Midterm 2

Friday, 11-8-95 10:30-11:20

Math 151

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Name ______________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–5.

k is a constant.

$\lim_{x\to c} f(x)$ exists.

$\lim_{x\to c} g(x)$ exists.

Conclusions:

1. $\lim_{x\to c} k = k$.
2. $\lim_{x\to c} x = c$.
3. $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim_{x\to c} f(x)$.
4. $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.
5. $\lim_{x\to c}(f(x) \cdot g(x)) = (\lim_{x\to c} f(x)) \cdot (\lim_{x\to c} g(x))$.

6. **The Composite Limit Theorem.** If $\lim_{x\to c} g(x) = L$ AND f is continuous at L, then

$$\lim_{x\to c}(f \circ g)(x) = f(\lim_{x\to c} g(x))$$

which could also be written as

$$\lim_{x\to c} f(g(x)) = f(L)$$

1. State the definition of

$$\lim_{x \to c} f(x) = L.$$

(2 *points*)

2. State the "limit" (not the $\epsilon,\ \delta$) definition of "$f(x)$ is continuous at c."
(2 *points*)

3. State the $\epsilon,\ \delta$ (not the "limit") definition of "$f(x)$ is continuous at c."
(2 *points*)

4. State the definition of "$f(x)$ is decreasing on S."
(2 *points*)

5. State the "Uniqueness of the Limit" theorem.
(2 *points*)

6. The following three questions are "ϵ, δ" proofs. Don't use the limit theorems on page one; use the definition of limit. Do all of them; I will grade them, 6 points each with partial credit, but then I will only count your best two answers in the total score, for a maximum score of 12 points.

6a. Prove, using the definition,

$$\lim_{x \to 4} 3x^2 - x = 44.$$

scratchwork:

formal proof:

6b. Prove, using the definition,

$$\lim_{x \to -2} -x^2 + 8x - 1 = -21.$$

scratchwork:

formal proof:

6c. Prove, using the definition,

$$\lim_{x\to 2} x^2 - 4x + 5 = 1.$$

scratchwork:

formal proof:

7. Compute the following limits any way you want. You don't have to explain your reasoning. Write either a real number after the "=," or "**DNE**" if the limit Does Not Exist. Some of the formulas involve "y" — treat it as a constant and express your answer in terms of y. (1 pt each)

- $\displaystyle\lim_{x\to 5} \frac{6}{4-x} =$
- $\displaystyle\lim_{x\to -3} \sqrt{4(x^2+16)} =$
- $\displaystyle\lim_{x\to 5} \frac{x^2-25}{\sqrt{x-5}} =$
- $\displaystyle\lim_{x\to 0^+} (\sqrt{x})^2 =$
- $\displaystyle\lim_{x\to 1} \frac{(y+x)^2-y^2}{x} =$
- $\displaystyle\lim_{x\to 0} \frac{xy}{|x|} =$

  (Hint: this limit exists for some y and not for others.)

8. For each of the five questions below, give an *example* of a function with all of the required properties. It will be good enough to give a formula, or to sketch a graph that is larger than a quarter. The five functions are unrelated to each other or to any other functions on this test. Try all five; I will count your best four answers. $^{2\ each}$

A. The function $f(x)$

- has domain $\mathbb{R}$,
- AND is continuous at $c = 0$,
- AND is not continuous at c if $c \neq 0$.

B. The function $g(x)$

- has domain $[-1, 1]$,
- AND $g(-1) = 4$,
- AND $g(1) = -2$,
- AND $\forall x \in [-1, 1],\ g(x) \neq 0$.

C. The function $h(x)$

- is a rational function, i.e., $h(x) = \frac{p(x)}{q(x)}$, for polynomials $p(x)$ and $q(x)$.
- AND $\lim\limits_{x \to 2} \dfrac{h(x)}{x-2} = 0$
- AND $\lim\limits_{x \to 4} h(x)$ does not exist.

(write a formula for this one, not just a graph.)

(problem 8, continued) D. The function $j(x)$

- does not have a limit at $c = 0$,
- AND $(j(x))^2$ does have a limit at $c = 0$,
- AND $(j(x))^2$ is not continuous at $c = 0$.

E. The function $z(x)$

- is a polynomial,
- AND $\exists\ B > 0$ such that $|\frac{1}{z(x)}| < B$ for all $x \in \mathbb{R}$.

9. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

__________ If $f(x)$ is continuous at every c in the open interval $(a,\ b)$, then $\lim_{x \to a^+} f(x)$ exists.

__________ $\epsilon > 0$ AND $M > 0$ AND $\delta = \min\{\frac{|M|}{2},\ \frac{\epsilon}{2(1+|M|)}\} \Rightarrow \delta > 0$.

__________ If $g(x)$ is continuous at c AND $\lim_{y \to g(c)} f(x) = L$, then $\lim_{x \to c} f(g(x)) = L$.

__________ $\lim_{x \to c} g(x) = 0$ AND $\lim_{x \to c} f(x) = 0 \Rightarrow \lim_{x \to c} \dfrac{f(x)}{g(x)} = 1$.

__________ $|x - y| \geq |x| - |y|$.

__________ If $(f(x))^2$ is continuous, then $f(x)$ is continuous.

__________ If $f(x)$ and $g(x)$ are continuous at c, and $f(x) = g(x)$ for every $x \in (c - \delta, c) \cup (c, c + \delta)$, then $f(c) = g(c)$.

__________ $\lim_{x \to c} f(x) = L \Rightarrow \lim_{x \to c} f(x + 1) = L + 1$.

10. Use the Limit Theorems #1–6 from the cover page **one at a time** to carefully calculate the following limits: (Justify each step with the numbered theorem; for example, I've already done "a.")$^{(1\ each)}$

a. $\lim_{x\to 2} 2x + 2 = \lim_{x\to 2} 2x + \lim_{x\to 2} 2$ (using #4)
$= 2\lim_{x\to 2} x + \lim_{x\to 2} 2$ (#3) $= 2\lim_{x\to 2} x + 2$ (#1)
$= 2 \cdot 2 + 2$ (#2) $= 6$.

b. $\lim_{x\to 2} x^2 - 4x + 5 =$

c. $\lim_{x\to -3} x^2(2x-1) =$

d. Use the fact that $f(x) = \frac{1}{x}$ is continuous at L if $L \neq 0$, and show carefully (as in b. and c.) that

$$(\lim_{x\to c} h(x) = N \text{ AND } \lim_{x\to c} g(x) = L \neq 0) \Rightarrow \lim_{x\to c} \frac{h(x)}{g(x)} = \frac{N}{L}.$$

e. Similarly, using the fact that $f(x) = \sqrt{x}$ is continuous at L if $L > 0$, show that $\sqrt{\frac{1}{x^2-1}}$ is continuous at every point in its domain. (in particular, what is the domain?)