Midterm 3
Friday, 5-25-95, 9:30-10:20
Math 133
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Name ______________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points.

Here are some facts you might need:

Definition 3.1 A sequence (a_n) is bounded (by B) if $\exists B \in \mathbb{R}$ such that $\forall n \in \mathbb{N},\ |a_n| \leq B$.

Definition 3.2 A sequence is monotonic if it is either increasing or decreasing.

Theorem 3.3 (Monotonic Sequence Theorem) A sequence (a_n) which is monotonic and bounded is convergent.

Theorem 3.4 (Ordinary Comparison Test) Suppose $\exists N$ s.t. $n \geq N \Rightarrow 0 \leq a_n \leq b_n$. Then $\sum_{i=1}^{\infty} b_i$ converges $\Rightarrow \sum_{i=1}^{\infty} a_i$ converges.

State the definition of the following sentences #1–7. (2 pts each):

1. a_n is an increasing sequence.

2. The sequence a_n converges to L. (a limit statement)

3. The sequence a_n converges to L. (use the ϵ, M definition of the limit statement)

4. The sequence a_n diverges to positive infinity. (a limit statement)

5. The sequence a_n diverges to positive infinity. (the formal definition of the limit statement)

6. The series $\sum_{i=1}^{\infty} a_i$ converges to S. (a limit statement)

7. $\sum_{i=1}^{\infty} a_i$ is a geometric series. (give a formula for the terms a_i)

8. State the formula for the length $|\vec{x}|$ of a vector $\vec{x} = (x_1,\ x_2) \in \mathbb{R}^2$ and state the triangle inequality for vector length. (2 pts)

Fill in the blanks to complete the following theorems. (2 pts each for #9–12)

9. **Theorem 3.5** (L'Hôpital's Rule for forms of type 0/0). Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then:________________________.

10. **Theorem 3.6** (Limit Comparison Test) Suppose $a_n \geq 0,\ b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

If ____________________, then ($\sum_{i=1}^{\infty} a_i$ converges $\iff$ $\sum_{i=1}^{\infty} b_i$ converges).

If ____________________, then ($\sum_{i=1}^{\infty} b_i$ converges $\Rightarrow$ $\sum_{i=1}^{\infty} a_i$ converges).

11. **Theorem 3.7** (Ratio Test) Let $\sum_{i=1}^{\infty} a_i$ be a series of positive terms, and suppose $\lim_{n\to\infty} \frac{a_{(n+1)}}{a_n} = \rho$.

If ____________________, the series converges.
If ____________________, the series diverges.
If ____________________, the test is inconclusive.

12. **Theorem 3.8** (Absolute Convergence Test) (State this theorem):

____.

(2 pts each for #13 a,b).

13. a. Use the limit definition of "$\sum_{i=1}^{\infty}$" to prove: $\sum_{i=1}^{\infty} k \cdot a_i = k \cdot \sum_{i=1}^{\infty} a_i$.

b. What other theorem is required to show that $\sum_{i=1}^{\infty}$ is a "linear operator"? (state the theorem, don't prove it.)

14. True or False. Write **T** or **F** next to each sentence. You need not explain your answer.$^{(1\ each)}$

The following questions assume that a_n and b_n are sequences, satisfying the hypothesis of the Ordinary Comparison Test (Thm. 3.4 on the cover page): $0 \leq a_n \leq b_n$ for large enough n. "$\sum$" will be used to abbrevate $\sum_{i=1}^{\infty}$.

______ If $\sum b_i$ diverges, then so does $\sum a_i$.

______ If $\sum a_i$ converges, then so does $\sum b_i$.

______ If $\sum a_i$ diverges, then so does $\sum b_i$.

______ If $\sum a_i$ converges, then so does $\sum a_i^2$.

______ If $\sum a_i$ converges, then so does $\sum \sqrt{a_i}$.

______ If $\sum a_i$ converges, then $\lim_{n\to\infty} a_n = 0$.

______ If $\lim_{n\to\infty} a_n = 0$, then $\sum a_i$ converges.

______ If $\sum a_i$ converges, then (a_n) is a decreasing sequence.

______ k is a constant. If $\sum a_i$ diverges, then $\sum k \cdot a_i$ diverges.

(2 pts each for #15–20. Best 5 of 6.) 15. Calculate $\lim_{x\to 0^+} \dfrac{\sin(x)}{\sqrt{x}}$.

16. Calculate $\lim_{x\to 0^+} x^{(x^2)}$. (Hint: $x^{(x^2)} = \exp(x^2 \ln(x))$).

17. Calculate $\lim_{x\to\infty} \dfrac{x^{12}}{e^{\left(\frac{x}{40}\right)}}$.

18. Use the Integral Test to determine whether $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{1+k^3}$ converges.

19. Use the Ratio Test to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ converges.

20. Use the Limit Comparison Test to determine whether $\displaystyle\sum_{i=1}^{\infty} \frac{n+3}{n^2 \cdot \sqrt{n}}$ converges.

21. (1 pt) Give an example of two sequences a_n, b_n such that $a_n \leq b_n$ and $\displaystyle\sum_{i=1}^{\infty} b_i$ converges and $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges.

22. (2 pts) Prove the Bounded Sum Theorem: If a_n is a sequence of positive numbers, and $\exists B \in \mathbb{R}$ such that $\forall n \in \mathbb{N}, \displaystyle\sum_{i=1}^{n} a_i \leq B$, then the series $\displaystyle\sum_{i=1}^{\infty} a_i$ converges.