following derivatives any way you want. You don't have to explain your reasoning, just write a **real number** or "DNE" if the function is not differentiable at the specified point. (1 pt each)

- $(\frac{d}{dx}f)(10) =$
- $(\frac{d}{dx}(f \circ g))(0) =$
- $(\frac{d^2}{dx^2}g)(4) =$
- $(\frac{d}{dx}\sqrt{g})(1) =$
- $(\frac{d}{dx}\sqrt{f})(0) =$

7. Compute the following derivative **functions** any way you want. You don't have to explain your reasoning, but do specify the domain of the derivative. (2 pts each)

- $\frac{d}{dx}(a_3x^3 + a_2x^2 + a_1x + a_0) =$
- $\frac{d}{dx}\frac{1}{x} =$
- $\frac{d}{dx}\sqrt{1 - x^2} =$
- $\frac{d}{dx}(x^2 \cdot \frac{d}{dx}(\sqrt{x + 1})) =$
- $\frac{d}{dx}(\sqrt{x^2 + 1}) =$

8. Using the definition of the derivative, prove that $h(x) = \sqrt{1 + x^2}$ is differentiable on $\mathbb{R}$. (3 pts)

9. For each of the questions below, give an *example* of a function or pair of functions with all of the required properties. It will be good enough to give a formula, or to sketch a graph that is larger than a quarter. These functions are unrelated to each other or to any other functions on this test. 2 *each*

A. The function $f(x)$

- is continuous on $\mathbb{R}$,
- AND is increasing on $\mathbb{R}$
- AND is not differentiable on $\mathbb{R}$.

B. The functions $g(x)$ and $h(x)$

- both have domain $\mathbb{R}$,
- AND $g(x)$ is not differentiable at 0
- AND $h(x)$ is not differentiable at 0,
- AND $(g \cdot h)(x)$ is differentiable at 0

C. The function $k(x)$

- is not a constant function
- AND is a polynomial that has a maximum value on $(0, 2)$

(write a formula for this one, not just a graph.)

(problem 9, continued) D. The functions $j(x)$ and $m(x)$

- both have derivatives which are positive for all x,
- AND $(j \cdot m)(x)$ has a negative derivative for some x.

10. a) (2 pts) Suppose $g(x)$ is differentiable at c, and $g(c) = 0$. Prove that the following limit exists:

$$\lim_{x \to c} \frac{g(x)}{x - c}$$

b) (2 pts) Give an example of a function g and a number c to show that weakening the hypothesis of the theorem to "$g(x)$ is continuous at c, $g(c) = 0$" is not sufficient to get the above result.

11. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

_____ $f(x) = x^3 \iff \frac{df(x)}{dx} = 3x^2$.

_____ If $f(x)$ is differentiable on (a, b) and $a < c < d < b$, then $f(x)$ is differentiable on (c, d).

_____ If $f(x)$ is differentiable, and $g(x)$ is not differentiable, then $(f \cdot g)(x)$ is not differentiable.

_____ If $f(c)$ is a maximum value of $f(x)$ on the closed interval I, then $f'(c) = 0$.

_____ If $|f(x)|$ is differentiable then $f(x)$ is differentiable.

_____ If $f(x)$ is differentiable then $|f(x)|$ is differentiable.

12. A rectangle of length $L(t)$ and height $H(t)$ is inscribed in a circle of radius 5. (this means all four corners of the rectangle are also points on the circle) The rate of change of the length is -2. Find the rate of change in the area $A(t)$ of the rectangle when its length is 6. (Hint: $A = L \cdot H$) (4 pts)