Name _____ Student ID# _____

Midterm 1. Friday, 10-18-96 12:30-1:20. Math 100. Adam Coffman.

This is a closed book exam. You may not work together, or use notes or a calculator. Clearly indicate your answers on this test. You may use the back of each page for scratchwork. This test is worth 50 points out of a quarter total of about 450.

Recall that the following axioms of arithmetic hold, where $\mathbb{F}$ stands for either the set of rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$, or complex numbers, $\mathbb{C}$:

1. The commutative law for $+$: $x + y = y + x$.
2. The associative law for $+$: $(x + y) + z = x + (y + z)$.
3. Additive identity: There is a number $0 \in \mathbb{F}$ such that $x + 0 = x$.
4. Additive inverse: For every $x \in \mathbb{F}$, there is a number $-x \in \mathbb{F}$ such that $x + (-x) = 0$.
5. The commutative law for $\cdot$: $x \cdot y = y \cdot x$.
6. The associative law for $\cdot$: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
7. Multiplicative identity: There is a number $1 \in \mathbb{F}$ such that $1 \neq 0$ AND $1 \cdot x = x$.
8. Multiplicative inverse: For every $x \in \mathbb{F}$ except $x = 0$, there is a number $\frac{1}{x} = x^{-1} \in \mathbb{F}$ such that $x \cdot \frac{1}{x} = 1$.
9. The distributive law: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Questions #1–10 one point each.

1. State the definition of the "difference," $x - y$, in terms of properties #1–9 on page 1.

2. State the definition of "x is a rational number," in terms of the set $\mathbb{Z}$ of integer numbers.

3. State the definition of "r is a root of the polynomial $ax^3 + bx^2 + cx + d$."

4. If z is the complex number $z = x + iy$, state the definition of its complex conjugate, $\bar{z}$.

5. State the equation describing the property "y is directly proportional to x, with constant of proportionality k."

6. State the quadratic equation defining the statement "x is a square root of b."

7. State the "quadratic formula" for the roots of the quadratic polynomial $ax^2 + bx + c$.

8. Under what condition on the coefficients a, b, c can the polynomial $ax^2 + bx + c$ be factored as $a(x - r)^2$?

9. Express 2.1×10^4 and 2.1×10^{-4} as decimals.

10. Which of the properties #1–9 on page 1 are not satisfied if the set $\mathbb{F}$ is the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$?

11. Simplify the following expressions until they are integers or rational numbers in lowest terms. $^{(1 each)}$

- _____ $\frac{7}{8} + \frac{5}{6}$.
- _____ $(\frac{3}{5} - 1)^2$.
- _____ $(-\frac{1}{27})^{1/3}$.
- _____ $(\frac{1}{27})^{-1/3}$.
- _____ i^4.

12. Solve the following equalities, that is, find **all** numbers $x \in \mathbb{R}$ that satisfy the given equation. You don't have to explain your reasoning or unnecessarily simplify your answer. The solutions x might involve both numbers and real constants such as m and b. (Hints: There will always be at least one real solution. Check your answers!) (Give an answer for each question; I will count your nine best answers out of twelve for one point each.)

- _____ $2x - \frac{4}{3} = 0$.
- _____ $mx + b = 0$, $m \neq 0$.
- _____ $mx + b = 2x + 3$, $m \neq 2$.
- _____ $\frac{5+x}{6} - \frac{10-x}{3} = 1$.
- _____ $x^3 - 10^{12} = 0$.
- _____ $x^4 - 10^{12} = 0$.
- _____ $\sqrt{x} = x^2$.
- _____ $\sqrt{-x} = m$, $m > 0$.
- _____ $\frac{1}{x} = 25x$.
- _____ $x^2 - 11x + 30 = 0$.
- _____ $x = \frac{2x+3}{x+1}$.
- _____ $x = \frac{2x^2+8x-10}{x-1}$.

13. Give examples of quadratic polynomials such that: $^{(1 each)}$

- _____ the polynomial has no real roots.
- _____ the polynomial has two real roots, and neither is rational.
- _____ the polynomial has roots $\frac{1}{4}$ and 5.
- _____ the polynomial has the same roots as $x^2 - 5x + 6$, but it has different coefficients.

14. True or False. Write **T** or **F** next to each sentence. The numbers and lower-case variables refer to elements of $\mathbb{R}$ unless otherwise specified. You need not explain your answer.$^{(1\ each)}$

_____ If $a^2 \in \mathbb{Q}$ then $a \in \mathbb{Q}$.

_____ If $a \in \mathbb{Q}$ then $a^2 \in \mathbb{Q}$.

_____ If $p = r$ and $q = s$ and $q \neq 0$ and $s \neq 0$ then $\frac{p}{q} = \frac{r}{s}$.

_____ If $\frac{p}{q} = \frac{r}{s}$ and $q \neq 0$ and $s \neq 0$ then $p = r$ and $q = s$.

_____ If $a \in \mathbb{R}, b \in \mathbb{N}, c \in \mathbb{N}$, then $a^b \cdot a^c = a^{b+c}$.

_____ If $a \in \mathbb{R}, b \in \mathbb{N}, c \in \mathbb{N}$, then $(a^b)^c = a^{(b^c)}$.

_____ If $a \in \mathbb{R}, b \in \mathbb{N}, c \in \mathbb{N}$, then $(a^b)^c = (a^c)^b$.

_____ If $a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{N}$, then $a^c \cdot b^c = (a \cdot b)^c$.

_____ If $a \in \mathbb{R}, b \in \mathbb{R}, c \in \mathbb{N}$, then $a^c + b^c = (a + b)^c$.

_____ If m and b are both irrational and $m \neq 0$, then the solution x of the equation $mx + b = 0$ must also be irrational.

_____ If $x \neq -3$, then $\left(\frac{7}{3+x}\right)^{-1} = \frac{3}{7} + \frac{x}{7}$.

_____ If $a^2 = b^2$ then $a = b$.

_____ The product of two odd integers is always odd.

_____ The sum of two odd integers is always odd.

_____ Every integer n is either three times an integer, $n = 3k$, or one greater than three times an integer, $n = 3k + 1$.

_____ $\frac{1+\sqrt{3}}{2}$ is a root of the polynomial $x^2 - 3x + 1$.

_____ $(a + b)(a^2 + ab + b^2) = a^3 + b^3$.

15. Here is a list of statements which could be used to justify steps in a proof. Some of them might get used twice in the proof below, or not at all. Before each line, write the reason why that line follows from the previous line. I have already written one of the answers.(5 *pts total*)

A. Associative law for $\cdot$

B. Multiply both sides by b

C. Commutative law for $\cdot$

D. Multiply both sides by d

E. Multiplicative identity

F. Multiply both sides by 1

G. Distributive Law

H. Hypothesis

I. Multiplicative inverse

Theorem. For a, b, c, and d real numbers, with $b \neq 0$ and $d \neq 0$, if $a \cdot b^{-1} = c \cdot d^{-1}$, then $a \cdot d = b \cdot c$.

Proof:

_____ $a \cdot b^{-1} = c \cdot d^{-1}$.

_____ $(a \cdot b^{-1}) \cdot b = (c \cdot d^{-1}) \cdot b$.

_____ $a \cdot (b^{-1} \cdot b) = c \cdot (d^{-1} \cdot b)$.

_____ $a \cdot (b \cdot b^{-1}) = c \cdot (b \cdot d^{-1})$.

_____ $a \cdot 1 = c \cdot (b \cdot d^{-1})$.

_____ $a \cdot 1 = (c \cdot b) \cdot d^{-1}$.

_____ $(a \cdot 1) \cdot d = ((c \cdot b) \cdot d^{-1}) \cdot d$.

_____ $(a \cdot 1) \cdot d = (c \cdot b) \cdot (d^{-1} \cdot d)$.

_____ $(1 \cdot a) \cdot d = (d \cdot d^{-1}) \cdot (b \cdot c)$.

_____ $(1 \cdot a) \cdot d = 1 \cdot (b \cdot c)$.

_____ $a \cdot d = b \cdot c$. ∎