Midterm 1
Monday, 4-14-95 9:30-10:20
Math 133
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Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points.

Here are some theorems you might need:

a and b are in the domain of the function ln. x and y are in the domain of the function exp.

$$\begin{aligned}
\ln(ab) &= \ln(a)+\ln(b) && (1)\\
\ln(a^x) &= x\ln(a) && (2)\\
\frac{d}{da}\ln(a) &= \frac{1}{a} && (3)\\
e^x\cdot e^y &= e^{x+y} && (4)\\
(e^x)^y &= e^{x\cdot y} && (5)\\
\frac{d}{dx}e^x &= e^x && (6)
\end{aligned}$$

Definition 0.1 $\cosh(x)=(e^x+e^{-x})/2$.

Definition 0.2 $\sinh(x)=(e^x-e^{-x})/2$.

Theorem 0.3 (the Inverse Function Theorem) If f is one-to-one, and differentiable at c with $f'(c)\neq 0$, then f^{-1} is differentiable at $f(c)$, with derivative

$$(\frac{d}{dx}f^{-1})(f(c))=\frac{1}{f'(c)}.$$

This is also written, when $x=f(c)$,

$$(\frac{d}{dx}f^{-1})(x)=\frac{1}{f'(f^{-1}(x))}.$$

1. State the definition, domain, and range of the following functions:$^{(1\ each)}$

$\ln(x) =$

$e^x =$

x^e (on the domain $(0, \infty)$) $=$

$3^x =$

$(1 - x^2)^{(-1/2)} =$

2.$^{(2\ pts)}$ a. For a real number x, the usual definition of x^2 is as the product $x \cdot x$. Show that this definition agrees with the definition in terms of the exp and ln functions.

b. Show that you get the same derivative, using the product rule on the first definition, and the chain rule on the new definition.

3. Show, by computing a lower sum and an upper sum with four rectangles, that $1 \leq \ln(5) \leq 3$. (of course it is excellent if you can get even better estimates than these.)$^{(2\ pts)}$

4. True or False. Write **T** or **F** next to each sentence. You need not explain your answer.(1 *each*)

______ $\ln(a+b) = \ln(a) \cdot \ln(b)$.

______ $\forall x \in \mathbb{R},\ \ln(e^x) = x$.

______ $e^{(x^2)} = (e^x)^2$.

______ $(\ln(x))^4 = 4\ln(x)$

______ $\forall x \in (-1,\infty),\ \ln(\frac{1}{1+x}) = -\ln(1+x)$

______ $\forall x \in (0,\infty),\ \ln(\frac{1}{1+x}) = -\ln(1+x)$

______ $(\cosh(x))^2 + (\sinh(x))^2 = 1$.

______ $\cosh(x)$ is increasing on $\mathbb{R}$.

______ $\forall x \in \mathbb{R},\ e^{-x} \leq e^x$.

5. (2 *each*) a. Prove formula (1) on the cover sheet. (Hint: start by calculating $\frac{d}{da}$ of the left side; then integrate the answer.)

b. Prove $e^0 = 1$.

c. Use formula (4) and $e^0 = 1$ to prove $e^{-x} = \frac{1}{e^x}$.

d. Show $\int \frac{1}{x}dx = \ln|x|$ on the domain $\{x \neq 0\}$.

e. Use the Inverse Function Theorem and (3) to prove (6).

6.$^{(2\ pts)}$ Show that $f(x) = x\sqrt{1-x}$ is not injective on $(-\infty, 1]$. (Hint: this means find two different numbers with the same f-value.)

7.$^{(2\ pts)}$ Give an example of a function which is injective and continuous on its domain, but not monotonic. (Hint: continuous, injective functions on an interval are monotonic.)

8.$^{(2\ pts)}$ Give an example of a function $f(x)$ such that $f(x)$ is injective but $g(x) = (f(x))^2$ is not injective.

9.$^{(5\ pts)}$ Let $f(x) = \ln(1 + \ln(x))$, on the domain $(1, \infty)$.
Show that $f(x)$ is injective. (Hint: monotonicity theorem)

Find the range of $f(x)$.

Calculate $f^{-1}(x)$ and its domain and range.

Check your answer by calculating the compositions $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

10.$^{(2\ pts)}$ Use the definition of 3^x to calculate $\frac{d}{dx}3^x$ and $\int_0^2 3^x dx$.

11. Give any one solution, y, to each of the following differential equations: $y = y(x)$ is a function of x, and derivatives are in x, using the abbreviation $y' = \frac{d}{dx}y(x)$. Some of the following questions put extra conditions that the function y must satisfy. g, m, and k are non-zero constants. It will help a lot if you clearly indicate your answer on this page and check it on another.$^{(2\ pts\ each;best\ 6)}$

- $y' = y$
- $y' = 2y,\ y(0) = k$
- $y'' = g,\ y(0) = 100,\ y'(0) = 1$
- $y'' = 2y$
- $y' = x(\ln(x) - 1)$
- $y' = x^2 y$
- $y' = x^2 y^4,\ y(1) = 1$
- $y' = \frac{g}{m} + ky$