Midterm 3
Friday, 12-2-94 9:30-10:20
Math 131
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Name ____________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of 400.

Hypotheses for theorems 1–7:
$k$ is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

1. Compute the following derivatives any way you want. You don't have to explain your reasoning. Write either a function value after the "=," or "**DNE**" if the derivative Does Not Exist. Some of the questions refer to the following functions. (*2 each, best 6*)

$$r(x) = \frac{1}{x} \qquad\qquad s(x) = x^2 + 1$$

.

$t(x)$ is an unknown but differentiable function with domain $\mathbb{R}$

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- $s'(2) =$
- $(s \cdot t)'(2) =$
- $(s \circ t)'(2) =$
- $(t \circ s)'(2) =$
- $r'(0) =$
- $\frac{d}{dx}((x-1) \cdot s(x))(3) =$
- $\frac{d^2}{dx^2}(s \cdot t)(2) =$

2. State the "$h \to 0$" definition of "$n(u)$ is differentiable at $z$." (hint: this is a *sentence*, not just a number. (3)

5. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer. (*1 each*)

____________ $f(x)$ continuous at $c \Rightarrow f(x)$ differentiable at $c$.

____________ $f(x)$ differentiable at $c \Rightarrow f(x)$ continuous at $c$.

____________ $f(x), g(x)$ differentiable AND $\frac{d}{dx}(f \circ g)(c) = 1 \Rightarrow g'(c) \neq 0$.

____________ If neither $f(x)$ nor $g(x)$ is differentiable at $c$, then $(f+g)(x)$ is also not differentiable at $c$.

____________ If $f(x)$ is differentiable, and $g(x)$ is not differentiable, then $(f+g)(x)$ is not differentiable.

6. Use the Derivative Theorems #1–7 from the cover page **one at a time** to carefully calculate the following derivatives: (Justify each step with the numbered theorem; here's an example of what to do:)$^{(2\ each)}$

a. $\frac{d}{dx}2x + 2 = \frac{d}{dx}2x + \frac{d}{dx}2$ (using #4)
$= 2\frac{d}{dx}x + \frac{d}{dx}2$ (#3) $= 2 \cdot 1 + \frac{d}{dx}2$ (#2)
$= 2 + 0$ (#1) $= 2$.

b. $\frac{d}{dx}3x^2 =$

c. $\frac{d}{dx}x^2(2x - 1) =$

d. $\frac{d}{du}\frac{1}{u} =$

7. $^{(2each)}$ a. Assume $\sqrt{x}$ is differentiable on the open interval $(0, \infty)$. Use the product rule and the fact that $\sqrt{x} \cdot \sqrt{x} = x$ for $x > 0$ to find the derivative of $\sqrt{x}$.

b. Use the fact that $\frac{d}{du}(\frac{1}{u}) = \frac{-1}{u^2}$, the Chain Rule (#7), and the Product Rule (#5) to prove the Quotient Rule (#6).

8. Consider the following

**Theorem.** $f(x)$ is continuous at $c$, AND
$g(x)$ is differentiable at $c$, AND $g(c) = 0$
$\Rightarrow (f \cdot g)(x)$ is differentiable, with $\frac{d}{dx}(f \cdot g)(c) = f(c) \cdot g'(c)$.

This theorem is true; **why** can't you just prove it with the Product Rule, saying that: $^{(1)}$

$$\frac{d}{dx}(f \cdot g)(c) = f(c) \cdot g'(c) + f'(c) \cdot g(c) = f(c) \cdot g'(c) + f'(c) \cdot 0 = f(c) \cdot g'(c)?$$

This theorem becomes false if any of the three hypotheses is omitted. In each of the following three questions, give an example of a function $f(x)$ and an example of a function $g(x)$ with the required properties. Hint: constant functions are differentiable and can be a good source of simple examples.

a. $^{(2 each)}$

- $f(x)$ is continuous at $c$
- AND $g(x)$ is differentiable at $c$
- AND $(f \cdot g)(x)$ is not differentiable at $c$.

b.

- $f(x)$ is continuous at $c$
- AND $g(x)$ is continuous at $c$
- AND $g(c) = 0$
- AND $(f \cdot g)(x)$ is not differentiable at $c$.

c.

- $f(c)$ exists
- $f(x)$ is not continuous at $c$
- AND $g(x)$ is differentiable at $c$
- AND $g(c) = 0$
- AND $(f \cdot g)(x)$ is not differentiable at $c$.

Here is a list of arguments used to justify statements. Some of them might get used twice in the proof below, or not at all; write the letter from the list next to the step that it justifies.$^{(1\ each)}$

A. By hypothesis, $f(x)$ is continuous at $c$.

B. By hypothesis, $g(c) = 0$.

C. Definition of derivative.

D. Definition of limit.

E. Theorem: $b \in \mathbb{R} \Rightarrow b - 0 = b$.

F. Definition of product of functions.

G. Theorem: $b \in \mathbb{R} \Rightarrow b \cdot 0 = 0$.

H. Product theorem for limits.

I. Constant multiple theorem for limits.

J. Product theorem for derivatives.

PROOF:

______________ $\frac{d}{dx}(f \cdot g)(c) = \lim_{x \to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c}$

______________ $= \lim_{x \to c} \frac{f(x) \cdot g(x) - f(c) \cdot g(c)}{x - c}$

______________ $= \lim_{x \to c} \frac{f(x) \cdot g(x) - f(c) \cdot 0}{x - c}$

______________ $= \lim_{x \to c} \frac{f(x) \cdot g(x) - 0}{x - c}$

______________ $= \lim_{x \to c} \frac{f(x) \cdot g(x)}{x - c}$

______________ $= (\lim_{x \to c} f(x)) \cdot (\lim_{x \to c} \frac{g(x)}{x - c})$

______________ $= (f(c)) \cdot (\lim_{x \to c} \frac{g(x)}{x - c})$

______________ $= (f(c)) \cdot (\lim_{x \to c} \frac{g(x) - 0}{x - c})$

______________ $= (f(c)) \cdot (\lim_{x \to c} \frac{g(x) - g(c)}{x - c})$

______________ $= (f(c)) \cdot (g'(c)).$

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