Midterm 1

Monday, 4-15-96 10:30-11:30 **Euler's birthday: 1707–1783.**

Math 153

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Name ______________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–7:

k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0.$
2. $\frac{d}{dx}x = 1.$
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x).$
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x).$
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\dfrac{f(x)}{g(x)}) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x).$

Theorem 2.1 (Monotonic Sequence Theorem) A sequence a_n which is increasing and bounded is convergent. A decreasing, bounded sequence is also convergent.

Theorem 2.2 (Pinching Theorem) If $a_n \leq b_n \leq c_n$ for $n \geq K$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem 2.3 If $\lim_{n\to\infty} c_n = c$, and $\{c_n\} \subseteq \text{Domain}(f)$, and $f(x)$ is continuous at c, then $\lim_{n\to\infty} f(c_n) = f(c)$.

In the following six questions, give only one definition, version of the required statement, or example. In the proofs, you may cite any theorem from the cover page. (2 pts each)

1. Define "a_n is a decreasing sequence."

2. Define "a_n is a bounded sequence."

3. State the ϵ, N definition of "a_n converges to L."

4. State the definition of "$\lim_{n\to\infty} a_n = +\infty$."

5. State the "0/0" form of L'Hôpital's Rule, for $\displaystyle\lim_{x\to c^+} \frac{f(x)}{g(x)}$.

6. a. Which equations define a "conic" curve in the x, y-plane? (State the general form of the polynomial equation)

b. Give an example of a conic which contains the point $(0,0)$ and does not contain any lines.

7. Use the Pinching Theorem to show that $\lim_{n\to\infty} \log(n) - \log(n+1) = 0$ and $\lim_{n\to\infty} \frac{2^n}{n!} = 0$. (2 pts each)

8. a. Calculate the following limits any way you want. Expressions in the variable n are sequences, and expressions in the variable x are functions with domain $\mathbb{R}$. Write either a number, or $+\infty$, or $-\infty$, or "None," if none of the previous three holds. (1.5 each)

- $\lim_{n\to\infty} \frac{(2n+1)^2}{(3n-1)^2} =$
- $\lim_{n\to\infty} \cos(\pi n) =$
- $\lim_{n\to\infty} \cos(2\pi n) =$
- $\lim_{n\to\infty} (2n)^{(1/n)} =$
- $\lim_{n\to\infty} \frac{5^{n+1}}{4^{2n-1}} =$
- $\lim_{x\to 1^+} \frac{5-3x^2}{1-x^2} =$
- $\lim_{x\to 1^+} \frac{d}{dx}((x-1)^{5/7}) =$
- $\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} =$
- $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sin x} =$
- $\lim_{x\to 0^+} (e^x + x)^{1/x} =$

9. (2 pts each) a. Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $0 < a < b$. Find a pair of points demonstrating that this curve not the graph of a function. However, the points (x, y) where $y \geq 0$ do form the graph of a function $f(x)$: find this function and state its domain.

b. Calculate the derivative $f'(x)$ and state the equations of all vertical tangent lines.

c. Show that $\lim_{x \to \infty} f(x) - \frac{b}{a}x = 0$.

d. On the same axes, sketch a graph of the hyperbola and the line $ay + bx = 0$.

10. True or False. Write **T** or **F** next to each sentence. The numbers n refer to natural numbers in the domain of a sequence. You need not explain your answer.$^{(1\ each)}$

______ If the function $f(x)$ defined on $\mathbb{R}$ satisfies $\lim_{x\to\infty} f(x) = L$ then the sequence $f(n)$ satisfies $\lim_{n\to\infty} f(n) = L$.

______ If $0 \leq b_n \leq c_n$ and the sequence c_n is decreasing and convergent, then b_n is convergent.

______ If b_n is an increasing sequence, then the sequence $c_n = (b_n + b_{n+1})$ is also increasing.

______ $\lim_{b\to\infty} \int_{-b}^{b} f(x)dx = 0 \implies \lim_{b\to\infty} \int_{0}^{b} f(x)dx = 0.$

______ If the sequence d_n is defined recursively by $d_1 = 2$ and $d_n = (d_{n-1})!$ for $n > 1$, then d_n is divergent.

11. (3 pts each) a. Use the ϵ, N definition of limit to prove $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

b. Use the M, N definition of limit to prove $\lim_{x\to\infty} x^3 + 1 = +\infty$.