Midterm 2
Wednesday, 5-8-96 10:30-11:30
Math 153
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Name ____________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Theorem 2.1 (Monotonic Sequence Theorem) A sequence a_n which is weakly increasing and bounded is convergent. A weakly decreasing, bounded sequence is also convergent.

Theorem 2.2 If Σa_n is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

Theorem 2.3 (Divergence Test) If $\lim_{n\to\infty} a_n \neq 0$, then Σa_n is a divergent series. (This is just the contrapositive of the previous theorem)

Theorem 2.4 (Basic Comparison Test) Let Σa_k be a series with non-negative terms.

- Σa_k converges if there exists a convergent series Σc_k with nonnegative terms such that $a_k \leq c_k$ for all k sufficiently large.
- Σa_k diverges if there exists a divergent series Σd_k with nonnegative terms such that $d_k \leq a_k$ for all k sufficiently large.

(first three problems 2 pts each)

1. Given a sequence a_i, define s_n, the sequence of partial sums of a_i.

2. State (in terms of $\lim_{n\to\infty}$) the definition of "$\sum_{i=1}^{\infty} a_i$ converges to L."

3. State (in terms of the ϵ, N definition of $\lim_{n\to\infty}$) the definition of "$\sum_{i=1}^{\infty} a_i$ converges to L."

4. State the definition of the harmonic series, and use integrals to show the harmonic series is divergent. (3 pts)

5. Prove: $\displaystyle\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $x \in (-1, 1)$. (3 pts)

6. A frog is at the bottom of a well which is thirty feet deep. Each day, it can climb up the vertical wall three feet, but at night, it slides back down two feet. How many days will it take the frog to escape the well? (1 pt)

7. State the definition of "$\sum_{i=1}^{\infty} a_i$ is absolutely convergent." (2 pts)

8. Use the Basic Comparison Test to show that an absolutely convergent series is a convergent series. (2 pts)

9. The operator $J : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $J(x, y) = (-y, x)$. (Geometrically, J is a rotation by $90°$ of the plane $\mathbb{R}^2$, but you will not need this fact. 3 pts)

a. Check both parts of the definition of linear operator to show that J is a linear operator.

b. Show that $J(J(x, y)) = -1 \cdot (x, y)$ and that $J(J(J(J(x, y)))) = (x, y)$.

c. If $r \in \mathbb{R}$ and $s \in \mathbb{R}$ are scalars, define $(r + sJ) : \mathbb{R}^2 \to \mathbb{R}^2$ as $(r + sJ)(x, y) = r \cdot (x, y) + s \cdot J(x, y)$. Show that $r + sJ$ is a linear operator and that the composition $(r - sJ)((r + sJ)(x, y))$ is just multiplication by some nonnegative real scalar.

10. Use the Monotonic Sequence Theorem to prove the first ("convergence") part of the Basic Comparison Test. (3 pts)

11. State whether each of the following series is convergent or divergent. Show all work and state which theorem you're using. (Use any theorem from class or the book; do not try to find an exact value for the infinite sum.) (3 each, best 5 of 6)

- $\displaystyle\sum_{i=0}^{\infty} \frac{10^i}{i!}$
- $\displaystyle\sum_{i=0}^{\infty} \frac{(2n+1)^2}{(3n-1)^2}$
- $\displaystyle\sum_{k=2}^{\infty} \frac{\log(k^{50})}{e^{\pi k}}$
- $\displaystyle\frac{2}{2} + \frac{4}{3^2} + \frac{8}{4^3} + \frac{16}{5^4} + \frac{32}{6^5} + \cdots$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$
- $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n+1}+1}$

12. True or False. Write **T** or **F** next to each sentence. The numbers i and n refer to natural numbers in the domain of a sequence, and the sequence a_i is *nonnegative*: $a_i \geq 0$. The symbol $\sum$ abbreviates $\sum_{i=1}^{\infty}$. $^{(1\ each)}$

______ If $\sum a_i$ converges, then so does $\sum a_i^2$.

______ If $\sum a_i$ converges, then so does $\sum \sqrt{a_i}$.

______ a_n is a decreasing sequence and $\lim_{n\to\infty} a_n = 0 \implies \sum a_i$ converges.

______ a_n is a decreasing sequence and $\lim_{n\to\infty} a_n = 0 \implies \sum(-1)^i a_i$ converges.

______ If $\sum a_i$ converges, then (a_n) is a decreasing sequence.

______ The set of weakly increasing functions with domain $\mathbb{R}$, where $f+g$ and $k \cdot f$ are defined as usual, forms a vector space.

______ The set of divergent series $\{(d_1, d_2, d_3, \dots) : \sum d_i \text{does not exist}\}$, where sums $c+d$ and scalar multiples $k \cdot d$ are defined as usual, forms a vector space.

______ The set of second-degree polynomials in two variables, $\{Ax^2 + Bxy + Cy^2 + Dx + Ey + F\}$, where addition and scalar multiplication are as usual for polynomials, forms a vector space.

13. Give an example of a sequence a_n and a function $f(x)$ with domain $\mathbb{R}$ such that $f(x) \geq 0$ on $[1, \infty)$, $f(n) = a_n$ for $n \in \mathbb{N}$, and $\int_0^\infty f(x)dx$ exists, but $\sum a_n$ diverges. (hint: the integral test says it's impossible if f is continuous and weakly decreasing. Try graphing the sequence first. 2 pts)

14. Give two examples of sequences $a_n \geq 0$ and $b_n \geq 0$ for which the ratio test is inconclusive ($\lambda = 1$), and where $\sum a_n$ is convergent and $\sum b_n$ is divergent. (2 pts)