Midterm 2

Thursday, 7-11-95 9:30-11:20

Math 131

Adam Coffman

Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 300.

Hypotheses for theorems 1–5.

k is a constant.

$\lim_{x\to c} f(x)$ exists.

$\lim_{x\to c} g(x)$ exists.

Conclusions:

1. $\lim_{x\to c} k = k$.
2. $\lim_{x\to c} x = c$.
3. $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim_{x\to c} f(x)$.
4. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
5. $\lim_{x\to c}(f(x) \cdot g(x)) = (\lim_{x\to c} f(x)) \cdot (\lim_{x\to c} g(x))$.

6. **The Composite Limit Theorem.** If $\lim_{x\to c} g(x) = L$ AND f is continuous at L, then

$$\lim_{x\to c}(f \circ g)(x) = f(\lim_{x\to c} g(x))$$

which could also be written as

$$\lim_{x\to c} f(g(x)) = f(L)$$

1. State the definition of

$$\lim_{t \to a} g(t) = M.$$

(1 *point*)

2. State the definition of the right-limit:

$$\lim_{x \to c^+} f(x) = L.$$

(1 *point*)

3. State the "limit" (not the ϵ, δ) definition of "$f(x)$ is continuous at c."
(1 *point*)

4. State the "limit" (not the ϵ, δ) definition of "$f(x)$ is continuous on the open interval (a, b)."
(1 *point*)

5. State the Intermediate Value Theorem.
(1 *point*)

6. The statement of the Composite Limit Theorem refers to functions $f(x)$ and $g(x)$. Assume $g(x)$ is continuous at c, and use the CLT to show $f \circ g$ is also continuous at c. (1 *point*)

7. The following three questions are "ϵ, δ" proofs. Don't use the limit theorems on page one; use the definition of limit. Do all of them; I will grade them, 6 points each with partial credit, but then I will only count your best two answers in the total score, for a maximum score of 12 points.

a. Prove, using the definition,

$$\lim_{x \to 1} x - 5 = -4.$$

scratchwork:

formal proof:

b. Prove, using the definition,

$$\lim_{x \to -2} -3x + 10 = 16.$$

scratchwork:

formal proof:

7c. Assume $k \neq 0$ and that $\lim_{x \to c} f(x) = L$. Prove, using the definition,

$$\lim_{x \to c} (k \cdot f(x)) = k \cdot L.$$

scratchwork:

formal proof:

8. Compute the following limits any way you want. You don't have to explain your reasoning. Write either a real number after the "=," or "**DNE**" if the limit Does Not Exist. Some of the formulas involve "y" — treat it as a constant and express your answer in terms of y. (1 pt each)

- $\lim_{x \to 4} \dfrac{6}{4 - x} =$
- $\lim_{x \to 3} \sqrt{4y((-x)^2 + 16)} =$
  (Hint: this limit exists for some y and not for others.)
- $\lim_{x \to 5} \left(\dfrac{x^2 - 25}{x - 5} \right)^2 =$
- $\lim_{x \to 0^-} [\![x^2]\!] =$
- $\lim_{h \to 0} \dfrac{(y + h)^3 - y^3}{h} =$
- $\lim_{h \to 0} (\lim_{x \to 0} \dfrac{x}{h}) =$
- $\lim_{x \to 0} y^2 + xy + x^2 =$

9. For each of the four questions below, give an *example* of a function with all of the required properties. It will be good enough to give a formula, or to sketch a graph that is larger than a quarter. The three functions are unrelated to each other or to any other functions on this test. [1 *each*]

A. The function $f(x)$

- has domain $(-1, 1)$,
- AND is not continuous at $c = 0$,
- AND is increasing. (recall this means $x < y \implies f(x) < f(y)$)

B. The function $g(x)$

- has domain $[-2, 2]$,
- AND $\lim_{x \to 1} g(x) = 1$
- AND $g(x)$ is not continuous at 1
- AND $\lim_{x \to -1^+} g(x) = 1$
- AND $g(x)$ is not right-continuous at -1.

C. The function $h(x)$

- satisfies $-2 < x < 2 \implies h(x) > 0$
- AND is a quotient of polynomial functions: $h(x) = \frac{p(x)}{q(x)}$
- AND $\lim_{x \to 3} h(x)$ does not exist.

(write a formula for this one, not just a graph.)

9 D. The function $k(x)$

- is increasing on $(0, 1)$
- is decreasing on $(1, 2)$
- is continuous on $(0, 3)$
- has a maximum value that is *not* $k(1)$.

10. Graph the following functions. Draw one set of axes for each function, here or on another paper. 1 pt each.

- $e(x) = \frac{10-x}{5}$
- $f(x) = \frac{x^2-9}{x+3}$
- $g(x) = \sqrt{|x|}$
- $h(x) = \frac{\sqrt{1-x^2}}{x^2}$
- $j(x) = 2[\![x]\!]$
- $k(x) = \begin{cases} x^3 & \text{if } x < -1 \\ x & \text{if } -1 < x < 1 \\ 1-x & \text{if } x \geq 1 \end{cases}$

10. Use the Limit Theorems #1–6 from the cover page **one at a time** to carefully calculate the following limits: (Justify each step with the numbered theorem; for example, I've already done "a.")$^{(1\ each)}$

a. $\lim_{x\to 2} 2x + 2 = \lim_{x\to 2} 2x + \lim_{x\to 2} 2$ (using #4)
$= 2\lim_{x\to 2} x + \lim_{x\to 2} 2$ (#3) $= 2\lim_{x\to 2} x + 2$ (#1)
$= 2 \cdot 2 + 2$ (#2) $= 6$.

b. $\lim_{x\to 2} x^2 - 4 =$

c. $\lim_{x\to -3} \dfrac{x^2-4}{2x-4} =$

d. Suppose $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = L$ and $\lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = M$. Use one of the limit theorems to show

$$\lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} = L + M.$$

e. Assume "$\cos(u)$" is continuous on $\mathbb{R}$. Find the following limit in terms of $\cos(L)$ for some constant L.
$\lim_{x\to 1}(\cos(10-2x)+1)^2 =$

11. Recall that the "maximum value" of the function f is a number $f(c)$ such that $f(x) \le f(c)$ for all x in the domain of f. Show that if f has a maximum value $f(c)$, and $h(x)$ is an increasing function with domain $\mathbb{R}$, then $h(f(c))$ is the maximum value of $h \circ f$.

12. Recall the following version of the "squeeze theorem:" if $0 < |x - c| < \delta \implies 0 \le f(x) \le g(x)$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} f(x) = 0$.

Use this theorem to show that if $0 \le \frac{F(x) - F(c)}{|x - c|} \le B$ for $0 < |x - c| < \delta$, then $F(x)$ is continuous at c.

13. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

__________ If $f(x)$ is continuous on (a, b) then so is $|f(x)|$.

__________ If $|f(x)|$ is continuous on (a, b) then so is $f(x)$.

__________ The domain of $f \circ g(x)$ is a subset of the domain of $g(x)$.

__________ If the maximum value of $f(x)$ is equal to its minumum value, then $f(x)$ is constant.

__________ $\lim_{x \to c} g(x) = 0 \iff \lim_{x \to c} |g(x)| = 0$.

__________ If $f(x)$ is not continuous at c, there is some function $g(x)$ which is continuous at c and such that $f(x) + g(x)$ is continuous at c.

__________ If $f(x)$ is not continuous at c, there is some function $g(x)$ which is continuous at c and such that $f(x) \cdot g(x)$ is continuous at c.

__________ If $\lim_{x \to c} f(x) = L$, then there exists a $\delta > 0$ such that $0 < |x - c| < \delta \implies L - 1 < f(x)$.

__________ If $\lim_{x \to c} f(x) = L$, then there exists a $\delta > 0$ such that $0 < |x - c| < \delta \implies L - 1 < f(x) < L$.