Midterm 1
Wednesday, 1-24-96 10:30-11:20
Math 152
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Name ____________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Hypotheses for theorems 1–7:
k is a constant.

$f(x)$ and $g(x)$ are differentiable functions, with derivatives $f'(x)$ and $g'(x)$.

The notation $\frac{d}{dx}$ is also used to mean "the derivative of".

Conclusions:

1. $\frac{d}{dx}k = 0$.
2. $\frac{d}{dx}x = 1$.
3. $\frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}f(x)$.
4. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$.
5. $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + f'(x)g(x)$.
6. $g(x) \neq 0 \Rightarrow \frac{d}{dx}(\dfrac{f(x)}{g(x)}) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.
7. $\frac{d}{dx}((f \circ g)(x)) = f'(g(x)) \cdot g'(x)$.

Theorem 0.1 The function $\sqrt{x}$ is continuous on $[0, \infty)$, and differentiable on $(0, \infty)$. The derivative function $\frac{d}{dx}\sqrt{x}$ is equal to $\frac{1}{2\sqrt{x}}$.

Answers to the following questions should be one sentence. (2 pts each)

1. State the Mean Value Theorem for the function $f(x)$ which is continuous on $[a, b]$ and differentiable on (a, b).

2. State the definition of "$f(x)$ has a maximum value on the set S."

3. State the definition of "$f(c)$ is a local maximum value of $f(x)$." (assume the domain of $f(x)$ is $\mathbb{R}$)

4. State the definition of "c is a critical point in the domain of $f(x)$."

5. Assume the interval $[a, b]$ is partitioned into subintervals by the set $\{a = x_0 < x_1 < x_2 < x_3 \ldots < x_{n-1} < x_n = b\}$. State the formula for an "Upper Sum" of the function $f(x)$ using this partition. (assume f is continuous on $[a, b]$. define all symbols used in the sum.)

6. State the definition of "$f(x)$ is decreasing on the set S."

7. State the definition of a "rectangle" in the plane $\mathbb{R}^2$. (fill in the following blank:)

$\{(x, y) \in \mathbb{R}^2$ such that ____________________________________$\}$

8. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.(1 *each*)

____________ Every polynomial function has either a local maximum value or a local minimum value.

____________ If $f(x)$ does not have a global maximum value on the set S then it does not have any local maximum values.

____________ If $f(x)$ does not have any local maximum values on the set S then it does not have a global maximum value.

____________ If $f'(x) > 0$ on $S \subseteq \mathbb{R}$ then $f(x)$ is increasing on S.

____________ If $f(x)$ has only one local maximum value $f(c)$ on the set S then $f(c)$ is the global maximum value on S.

____________ If $f(x)$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is differentiable on (a, b).

____________ If $f(x)$ is continuous and positive on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is increasing on (a, b).

____________ If $f(x)$ is continuous and increasing on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is nonnegative on (a, b).

____________ If $f(c)$ is the global minimum value of $f(x)$ on the set S and $g(d)$ is the global minimum value of $g(x)$ on S, then $f(c) + g(d)$ is the global minimum value of $(f + g)(x)$ on S.

____________ If $f''(c) > 0$ then $f(c)$ is a local minimum value.

____________ If $f(a) = 0$ and $f(b) = 0$ and $f(x)$ is differentiable on (a, b), then there is a critical point $c \in (a, b)$.

____________ If c is a critical point of $f(x)$ and $f''(c)$ exists and is non-zero, then $f(c)$ is a local extreme value.

____________ If $f(x)$ is continuous on $[a, b]$ then $\int_a^b f(t)dt + \int_b^a f(t)dt = 0$.

____________ $\int_a^b f(t)dt = 0 \iff \int_a^b (f(t))^2 dt = 0$.

9. A circle of radius $r > 0$ is centered at the origin $O = (0, 0)$. A line segment with endpoints $A = (x, 0)$ and $B = (0, y)$ on the positive coordinate axes is tangent to the circle at point P. Find the length of line segment AB as a function $f(x)$ and the minimum value of the function. Using the following hints might be worth partial credit. (7 pts)

- Draw a picture of the circle, the line segment, and the radius OP.
- State the domain of $f(x)$.
- $AB = AP + PB$. Express these lengths in terms of r and x (not y), using the Pythagorean Theorem for AP and the fact that $\triangle OPA$ and $\triangle OPB$ are similar, so $\frac{AP}{OP} = \frac{OP}{BP}$.
- Find the critical points $x = c$, and use the first derivative test to find the minimum value $f(c)$. Both c and $f(c)$ should be simple expressions depending on r.
- The derivative theorems on the cover page might be useful.

(5 pts each)

10. Use the Mean Value Theorem to prove that any function $f(x)$ such that $f'(x) = 0$ on (a, b) is constant on (a, b).

11. Consider a continuous function $f(x)$ which is positive on the interval $[a, b]$. Let P be the partition $\{a < b\}$, so that the Lower Sum $L_f(P)$ for $f(x)$ is the area of one rectangle. Let $Q = \{a < c < b\}$ be a partition so that the lower sum $L_f(Q)$ is the sum of the areas of two rectangles. Show that $L_f(P) \leq L_f(Q)$.

12. Recall that the "Area function" takes as input some subsets of the plane and gives as output a nonnegative real number, written Area($\mathcal{A}$) for $\mathcal{A} \subseteq \mathbb{R}^2$. This function has the following properties:

- If $\mathcal{A}$ and $\mathcal{B}$ are both in the domain of the area function, and $\mathcal{A} \cap \mathcal{B} = \emptyset$, then the area of $\mathcal{A} \cup \mathcal{B}$ is also defined and is equal to Area($\mathcal{A}$)+Area($\mathcal{B}$).
- If $\mathcal{A}$ and $\mathcal{B}$ are both in the domain of the area function, and $\mathcal{A} \subseteq \mathcal{B}$, then the set $\mathcal{B} \setminus \mathcal{A} = \{(x, y) \in \mathcal{B} \text{ AND} (x, y) \notin \mathcal{A}\}$ is also in the domain of the area function.

Show that if $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \cap \mathcal{B}$ all are in the domain of the area function, then so is $\mathcal{A} \cup \mathcal{B}$ and Area($\mathcal{A} \cup \mathcal{B}$) =Area($\mathcal{A}$)+Area($\mathcal{B}$)−Area($\mathcal{A} \cap \mathcal{B}$).