Name ______________________________ Student ID# _______________

Midterm 1. Friday, 6-28-96 9:30-11:20. Math 131. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 300.

A *field* is any set $\mathbb{F}$, together with two operations $\cdot$ and $+$, subject to the following axioms:

1. The commutative law for $+$: $x + y = y + x$.
2. The associative law for $+$: $(x + y) + z = x + (y + z)$.
3. Additive identity: There is an element $0 \in \mathbb{F}$ such that $x + 0 = x$.
4. Additive inverse: For every $x \in \mathbb{F}$, there is an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$.
5. The commutative law for $\cdot$: $x \cdot y = y \cdot x$.
6. The associative law for $\cdot$: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
7. Multiplicative identity: There is an element $1 \in \mathbb{F}$ such that $1 \neq 0$ AND $1 \cdot x = x$.
8. Multiplicative inverse: For every $x \in \mathbb{F}$, $x \neq 0 \Rightarrow$ There is an element $\frac{1}{x} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$.
9. The distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$.

$\mathbb{F}$ is an *ordered field* if it contains a subset of positive numbers $P = \{x > 0\} \subseteq \mathbb{F}$ such that:

- Additive Order: $(x > 0$ AND $y > 0) \Rightarrow (x + y) > 0$.
- Multiplicative Order: $(x > 0$ AND $y > 0) \Rightarrow (x \cdot y) > 0$.
- Trichotomy: Every $x \in \mathbb{F}$ has *exactly* one of the following properties:
  - ⋄ $x > 0$,
  - ⋄ $x = 0$,
  - ⋄ $-x > 0$.

Note that any ordered field contains the natural numbers, $\{1, 2 = 1 + 1, 3 = 1 + 1 + 1, 4 = 1 + 1 + 1 + 1, \ldots\}$, the integers $\mathbb{Z}$, which are the natural numbers, together with their additive inverses and 0, and the field of rational numbers, $\mathbb{Q}$.

The real numbers $\mathbb{R}$ form an ordered field which also satisfies the Completeness Axiom, which states that any nonempty set $S \subseteq \mathbb{R}$ which has a real upper bound also has a real least upper bound.

1. State the definition of $|x|$.$^{(1\ point)}$

2. State the Triangle Inequality.$^{(1)}$

3. State the definition of "$f(x)$ is an increasing function."$^{(1)}$

4. $^{(1)}$ State the formula for the distance between two numbers $x,\ y \in \mathbb{R}$.

5. $^{(1)}$ State the formula for the distance between two points (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^2$.

6. $^{(1)}$ State the definition of "b is an upper bound of the set $S \subseteq \mathbb{R}$."

7. $^{(1)}$ The function $f(x)$ has domain D. State the definition of the "range of $f(x)$."

8. $^{(1)}$ State the definition of the composition function $h \circ g$.

9. The set $L = \{(x, y) \in \mathbb{R}^2 : Ax + By + C = 0\}$ is a line in the plane.$^{(3)}$

- L is vertical if ______________.
- If L isn't vertical, its slope is ______________.
- If L isn't vertical, it intersects the line $x+1 = 0$ at the point ______________.

10. The set $\{(x, y) \in \mathbb{R}^2 : y = ax^2 + bx + c,\ a \neq 0\}$ is a parabola. Find the points of intersection of the parabola and the x-axis, $\{y = 0\}$, and state conditions on a, b, and c determining the number of points of intersection (twice, once, or $\emptyset$).$^{(2)}$

11. Solve the following equations or inequalities, that is, find the subset of $\mathbb{R}$ consisting of all elements x with the required property or properties. You don't have to explain your reasoning, but do **write your answer in interval notation**. $^{(1 each)}$

- ____________ $-6 < -2x < 8$.
- ____________ $x^3 \leq 8$.
- ____________ $|x - c| < 0$.
- ____________ $|x| + 1 = |x + 1|$.
- ____________ $\frac{1}{x} < b$, $b > 0$.
- ____________ $\frac{1}{x} < b$, $b < 0$.
- ____________ $|-2x - 4| < \epsilon$, $\epsilon > 0$.
- ____________ $x + 2 < 0$ AND $\frac{x}{2} - 3 \geq 2$.
- ____________ $0 < \sqrt{x} < x$.
- ____________ $-1 < x < 3$ OR $(2 < x \leq 4$ AND $2x \geq 5)$.
- ____________ $(-1 < x < 3$ OR $2 < x \leq 4)$ AND $2x \geq 5$.
- ____________ $|\frac{1}{x} - 1| < 1$. (Hint: the answer is an open ray.)

12. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

______________ The quotient of non-zero irrational numbers is irrational.

______________ $(c < 0 \text{ AND } |x - c| < -c) \Rightarrow x < 0$.

______________ $x < 2 \Rightarrow x < 10$.

______________ $|x - 5| < \epsilon/3 \Rightarrow |3x - 15| < \epsilon$.

______________ $|x - 5| < \epsilon/10 \Rightarrow |3x - 15| < \epsilon$.

______________ $|x - 5| < \epsilon \Rightarrow |x - 5| \cdot |x - 2| < \epsilon$.

______________ $a \neq 0, b \neq 0, c \neq 0 \Rightarrow a/(b/c) = (a/b)/c$.

______________ If S is a finite, nonempty set of numbers, then l.u.b.$S \in S$.

______________ $a^2 + b^2 = 0 \Rightarrow a = 0$.

13. Graph the following functions below, or on the other side of the page.$^{(1\ each)}$

- $f(x) = |x - 2|$
- $f(x) = 1 + \frac{1}{x}$
- $f(x) = \sqrt{4 - x^2}$
- $f(x) = |x(x - 2)|$

14. $^{(2)}$Recall $x < y$ means that the difference $y - x$ is in the set of positive numbers, as described on page 1: $y - x > 0$. Prove, justifying each step, that if $c < 0$ and $x < y$, then $cy < cx$.

15. $^{(2)}$Use the Triangle inequality and the fact that $a = a - b + b$ to prove $|a| - |b| \leq |a - b|$.

16. $^{(1)}$We discussed in class why there's no smallest positive number. Explain why there's no biggest positive number.

17. Consider the following functions:

$$\begin{aligned} f(x) &= \sqrt{x} - 1 \\ g(x) &= (x+1)^2 \end{aligned}$$

a. What are the domains of f and g?$^{(1)}$

b. Compute the following sums, products, or compositions, and explicitly state their domains.$^{(1 each)}$ (Hint: you don't have to simplify your answer.)

- $(f + g)(x) =$
- $(\frac{f}{g})(x) =$
- $(\frac{g}{f})(x) =$
- $(f \circ g)(x) =$
- $(g \circ f)(x) =$
- $(f \circ f)(x) =$