Midterm 1
Friday, 10-21-94 9:30-10:20
Math 131
Adam Coffman

Name ________________________ Student ID# ____________

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of 400.

A *field* is any set $\mathbb{F}$, together with two binary operations $\cdot$ and $+$ which are defined by an addition table and a multiplication table, and which are subject to the following axioms:

1. The commutative law for $+$ : $x + y = y + x$.
2. The associative law for $+$ : $(x + y) + z = x + (y + z)$.
3. Additive identity: $\exists 0 \in \mathbb{F}$ such that $x + 0 = x$.
4. Additive inverse: $\forall x \in \mathbb{F},\ \exists -x \in \mathbb{F}$ such that $x + (-x) = 0$.
5. The commutative law for $\cdot$ : $x \cdot y = y \cdot x$.
6. The associative law for $\cdot$ : $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
7. Multiplicative identity: $\exists 1 \in \mathbb{F}$ such that $1 \neq 0$ AND $1 \cdot x = x$.
8. Multiplicative inverse: $\forall x \in \mathbb{F},\ x \neq 0 \Rightarrow \exists x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1$.
9. The distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$.

$\mathbb{F}$ is an *ordered field* if it contains a subset $P \subseteq \mathbb{F}$ such that:

- Additive Order: $(x \in P$ AND $y \in P) \Rightarrow (x + y) \in P$.
- Multiplicative Order: $(x \in P$ AND $y \in P) \Rightarrow (x \cdot y) \in P$.
- Trichotomy: Every $x \in \mathbb{F}$ has *exactly* one of the following properties:
  - ◇ $x \in P$,
  - ◇ $-x \in P$,
  - ◇ $x = 0$.

1. State the Triangle Inequality for elements in an ordered field.(2)

2. Give an example of integers $a$ and $b$ where the Triangle Inequality is a strict inequality (**show** that the two sides of the inequality are not equal).(2)

$a =$ ____________

$b =$ ____________

3. Solve the following inequalities, that is, find the subset of $\mathbb{R}$ consisting of elements $x$ with the required property or properties. You don't have to explain your reasoning, but do **write your answer in interval notation**. (1.5 *each; I will count your best* 6)

- ____________ $x + 14 < 9$.
- ____________ $6 < -2x < 8$.
- ____________ $x^2 \leq 4$. Hint: $x = -3$ does not have this property.
- ____________ $|x| > -1$.
- ____________ $x \neq |x|$.
- ____________ $0 < |x - c|$.
- ____________ $|f(x) - L| < 0.01$. Solve for $f(x)$.
- ____________ $x \in (-1, 3)$ AND $x \in (2, 4]$.
- ____________ $x \in (-1, 3)$ OR $x \in (2, 4]$.

Graph the following functions. Indicate where the graph intersects the $x$-axis and the $y$-axis. (plug in $y = 0$ and $x = 0$, respectively, and plot these ordered pairs on your graph) Each graph should be larger than a quarter.$^{(2\ each)}$

4. $y = \frac{x}{2} + 6$.

5. $y = (x - 2)^2$. (Hint: expanding the polynomial won't help any.)

6. $y = |x| - 1$

7. Graph the line through the points $(4, 5)$ and $(-1, 3)$. What is its slope?

8. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

____________ $0 < a < b \Rightarrow a^2 < ab$.

____________ $(a \in \mathbb{Q}$ AND $b \in \mathbb{Q}$ AND $a < b) \Rightarrow \exists c \in \mathbb{Q}$ such that $a < c < b$.

____________ The distance from (3,4) to (2,7) is 10.

____________ $27 < 4$ AND $4 + 6 = 10$.

____________ $27 < 4$ OR $4 + 6 = 10$.

____________ $(c > 0$ AND $|x - c| < \frac{c}{2}) \Rightarrow x > 0$.

____________ $(A \subseteq B$ AND $B \cap C = \emptyset) \Rightarrow A \cap C = \emptyset$.

____________ $(4, 5] \cap [4\frac{1}{2}, 7)$ is a closed interval.

____________ $(4, 5] \cup [4\frac{1}{2}, 7)$ is a closed interval.

____________ $x \in \mathbb{Z} \Rightarrow x^{-1} \in \mathbb{Q}$. (Hint: false.)

____________ $0 < a \Rightarrow \sqrt{a} < a^2$.

____________ The set of four-digit positive integers has an upper bound.

____________ The set of positive integers divisible by ten has an upper bound.

____________ $x \leq y \Rightarrow x = y$.

____________ $x = y \Rightarrow x \leq y$.

____________ $(b \neq 0$ AND $d \neq 0) \Rightarrow \frac{a}{b} + \frac{c}{d} = \frac{a+c}{bd}$.

9. Adam goes to the grocery store with a fifty-cent coupon for cat food. He knows, from previous experience, that Cashier A will enter the price of the cat food, subtract the coupon, and then add 8.75% sales tax. He also knows that Cashier B will do this in a different order, adding sales tax and *then* subtracting the fifty cents from the coupon. Don't do any calculations, but try to explain in your own words why Adam would rather conduct his transaction with Cashier A, and which of the axioms is (are) relevant to your discussion.$^{(2)}$

10. Define the absolute value of $x$. $^{(2\ pts)}$

11. Let $S$ be a subset of the real numbers. Give a definition of the statement "$b$ is an upper bound for $S$."$^{(2)}$

12. Here are a list of axioms and definitions. Some of them might get used twice in the proof below, or not at all; write the letter from the list next to the step that it justifies.(1 *each*)

A. Commutative law for $\cdot$

B. Multiplicative identity

C. Definition of "$y$ is a multiplicative inverse of $x$."

D. Multiply both sides by $y$

E. Definition of "$z$ is a multiplicative inverse of $x$."

F. "$z = y$" is the definition of "$x^{-1}$ is unique."

G. Distributive Law

H. Multiplicative Order

I. Associative law for $\cdot$

**Theorem.** $x \neq 0 \Rightarrow x^{-1}$ is unique.
PROOF: Suppose that $y$ and $z$ are two multiplicative inverses of $x$.

____________ $x \cdot z = 1.$

____________ $y \cdot (x \cdot z) = y \cdot 1.$

____________ $(y \cdot x) \cdot z = y \cdot 1.$

____________ $(x \cdot y) \cdot z = y \cdot 1.$

____________ $(x \cdot y) \cdot z = y.$

____________ $1 \cdot z = y.$

____________ $z = y.$

____________ This is the conclusion we wanted to prove. ∎