Midterm 2

Friday, 2-16-96 10:30-11:20

Math 152

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Name ______________________ Student ID# __________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–7:

k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Theorem 2.1 The function $\sqrt{x}$ is continuous on $[0, \infty)$, and differentiable on $(0, \infty)$. The derivative function $\frac{d}{dx}\sqrt{x}$ is equal to $\frac{1}{2\sqrt{x}}$.

Theorem 2.2 If $f(x)$ and $g(x)$ are differentiable on (a, b), and $f'(x) = g'(x)$ on (a, b), then $\exists C \in \mathbb{R}$ such that $f(x) = g(x) + C$.

Theorem 2.3 If $f(x)$ is continuous on $[a, b]$, then the function

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), with derivative $F'(x) = f(x)$.

Theorem 2.4 If $0 \leq f(x)$ on $[a, b]$ and $f(x)$ is integrable, then $0 \leq \int_a^b f(x)dx$.

In the following questions, give only one definition, version of the required statement, or example. In the proofs, you may cite any theorem from the cover page.

1. State the Fundamental Theorem of Calculus for the function $f(x)$ which is continuous on $[a, b]$. $^{(2pts)}$

2. Prove the Fundamental Theorem of Calculus.$^{(3pts)}$

3. State the definition of "$f(x)$ is a one-to-one function."$^{(2pts)}$

4. State the definition of the "range" of a function $g(x)$. $^{(2pts)}$

5. Use the Fundamental Theorem of Calculus to prove that a function $f(x)$, with domain $\mathbb{R}$ and which has a positive, continuous derivative $f'(x)$ on $\mathbb{R}$, is an increasing function.$^{(4pts)}$

6. State and prove the formula for "Integration by Parts" (for indefinite integrals)$^{(2pts)}$

7. $^{(4pts)}$ For $f(x)$ integrable on $[a,b]$, prove

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

8. Give an example of a function $f(x)$ which is integrable on $[0,2]$ but not continuous. Sketch both $f(x)$ and $F(x) = \int_0^x f(t)dt$. $^{(2pts)}$

9. Give an example of a function which is one-to-one but neither increasing nor decreasing on its domain.$^{(2pts)}$

10. Give an example of a function $f(x)$ which is decreasing, but $g(x) = f(x) + x$ is increasing.$^{(2pts)}$

11. State the definition of the function $\log(x)$, state its domain and range, and sketch a graph of $\log(x)$ (larger than a quarter). Prove that $2 < \log(10) < 3$.$^{(4pts)}$

12. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

____________ $\int_{-2}^{1} \frac{1}{x^2} dx = \frac{-1}{1} - \frac{-1}{-2} = -\frac{3}{2}$.

____________ $\log(a+b) = \log(a) \cdot \log(b)$.

____________ $(\log(x))^4 = 4\log(x)$.

____________ $\int_a^c f(x)dx - \int_a^b f(x)dx = \int_b^c f(x)dx$, (assuming all of these integrals exist).

13. Evaluate the following definite and indefinite integrals any way you want. (I recommend the methods of parts, substitution, guessing, and checking your answer. I do not recommend calculating l.u.b.'s of lower sums, leaving these blank, unnecessarily simplifying your answer, or cheating.) **3 points each; do all five and I will count your best four answers.**

a.) $\int \log(x)dx =$

b.) $\int \frac{x}{1-x^2} dx =$

c.) $\int_e^{e^2} \frac{1}{8x} dx =$

d.) $\int \frac{\log(x+a)}{x+a} dx =$

e.) $\int_0^a y\sqrt{a^2 - y^2} dy =$

14. I've graphed the function $f(x) = \frac{2\sqrt{x}}{1+x}$. The volume of the region obtained by revolving the area under the curve around the x-axis is approximated by discs centered on the x-axis, with radius $f(x)$. The exact formula is $\int_0^3 \pi(f(x))^2 dx$.

a. for partial credit (2 pts), give an upper bound for the volume, accurate to within 25% of the exact volume.

b. for full credit (5 pts), calculate the exact volume. (Hints: when I tried this problem, I used a $u = x + 1$ substitution, and got an answer involving the logarithm of an integer.)