Name ________________________ Student ID# ____________

Midterm 3. Wednesday, 6-4-97 12:30-1:20. Math 102. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

There are no questions on this page

$\mathbb{N}$ refers to the set of natural numbers $\{1, 2, 3, 4, 5, 6, 7, \dots\}$.

Theorem 1.1 For sequences a_k and b_k,

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k.$$

Theorem 1.2

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Theorem 1.3 For $r \neq 0$ and $r \neq 1$,

$$\sum_{k=1}^{n} r^{(k-1)} = \frac{1 - r^n}{1 - r}.$$

(#1–11 2 pts each)

1. For $n \in \mathbb{N}$, state the definition of the factorial

$$n! =$$

2. For $n \in \mathbb{N}$ and $j \in \mathbb{N}$ with $j < n$, state the definition of the binomial coefficient

$$\left(\begin{array}{c} n \\ j \end{array} \right) =$$

3. The "Principle of Mathematical Induction" is used to prove statements S_1, S_2, S_3, ... , about natural numbers. State the two steps in such a proof:

Step 1:

Step 2:

4. State the definition of a sequence.

5. By writing out the first several terms, give an example of an arithmetic sequence with common difference 3:

6. By writing out the first several terms, give an example of a geometric sequence with common ratio 2:

7. If a_k is a sequence, state the definition of the series

$$\sum_{k=1}^{n} a_k =$$

8. Let c be a constant. Use the definition of the Sigma notation to show that

$$\sum_{k=1}^{n}(c \cdot a_k) = c \cdot \left(\sum_{k=1}^{n} a_k \right).$$

9. Here are the first few terms of the "Fibonacci sequence" F_n. Write down the next five:

$$1, 1, 2, 3, 5,$$

10. Here are the first three rows of "Pascal's triangle." Write down the next five rows.

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

11. If a_1, a_2, ... , a_{10} are ten numbers, state the formula for their average using Sigma notation.

12. Here are some "recursive formulas" for sequences. Write down the first five ($k = 1, 2, 3, 4, 5$) terms of the sequence. Some of the expressions may be complicated fractions or exponents, which you need not simplify. (1.5 pts each)

- $\begin{cases} a_1 = 6 \\ a_k = \frac{1}{2} a_{(k-1)} \end{cases}$
- $\begin{cases} a_1 = 2 \\ a_k = (a_{(k-1)})^2 \end{cases}$
- $\begin{cases} a_1 = 1 \\ a_k = \frac{1}{k} \cdot (a_{(k-1)}) \end{cases}$
- $\begin{cases} a_1 = 2 \\ a_k = (a_{(k-1)})! \end{cases}$
- $\begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_k = a_{(k-2)} \end{cases}$

13. Use the definition of the Sigma notation (not the theorems from page 1) to calculate the following numbers. Simplify your answer. (1 pt each)

- $\displaystyle\sum_{k=1}^{4} 3k - 1 =$
- $\displaystyle\sum_{i=2}^{5} 10^i =$
- $\displaystyle\sum_{b=1}^{5} (-1)^b \cdot b^2 =$
- $\displaystyle\sum_{c=10}^{13} (13 - c) =$

14. (1 pt) Expand $(a + b)^7 =$

15. (2 pts) Prove the following statement about $n \in \mathbb{N}$ by mathematical induction:

$$n^2 - n \geq 0.$$

16. (1.5 pts each) Use the Theorems from page 1 to calculate the following sums. You do not have to show your work.

- $\displaystyle\sum_{k=1}^{19}(3k-1) =$
- $\displaystyle\sum_{k=1}^{10} 2^{k-1} =$
- $\displaystyle\sum_{k=1}^{10} 2^{k} =$
- $\displaystyle\sum_{i=1}^{91} 10^{(1-i)} =$
- $\displaystyle\sum_{b=1}^{n}\left(\left(\sum_{k=1}^{b} k\right) - \frac{b^2}{2}\right) =$

17. True or False. Write **T** or **F** next to each sentence. You need not explain your answer. m and n are natural numbers, and a_k and b_k are sequences (also assume a_0 is defined). $^{(1\ each)}$

________ $\displaystyle\sum_{k=1}^{n}(a_k \cdot b_k) = \left(\sum_{k=1}^{n} a_k\right) \cdot \left(\sum_{k=1}^{n} b_k\right).$

________ $\displaystyle\sum_{k=1}^{n} a_{(k-1)} = \left(\sum_{k=1}^{n} a_k\right) - n.$

________ $\displaystyle\sum_{k=1}^{n} a_{(k-1)} = \sum_{k=0}^{n-1} a_k.$

________ $\displaystyle\sum_{k=0}^{n} a_k = \sum_{k=0}^{n} a_{(n-k)}.$

________ $\displaystyle\sum_{k=1}^{n}(a_k - a_{(k-1)}) = a_n - a_0.$

________ $\displaystyle\sum_{k=1}^{m} a_k + \sum_{k=1}^{n} a_k = \sum_{k=1}^{m+n} a_k.$