Midterm 1
Wednesday, 01-25-95 9:30-10:20
Math 132
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Name __________________________ Student ID# _____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of 400.

1. Fill in the Blank. Complete the following sentences by writing in a correct answer. The sentences are in no particular order; their numbering is not related to their numbering in the book or on the review sheet. (1 each)

Definition 0.1 c is a *critical point* for $f(x)$ on the interval I if

- c is __________________________, OR
- c is a *singular point* for $f(x)$, meaning that __________________________, OR
- c is a *stationary point* for $f(x)$, meaning that __________________________.

Definition 0.2 The function $f(x)$ has a *maximum value* on the set S if $\exists c \in S$ such that $x \in S \Rightarrow$ __________________________.

Definition 0.3 The function $f(x)$ has a *minimum value* on the set S if ____.

Definition 0.4 $f(x)$ is *increasing* on the set S if __________________________.

Theorem 0.5 (Second Derivative Test for Local Extrema) Suppose $\exists \delta > 0$ such that $f(x)$ is differentiable on $(c - \delta, c + \delta)$, that its derivative $f'(x)$ is also differentiable, (with derivative $f''(x)$), and that c is a __________________________.

- if __________________________ then $f(c)$ is a local maximum value.
- if __________________________ then $f(c)$ is a local minimum value.

Theorem 0.6 (Max/Min Existence) If $f(x)$ is __________________________, then $f(x)$ has a maximum value $f(c)$ and a minimum value $f(d)$ on that interval.

True or False. Write **T** or **F** next to each sentence. You need not explain your answer.$^{(1\ each)}$

____________ $0 < a < b \Rightarrow a^2 < ab$.

____________ $f(x)$ increasing $\Longrightarrow$ $f(x)$ differentiable.

____________ $f(x)$ decreasing on (a,b) $\Longrightarrow$ $f(x)$ decreasing on $[a,b]$.

____________ $f(x) = \frac{1}{x}$ is decreasing on the set $(0,\infty)$

____________ $f(x) = \frac{1}{x}$ is decreasing on the set $(-\infty,0) \cup (0,\infty)$

____________ c cannot be both a critical point and an inflection point of $f(x)$

____________ $f(x)$ bounded above on S $\Longrightarrow$ $f(x)$ has a maximum value on S.

____________ $f(x)$ has a maximum value on S $\Longrightarrow$ $f(x)$ bounded above on S.

____________ If $f(x)$ is continuous on (0,2), AND $x < 1 \Longrightarrow f(x) > 0$, then $\lim\limits_{x\to 1^+} f(x) > 0$.

____________ If $f(x)$ is increasing, then $|f(x)|$ is increasing.

____________ If $f(c)$ is the maximum value of f on S, and $g(c)$ is the maximum value of g on S, then $f(c) + g(c)$ is the maximum value of $(f+g)$ on S.

____________ If $f(c)$ is the minimum value of f on S, and $g(d)$ is the minimum value of g on S, then $f(c) + g(d)$ is the minimum value of $(f+g)$ on S.

(3 each)

2. Consider the function $f(x) = x^2$ on the interval $[-1, 10]$. Find the number c whose existence is guaranteed by the Mean Value Theorem.

Hint: **Theorem 0.7** (Mean Value Theorem) If $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

3. Suppose $f(x)$ is increasing on the open interval $(0, 1)$. Show, without using the Critical Point Theorem, that $f(x)$ does not have a minimum value on $(0, 1)$. (Hint: suppose toward a contradiction that $f(c)$ is a minimum value.)

4. Show, using the limit definition of derivative, that the following function $f(x)$ is differentiable at 0, and write down the derivative $f'(x)$. Use the definition again to show that $f'(x)$ is not differentiable at 0. (Hint: compare the left-side limits to the right-side limits.)

$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$. $\qquad f'(x) = \begin{cases} & \\ & \end{cases}$.

In the following questions, give a formula or a clearly drawn sketch. (2 each)

5. Give an example of a function $f(x)$ such that $f(x)$ has a stationary point $c = 1$ but $f(x)$ is increasing on the interval $(0, 2)$.

6. Give an example of a function $f(x)$ on the interval (0,1) such that $f(x)$ is increasing and $(f(x))^2$ is decreasing.

7. Give an example of a function $f(x)$ on $[0, 1]$ which is bounded but does not have a maximum value.

8. Give an example of a function which is continuous on $[0, 3]$, concave up on $(0, 1)$, concave down on $(1, 3)$, increasing on $[0, 2]$, and decreasing on $[2, 3]$.

9. If $f(x)$ has a minimum value $f(c)$, calculate the slope of the line through distinct points $(x, f(x))$ and $(c, f(c))$ on the graph. Carefully show (justify steps) that this slope is negative for $x < c$ and positive for $x > c$.

10. A cylinder fits inside a sphere of radius r. The base of the cylinder is a circle of radius x, $0 \leq x \leq r$. The volume of the cylinder is given by the function $2\pi x^2\sqrt{r^2 - x^2}$. In terms of r, find the cylinder radii x such that the volume is a minimum and a maximum. Also find these extreme values. (5 points)

Hint: the following formula may be useful in one of the terms from the product rule.

$$\frac{d}{dx}\sqrt{r^2 - x^2} = \frac{-x}{\sqrt{r^2 - x^2}}.$$

11. Since you don't have enough room on this side of the page to draw a big enough graph, use the reverse side to sketch a graph of the function:

$$f(x) = \frac{x}{1 + x^2}$$

Indicate any local extrema and inflection points. (4 points)

Hint: here's the quotient rule for derivatives:

$$g(x) \neq 0 \Rightarrow \frac{d}{dx}\left(\frac{h(x)}{g(x)}\right) = \frac{h'(x)g(x) - h(x)g'(x)}{(g(x))^2}$$

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