Name ____________________________ Student ID# ______________

Midterm 1. Wednesday, 10-18-95 10:30-11:20. Math 151. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

A *field* is any set $\mathbb{F}$, together with two operations $\cdot$ and $+$, subject to the following axioms:

1. The commutative law for $+$: $x + y = y + x$.
2. The associative law for $+$: $(x + y) + z = x + (y + z)$.
3. Additive identity: $\exists 0 \in \mathbb{F}$ such that $x + 0 = x$.
4. Additive inverse: $\forall x \in \mathbb{F},\ \exists -x \in \mathbb{F}$ such that $x + (-x) = 0$.
5. The commutative law for $\cdot$: $x \cdot y = y \cdot x$.
6. The associative law for $\cdot$: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
7. Multiplicative identity: $\exists 1 \in \mathbb{F}$ such that $1 \neq 0$ AND $1 \cdot x = x$.
8. Multiplicative inverse: $\forall x \in \mathbb{F},\ x \neq 0 \Rightarrow \exists \frac{1}{x} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$.
9. The distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$.

$\mathbb{F}$ is an *ordered field* if it contains a subset of positive numbers $\{x > 0\} \subseteq \mathbb{F}$ such that:

- Additive Order: $(x > 0$ AND $y > 0) \Rightarrow (x + y) > 0$.
- Multiplicative Order: $(x > 0$ AND $y > 0) \Rightarrow (x \cdot y) > 0$.
- Trichotomy: Every $x \in \mathbb{F}$ has *exactly* one of the following properties:
  - ◇ $x > 0$,
  - ◇ $x = 0$,
  - ◇ $-x > 0$.

Note that any ordered field contains the natural numbers,

$$\{1, 2 = 1 + 1, 3 = 1 + 1 + 1, 4 = 1 + 1 + 1 + 1, \ldots\},$$

and the integers $\mathbb{Z}$, which are the natural numbers, together with their additive inverses and 0.

The real numbers $\mathbb{R}$ also satisfy the Least Upper Bound Axiom, which states that any nonempty set $S \subseteq \mathbb{R}$ which has a real upper bound also has a real least upper bound.

1. State the definition of $|x|$.$^{(1\ point)}$

2. State the Triangle Inequality.$^{(1)}$

3. Give an example of integers a and b where the Triangle Inequality is a strict inequality. (this means: show, by plugging your integers into both sides of the triangle inequality and calculating, that you get different answers.) $^{(1)}$

$a =$ ________

$b =$ ________

4. Prove the Triangle Inequality.$^{(2)}$

5. Solve the following inequalities, that is, find the subset of $\mathbb{R}$ consisting of all elements x with the required property or properties. You don't have to explain your reasoning, but do **write your answer in interval notation**. $^{(1each)}$

- ________________ $6 < -2x < 8$.
- ________________ $x^2 \leq 4$.
- ________________ $|x| > -1$.
- ________________ $x \neq |x|$.
- ________________ $0 < |x - c|$.
- ________________ $|f(x) - L| < 0.02$. Solve for $f(x)$.
- ________________ $x^2 - 2x < 0$ AND $x \notin \mathbb{Z}$.
- ________________ $\sqrt{x} < 1$.
- ________________ $|2 - x| \leq |x| + 2$.

6. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

______________ $0 < a < b \Rightarrow a^2 < ab$.

______________ $(c > 0 \text{ AND } |x - c| < \frac{c}{2}) \Rightarrow x > 0$.

______________ $(4, 5] \cap [4\frac{1}{2}, 7)$ is a closed interval.

______________ $(4, 5] \cup [4\frac{1}{2}, 7)$ is a closed interval.

______________ $0 < a \Rightarrow \sqrt{a} < a^2$.

______________ The set of five-digit positive integers has an upper bound.

______________ The set $S = \{x \in \mathbb{Z} : x \geq 0 \text{ AND } \frac{x}{10} \in \mathbb{Z}\}$ has an upper bound.

______________ $x \leq y \Rightarrow x = y$.

______________ $x = y \Rightarrow x \leq y$.

______________ $(b \neq 0 \text{ AND } d \neq 0) \Rightarrow \frac{a}{b} + \frac{c}{d} = \frac{a+c}{bd}$.

7. State the definition of "x is a rational number."$^{(1)}$

8. Prove, using the definition you just gave and the technique of contradiction, that the product of a non-zero rational number and an irrational number is irrational.$^{(2)}$

9. Adam goes to the grocery store with a fifty-cent coupon for cat food. He knows, from previous experience, that Cashier A will enter the price of the cat food, subtract the coupon, and then add 8.75% sales tax. He also knows that Cashier B will do this in a different order, adding sales tax and *then* subtracting the fifty cents from the coupon. Don't do any calculations with numbers (for example, the price of cat food is more than 50 cents but you don't know the exact amount), but try to explain in your own words why Adam would rather conduct his transaction with Cashier A, and which of the axioms is (are) relevant to your discussion.$^{(2)}$

10. Let S be a subset of the real numbers. Give a definition of the statement "b is an upper bound for S."$^{(2)}$

11. Let S be a nonempty subset of the real numbers. Give a definition of the statement "B is a least upper bound for S."$^{(2)}$

12. Here are a list of axioms and definitions. Some of them might get used twice in the proof below, or not at all; write the letter from the list next to the step that it justifies. I have already written one of the answers.$^{(1\ each)}$

A. Commutative law for $\cdot$

B. Multiplicative identity

C. Definition of "y is a multiplicative inverse of x."

D. Multiply both sides by $\frac{1}{x}$

E. Definition of "$\frac{1}{x}$ is a multiplicative inverse of x."

F. "$y = \frac{1}{x}$" is the definition of "$\frac{1}{x}$ is unique."

G. Distributive Law

H. Multiplicative Order

I. Associative law for $\cdot$

Theorem. $x \neq 0 \Rightarrow$ the multiplicative inverse $\frac{1}{x}$ is unique.

PROOF: Suppose that, in addition to the element $\frac{1}{x}$ we already know about, there is some other number y that is a multiplicative inverse for x.

_______________ $x \cdot y = 1$.

_______________ $\frac{1}{x} \cdot (x \cdot y) = \frac{1}{x} \cdot 1$.

_______________ $(\frac{1}{x} \cdot x) \cdot y = \frac{1}{x} \cdot 1$.

_______________ $(x \cdot \frac{1}{x}) \cdot y = \frac{1}{x} \cdot 1$.

_______________ $(x \cdot \frac{1}{x}) \cdot y = \frac{1}{x}$.

_______________ $1 \cdot y = \frac{1}{x}$.

_______________ $y = \frac{1}{x}$.

_______________ This is the conclusion we wanted to prove. ∎

13. Explain, in one or two English sentences, or with a formal proof, why $-(-x) = x$. Hint: use uniqueness of additive inverses.$^{(1)}$

14. Explain why, given any real number $c \in \mathbb{R}$, there is an integer n such that $c < n$. Hint: I proved in class that the set of integers $\mathbb{Z}$ does not have an upper bound.$^{(1)}$

15. Consider the following functions:

$$\begin{aligned} f(x) &= \sqrt{x-1} \\ g(x) &= x^2 + 1 \\ h(x) &= \frac{1}{x+2} \end{aligned}$$

a. What are the domains of f, g, and h?$^{(1)}$

b. Compute the following sums, products, or compositions, and explicitly state their domains.$^{(1 each)}$

- $(f \cdot h)(x) =$
- $(g + h)(x) =$
- $(f \circ g)(x) =$
- $(g \circ f)(x) =$
- $(f \circ h)(x) =$
- $(h \circ f)(x) =$
- $(g \circ (f + h))(x) =$