A Classification of Quadratically Parametrized Maps of the Real Projective Plane

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0 Preface — added in 2018

I've re-typeset using LaTeX— and lightly edited — my 1991 University of Michigan B.S. Honors Thesis. I've also added new Sections 12 and 13 to give some updates.

1 Introduction

An important application of algebraic geometry today is its use in computer graphics and the computer-aided design (CAD) of machines such as automobiles. Design plans are a representation of solid, three-dimensional objects and their boundaries, and, for such plans to be manipulated and displayed by a computer, the objects must be described mathematically. Often, thin sheets of metal, plastic, or glass are approximated by two-dimensional surfaces in three-dimensional space. Such a surface can be represented by three equations in two independent parameters, or by one implicit equation in three variables. Either way, the number of each type of equation is determined by the surface's dimension, and the dimension of the ambient space. An implicit equation of a surface is, generally, a more compact description, and some properties of the surface such as symmetry may be more evident than in the parametric description. Parameters, however, can be a more natural way for a CAD program to graphically plot a surface, and parametric equations arise often in applications.

Example 1.1. Consider, for example, the sphere in space. It is two-dimensional, so we can parametrize a sphere with two independent parameters, called θ and ϕ.

$$\begin{aligned} x &= \cos\theta\sin\phi \\ y &= \cos\theta\cos\phi \\ z &= \sin\theta. \end{aligned}$$

This is a familiar parametrization, where θ and ϕ are angles of latitude and longitude. The radius of the sphere is 1 and any point on the sphere can be described by values of θ and ϕ between 0 and 2π. There is another less obvious, but computationally simpler parametrization.

$$\begin{aligned} x &= \frac{1-s^2}{1+s^2}\frac{2t}{1+t^2} \\ y &= \frac{1-s^2}{1+s^2}\frac{1-t^2}{1+t^2} \\ z &= \frac{2s}{1+s^2}. \end{aligned} \tag{1.1}$$

Notice that the quotients which appear in the place of the cosines and sines in the previous parametrization satisfy the same algebraic relation as the cosines and sines: $(\cos\theta)^2 + (\sin\theta)^2 = 1$.

The s, t parametrization is described as "fourth degree," meaning that the maximum degree of the numerators and denominators is 4 (from the s^2t^2 term in (1.1)). It turns out that this is more than necessary; there is a similar parametrization of the sphere with second-degree polynomials in the numerator (Example 8.8).

The implicit equation in x, y, z describing the sphere is, of course, the quadratic equation $x^2 + y^2 + z^2 = 1$. Surfaces with quadratic implicit equations are called quadric surfaces, which are analogous to conic sections in the plane.

Elimination theory deals with the connection between the two ways of describing a surface — parametric or implicit. It makes use of linear algebra and matrices, and also commutative algebra and ideal theory.

If we substitute the s, t or the θ, ϕ parametrizations of x, y, and z into the implicit equation $x^2+y^2+z^2 = 1$, then we see that the implicit equation is identically satisfied. Any values for the parameters lead to values for x, y, and z that satisfy the implicit relation.

Example 1.2. To demonstrate a method for deriving an implicit equation from a set of parametric equations, consider a four-dimensional space with coordinates x, y, z, t and three parameters u, v, w. The set of points described by these parameters forms a three-dimensional object in the 4-dimensional space (such a thing is called a "hypersurface"; in general, a set a points that is the solution to a system of polynomial equations is called an "algebraic variety").

$$x = uv \qquad y = uw \qquad z = vw \qquad t = u^2 + v^2 + w^2.$$

To get an implicit equation in x, y, z, t that describes this hypersurface, solve for each variable in terms of x, y, z, t:

$$\begin{aligned}
u &= \frac{x}{v} && (1.2)\\
w &= \frac{y}{u} = \frac{yv}{x} && (1.3)\\
v &= \frac{z}{w} = \frac{zx}{yv} && (1.4)\\
v^2 &= \frac{zx}{y} && (1.5)\\
u^2 &= \frac{x^2}{v^2} = \frac{x^2y}{zx} = \frac{xy}{z} && (1.6)\\
w^2 &= \frac{y^2}{u^2} = \frac{y^2v^2}{x^2} = \frac{yz}{x} && (1.7)\\
t &= u^2+v^2+w^2 = \frac{xy}{z}+\frac{zx}{y}+\frac{yz}{x} && (1.8)\\
xyzt &= x^2y^2+x^2z^2+y^2z^2. && (1.9)
\end{aligned}$$

Step (1.9) is the desired implicit equation. It is of fourth degree in x, y, z, t.

A mathematician may become proficient with such computations by experience, but such an ad hoc manipulation of variables until all parameters are "eliminated" does not lend itself to automation. In several cases, there is a way to solve the system of parametric equations algorithmically, by entering their coefficients into a special matrix and taking the determinant. The determinant is the "resultant" of the equations and the theory behind the algorithm is called "elimination theory." The example used above is an example of an algebraic variety described by quadratic expressions in the parameters. Quadratic equations are the simplest non-linear kind of parametric equations, and occur in CAD when describing surfaces. That the implicit equation derived in Example 1.2 is necessarily fourth degree and irreducible (cannot be factored) can be proven by the methods of algebraic geometry.

The geometry of such an object is also very interesting. The "cross-section" formed by fixing $t = 1$ in (1.9), which intersects the hypersurface with the 3-dimensional space $\{(x, y, z, 1)\}$, describes a "Roman Surface," which is the image of a parametric representation of the real projective plane. Sets of quadratic parametric equations can lead to varieties which are the image of a non-orientable surface, and there are many ways to describe images of the real projective plane in three-dimensional Euclidean space. A classification of the parametrization coefficients leads to a classification of quadratically parametrized maps of the real projective plane into real projective 3-space, and such a classification uses linear algebra and algebraic geometry.

The goal of this classification is to establish some way to anticipate the geometric behavior of a set of quadratic parametric equations.

2 The Real Projective Plane

Some terminology is presented here for reference in later Sections. The following definition of real projective space is as a set defined by a larger set and an equivalence relation. Consider $\mathbb{R}^n \setminus \{\mathbf{0}\}$, and an equivalence relation $\sim$, where for vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $\mathbf{x} \sim \mathbf{y} \iff \mathbf{x} = \lambda\mathbf{y}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$, that is, vectors are $\sim$-equivalent if and only if one is a non-zero scalar multiple of the other.

Each line through the origin is now a point in the set of equivalence classes. The resulting set is a $(n-1)$-manifold, called "real projective $(n-1)$-space," or $\mathbb{R}P^{(n-1)}$. The "Real Projective Plane" is this object when $n = 3$. It is a 2-manifold, or a "surface," and with this in mind, there are algebraic and topological definitions analogous to the one given above.

The quotient map $p : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n$ sends real coordinates to "projective coordinates":

$$p(x_0, x_1, \ldots, x_n) = [x_0 : x_1 : \ldots : x_n], \tag{2.1}$$

and the projective coordinates are thought of as the n independent ratios of the $n + 1$ coordinates. For a chosen $0 \leq i \leq n$, the corresponding "affine subset" of the coordinates is the set of all points where $x_i \neq 0$ and this subset is homeomorphic to $\mathbb{R}^n$. The complement of this set is a $(n-1)$-manifold termed a "hyperplane at infinity," corresponding to $x_i = 0$. For example, the set of projective coordinates $[x_0 : x_1]$ contains a copy of $\mathbb{R}$ plus the point $[0 : 1]$, a single "point at infinity." Such a set is homeomorphic to a circle, and can be called the real projective line.

Consider the set of lines through the origin of three-dimensional space. The unit sphere contains two representative points from each line. To reduce the number of representative points to one, and thus get a surface which has the same topology as $\mathbb{R}P^2$, simplify identify each point with its opposite point. This is the "antipodal map" — one hemisphere is mapped onto the other. For points on the equator, however, the identification poses some problems, which we solve by making another identification. The equator can be collapsed into a diameter by a projection. The diameter can then be folded in half, away from the hemisphere. In terms of points $(x, y, 0)$ on the unit circle, $(x, y, 0) \mapsto (x, 0, 0) \mapsto (0, 0, |x|)$. In this way, the points $(x, y, 0)$ and $(-x, -y, 0)$ are both mapped to $(0, 0, |x|)$, but so are $(x, -y, 0)$ and $(-x, y, 0)$, which were supposed to be identified with each other anyway. This phenomenon is one of self-intersection, and the geometric object resulting from subjecting the equator of a hemisphere to the above operations was named by Steiner a "cross-cap," and it is one of the earliest geometric representations of the real projective plane.

There is no way to map $\mathbb{R}P^2$ into three-dimensional space in a continuous, injective way. This is a consequence of the fact that it does not admit an orientation. Its non-orientability can be thought of as inherited from a Möbius strip it contains, or it can be seen by looking at an illustration or model of the cross-cap surface.

There are other representations of the Real Projective Plane in 3-space, and some are displayed in Section 11, the Appendix of illustrations. (See also [A], [F]). They can be described once some terminology is introduced.

Results of Whitney describe smooth maps from the plane into space. The canonical embedding is the Euclidean plane, a surface of zero curvature and infinite extent in two dimensions, which no self-intersections. Under a suitable choice of location of the origin in a three-dimensional real vector space, the plane is a two-dimensional subspace. Topologically equivalent surfaces that are embeddings of the plane include graphs of maps from 2-space to the real numbers, such as $z = f(x, y) = x^2 + y^2$. Considering differentiable maps that send a plane of coordinates (parameters) into 3-space and that are not necessarily injective, different cases may arise. The self-intersection set (points with an inverse image containing more than one point) may be a 1-dimensional curve, or a 2-dimensional region, as in Example 1.1, where the parametric equations are periodic and the image is a sphere.

If the self-intersection set is one-dimensional, then it may have no boundary (as in the Klein bottle) or it may contain points of its boundary. An example of a geometric object that is the image of a planar region, and that has a self-intersection set with boundary, can be described as follows.

Example 2.1. The images of parametric curves, $\{(x, x^2, 0)\}$ and $\{(x, x^2, 1)\}$, are parabolas in space. Connect each point on one parabola to a point on the other using the line determined by $(x, x^2, 0)$ and $(-x, x^2, 1)$. This is a one-parameter family of lines, and it is the image of a plane mapped to three-dimensional space. The self-intersection set is the open ray $\{(0, y, 0.5) : y > 0\}$. This object, or anything homeomorphic to a region around $(0, 0, 0.5)$, is called a <u>Whitney umbrella</u> (several Whitney umbrellas will appear in Section 11, the Appendix of illustrations). The singularity point $(0, 0, 0.5)$ is called a <u>pinch point</u>. The line containing the ray is the <u>double line</u> and the opposite ray $\{(0, y, 0.5), y < 0\}$ is a <u>whisker</u> or <u>handle</u> of the umbrella.

The pinch point in Example 2.1 is undisturbed by small perturbations of the construction. If the lines connect $(x, x^2, 0)$ to $(-x + \varepsilon, (-x + \varepsilon)^2, 1)$, for example, then the pinch point moves in space and the neighborhood around it is distorted by a small amount, but its local topological shape is not changed.

The general definition of a <u>singularity</u> of a map from a plane to space is the image of a point where the Jacobian matrix of the mapping function has rank < 2. A singularity of an implicitly defined surface in space is a point where the implicit equation has gradient vector $\mathbf{0}$. We will use the word "singularity" to refer to either notion, although these are generally different sets of points.

Example 2.2. There are examples of singularities other than the pinch point of the Whitney umbrella. If, to the umbrella constructed in Example 2.1, the plane $\{(x,0,z)\}$ is added, then the point $(0,0,0.5)$ is a different kind of singularity, and the line $\{(0,0,z)\}$ becomes another self-intersection set. See Figure 9. in Section 11, the Appendix of illustrations. If, however, the new plane is perturbed by some amount ε in the positive y-direction, then the plane does not intersect the pinch point of the umbrella, the pinch point singularity is restored, and the triple point $(0,\varepsilon,0.5)$ occurs.

The behavior exhibited in Example 2.2 is expected — under small deformations, only singularities that look like pinch points are unchanged. By Whitney's theory of singularities, any other type is "unstable" under deformations; it is lost after most perturbations and either no singularity is left, or a pinch point remains.

Pinch points and more complicated singularities can occur when using quadratic parametric formulas to describe surfaces.

3 Quadratic Parametric Maps

Parameters occur naturally in the study of surfaces. More generally, parametric equations define a mapping from $\mathbb{F}^n$ (parameter space) to $\mathbb{F}^m$ (the target space, or the ambient space of the image of $\mathbb{F}^n$; $\mathbb{F}$ is a field which for current purposes will be either $\mathbb{R}$ or $\mathbb{C}$). A parametric representation of an object is easy for a computer to plot (for $\mathbb{F}=\mathbb{R}$, $n=1,2$, $m=2,3$), and it is the most convenient description of geometrical objects for many applications.

Example 3.1. Consider $\mathbb{R}^3$ as a space of parameters (u_0,u_1,u_2), and a map μ taking (u_0,u_1,u_2) to:

$$\begin{aligned} & (u_0^2+u_1^2+u_2^2, u_1u_2, u_0u_2, u_0u_1) \\ = \; & (x_0,x_1,x_2,x_3)=\mathbf{x}\in\mathbb{R}^4. \end{aligned}$$

Each x_i is a homogeneous, quadratic polynomial in the three parameters. These are the same equations as in Example 1.2, but using different variable letters. The homogeneity of the parametric polynomials suggests projective coordinates in both the parameter and the ambient spaces.

The map μ followed by p (from (2.1)) sends $\mathbb{R}^3\setminus\{\mathbf{0}\}$ to $\mathbb{R}P^3$. Let $m=p\circ\mu$; then

$$\begin{aligned} m(u_0,u_1,u_2) &= m(\lambda(u_0,u_2,u_2)) \\ &= [u_0^2+u_1^2+u_2^2 : u_1u_2 : u_0u_2 : u_0u_1]. \end{aligned} \tag{3.1}$$

By (3.1), it is clear that the domain might as well be $\mathbb{R}P^2$ instead of $\mathbb{R}^3$. Redefine $m : \mathbb{R}P^2 \to \mathbb{R}P^3$:

$$m([u_0 : u_1 : u_2]) = [u_0^2 + u_1^2 + u_2^2 : u_1u_2 : u_0u_2 : u_0u_1].$$

Since m is continuous and $\mathbb{R}P^2$ is compact, the image is compact in the topology of the target space $\mathbb{R}P^3$. However, the intersection of the image with an affine subset $\mathbb{R}^3$ need not be compact in the standard metric topology of $\mathbb{R}^3$.

The map m from Example 3.1 is a specific instance of a homogeneous quadratic map, and the generalization is as follows: u_0^2, u_1^2, u_2^2, u_1u_2, u_0u_2, u_0u_1 are the only six quadratic combinations of u_0, u_1, and u_2, so any homogeneous quadratic polynomial in these three variables is a linear combination of u_0^2, u_1^2, u_2^2, u_1u_2, u_0u_2, and u_0u_1. A 6×4 matrix $\mathbf{P}$ can be constructed with the following property:

$$(u_0^2, u_1^2, u_2^2, u_1u_2, u_0u_2, u_0u_1)\mathbf{P} = (x_0, x_1, x_2, x_3). \tag{3.2}$$

In Example 3.1,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.3}$$

$\mathbf{P}$ is called the coefficient matrix. The points (x_0, x_1, x_2, x_3) in the image of the map (3.2) satisfy a homogeneous polynomial equation as in (1.9). The set of all solutions (x_0, x_1, x_2, x_3) of the implicit equation forms a hypersurface in $\mathbb{R}^4$, which is mapped by p to a set of points $[x_0 : x_1 : x_2 : x_3]$ forming a surface in $\mathbb{R}P^3$.

4 Equivalence of Coefficient Matrices

Depending on the application, it is natural to have a notion of equivalence of parametrizations. Given two sets of parametric equations, or two implicit equations, they should be equivalent if they describe the same geometric set of points, or if some "nice" (linear, isometric, etc.) transformation relates the two sets. In the context of computer graphics, if a surface is of more interest than any particular equation that generates it, or if its position and orientation in space are subordinate to its shape, then the right notion of equivalence should be not only intuitive, but can lead to a simplification of computation, as in the sphere in Example 1.1.

For quadratic parametric maps as in Section 3, first assume the coefficient matrix $\mathbf{P}$ has maximal rank, that is, rank $\mathbf{P} = 4$. The kernel of the $\mathbb{F}$-linear transformation corresponding to $\mathbf{P}$ has dimension 2 (the left kernel is a set of row vectors in $\mathbb{F}^6$), and two basis elements for the kernel can (in the sparse cases we will consider) often be determined by inspection of $\mathbf{P}$. In the example (3.3), $(1, -1, 0, 0, 0, 0)$ and $(0, 1, -1, 0, 0, 0)$ are two row vectors in the kernel and their span can be written as $(\lambda, \mu - \lambda, -\mu, 0, 0, 0)$.

Notation 4.1. The isomorphism w takes symmetric three-by-three matrices with $\mathbb{F}$ entries to row vectors with six components:

$$w : \begin{bmatrix} a & f & e \\ f & b & d \\ e & d & c \end{bmatrix} \mapsto (a, b, c, d, e, f).$$

Kernels of coefficient matrices are mapped by w^{-1} onto two-dimensional subspaces of the space of symmetric matrices (such a subspace is called a pencil). Again from the (3.3) example,

$$w^{-1}(\lambda, \mu - \lambda, -\mu, 0, 0, 0) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu - \lambda & 0 \\ 0 & 0 & -\mu \end{bmatrix}. \tag{4.1}$$

If $\mathbf{u}$ is the row vector (u_0, u_1, u_2), then

$$\mathbf{u}^T\mathbf{u} = \begin{bmatrix} u_0^2 & u_0u_1 & u_0u_2 \\ u_0u_1 & u_1^2 & u_1u_2 \\ u_0u_2 & u_1u_2 & u_2^2 \end{bmatrix}$$

and

$$w(\mathbf{u}^T\mathbf{u}) = (u_0^2, u_1^2, u_2^2, u_1u_2, u_0u_2, u_0u_1).$$

Notation 4.1 makes a linear change of coordinates in the parameter space easy to express for quadratic maps. Let $\mathbf{A}$ be a nonsingular three-by-three matrix mapping $\mathbf{u}$ to $\mathbf{uA}$. By elementary facts about the transpose operator ^{T},

$$(\mathbf{uA})^T(\mathbf{uA}) = \mathbf{A}^T\mathbf{u}^T\mathbf{uA}.$$

A linear change of coordinates in the target space is implemented with a nonsingular four-by-four matrix $\mathbf{B}$, again acting on $\mathbf{x}$ from the right side.

Two quadratic maps are equivalent means: their parametric coordinates and image coordinates are $\mathbb{F}$-linear transformations of each other:

Notation 4.2. On the set of 6×4 coefficient matrices, define the equivalence relation $\mathbf{P} \sim \mathbf{Q}$ to mean that there are some $\mathbf{A}$ and $\mathbf{B}$ as above so that

$$w(\mathbf{u}^T\mathbf{u})\mathbf{Q} = w(\mathbf{A}^T\mathbf{u}^T\mathbf{uA})\mathbf{PB}.$$

Given two coefficient matrices, then, a concrete way to check equivalence would be desirable.

There is a more convenient statement of $w(\mathbf{A}^T\mathbf{u}^T\mathbf{u}\mathbf{A})$. Let $\mathbf{A} = \begin{bmatrix} b & c & d \\ e & f & g \\ h & i & j \end{bmatrix}$.

Then $w(\mathbf{A}^T\mathbf{u}^T\mathbf{u}\mathbf{A}) = w(\mathbf{u}^T\mathbf{u})\mathbf{R}(\mathbf{A})$, where

$$\mathbf{R}(\mathbf{A}) = \begin{bmatrix} b^2 & c^2 & d^2 & cd & bd & bc \\ e^2 & f^2 & g^2 & fg & eg & ef \\ h^2 & i^2 & j^2 & ij & hj & hi \\ 2eh & 2fi & 2gj & fj+gi & ej+gh & ei+fh \\ 2bh & 2ci & 2dj & cj+di & bj+dh & bi+ch \\ 2be & 2cf & 2dg & cg+df & bg+de & bf+ce \end{bmatrix}.$$

A computation ([Derive]) shows $\det(\mathbf{R}(\mathbf{A})) = \det(\mathbf{A})^4$, and since $\mathbf{A}$ is assumed to be nonsingular, $\mathbf{R}(\mathbf{A})$ is also nonsingular, with $(\mathbf{R}(\mathbf{A}))^{-1} = \mathbf{R}(\mathbf{A}^{-1})$.

So, Notation 4.2 can be restated as follows:

$$\mathbf{P} \sim \mathbf{Q} \iff w(\mathbf{u}^T\mathbf{u})\mathbf{Q} = w(\mathbf{u}^T\mathbf{u})\mathbf{R}(\mathbf{A})\mathbf{P}\mathbf{B}.$$

To determine the equivalence of $\mathbf{P}$ and $\mathbf{Q}$, it suffices to examine the pencils of matrices that are the images of $\ker(\mathbf{P})$ and $\ker(\mathbf{Q})$ under w^{-1}.

Theorem 4.3. $\mathbf{P} \sim \mathbf{Q}$ *if and only if the corresponding pencils are congruent (over* $\mathbb{F}$*).*

Proof. 1. Suppose there are two pencils S and T of symmetric three-by-three matrices with entries that are linear combinations of λ and μ, and that there exists a nonsingular $\mathbf{A}$ that relates them, as sets, by congruence:

$$\mathbf{S} \in S \implies \mathbf{A}^T\mathbf{S}\mathbf{A} \in T \quad \text{and} \quad \mathbf{T} \in T \implies (\mathbf{A}^{-1})^T\mathbf{T}\mathbf{A}^{-1} \in S.$$

Now, choose a $\mathbf{Q}$ and $\mathbf{P}$ such that $\ker(\mathbf{Q}) = w(S) = \{w(\mathbf{S}) : \mathbf{S} \in S\}$ and $\ker(\mathbf{P}) = w(T)$. By construction, $\mathbf{R}(\mathbf{A})\mathbf{P}$ and $\mathbf{Q}$ have the same kernel, so they must be related by a nonsingular transformation $\mathbf{B}$ as in Notation 4.2; this proves $\mathbf{P} \sim \mathbf{Q}$.

2. Suppose that $\mathbf{Q} = \mathbf{R}(\mathbf{A})\mathbf{P}\mathbf{B}$, with $\ker(\mathbf{Q}) = \ker(\mathbf{R}(\mathbf{A})\mathbf{P}\mathbf{B}) =$ the pencil S. Then $\mathbf{A}^T S\mathbf{A} = \ker(\mathbf{P}\mathbf{B}) = \ker(\mathbf{P})$, and the two kernels are related by congruence. ∎

The problem of classifying coordinate matrices has been reduced to finding congruence classes of pencils of matrices.

5 Characteristic of a Pencil

Notation 5.1. For a complex pencil $S = \{\lambda\mathbf{L} + \mu\mathbf{M}\}$ spanned by three-by-three, independent, complex, symmetric matrices $\mathbf{L}$ and $\mathbf{M}$, define

$$d_S(\lambda, \mu) = \det(\lambda\mathbf{L} + \mu\mathbf{M}),$$

a polynomial of degree at most three in λ and degree at most three in μ.

Notation 5.2. In the set of 3×3 complex symmetric matrices, define Ω to be the set of singular matrices (the chordal variety), and V to be the subset of Ω consisting of matrices of rank less than two (the Veronese variety).

Given S as in Notation 5.1, the intersection with the chordal variety is:

$$S \cap \Omega = \{\lambda\mathbf{L} + \mu\mathbf{M} : d_S(\lambda, \mu) = 0\}.$$

By the linearity properties of the determinant function, if (λ, μ) is a pair that satisfies $d_S(\lambda, \mu) = 0$, then $d_S(\kappa\lambda, \kappa\mu) = 0$, so $[\lambda : \mu]$ is called a root ratio of S with respect to $\mathbf{L}$ and $\mathbf{M}$. The pair $[0 : 0]$ is to be disregarded.

Notation 5.3. The intersection character, or characteristic of S is the ordered pair (ρ, v), where ρ is the number of distinct root ratios of S, and v is the number of distinct root ratios of S that correspond to $\lambda\mathbf{L} + \mu\mathbf{M}$ in V.

Solving for λ in the polynomial equation $d_S(\lambda, \mu) = 0$ gives at most three root ratios, unless the polynomial is identically zero.

Proposition 5.4. *Using elementary facts about determinants, it can be shown that there are exactly eight mutually exclusive possibilities for (ρ, v).*

1. *$d_S(\lambda, \mu)$ is identically zero. Any $[\lambda : \mu]$ is a root ratio. This corresponds to the case where S is contained in Ω.*
   - *(a) $(\rho, v) = (\infty, 0)$. S does not intersect V; $\lambda\mathbf{L} + \mu\mathbf{M}$ always has rank exactly two.*
   - *(b) $(\rho, v) = (\infty, 1)$.*
   - *(c) $(\rho, v) = (\infty, 2)$. There can be no more than two intersections with V.*
2. *$d_S(\lambda, \mu)$ has one root ratio (of multiplicity three).*
   - *(a) $(\rho, v) = (1, 0)$. The root ratio $[\lambda : \mu]$ corresponds to $\lambda\mathbf{L} + \mu\mathbf{M}$ with rank 2.*
   - *(b) $(\rho, v) = (1, 1)$. The root ratio $[\lambda : \mu]$ corresponds to $\lambda\mathbf{L} + \mu\mathbf{M}$ with rank 1.*

3. $d_S(\lambda, \mu)$ *has two distinct root ratios.*

   (a) $(\rho, v) = (2, 0)$. *Both root ratios correspond to matrices with rank* 2.

   (b) $(\rho, v) = (2, 1)$. *The double root ratio corresponds to* $\lambda\mathbf{L} + \mu\mathbf{M}$ *with rank* 1.

4. $d_S(\lambda, \mu)$ *has three distinct root ratios.*

   (a) $(\rho, v) = (3, 0)$. *None can be in the Veronese variety.*

∎

Example 5.5. The following list of pencils shows that each case from Proposition 5.4 can occur.

- 1.a. $\begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & \mu \\ \lambda & \mu & 0 \end{bmatrix}$ 1.b. $\begin{bmatrix} \lambda & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 1.c. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & \mu \\ 0 & \mu & -\lambda \end{bmatrix}$
- 2.a. $\begin{bmatrix} 0 & \lambda & \mu \\ \lambda & 0 & 0 \\ \mu & 0 & -\lambda \end{bmatrix}$ 2.b. $\begin{bmatrix} \lambda & 0 & \mu \\ 0 & \mu & 0 \\ \mu & 0 & 0 \end{bmatrix}$
- 3.a. $\begin{bmatrix} 0 & \lambda & -\lambda \\ \lambda & 0 & \mu \\ -\lambda & \mu & 0 \end{bmatrix}$ 3.b. $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}$
- 4.a. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu - \lambda & 0 \\ 0 & 0 & -\mu \end{bmatrix}$ (This was seen in Equation (4.1).)

Remark 5.6. Theorem 4.3 and Proposition 5.4 were known in classical complex projective geometry, although different terminology has been used. [A] gives cases 1.a., 2.a., 3.a., and 4.a. as representatives determined only by ρ, describing equivalence classes "under homography of 3-pencils of conics without fixed points in complex projective space."

Lemma 5.7. *The characteristic* (ρ, v) *of* S *does not depend on the* $\mathbf{L}$ *and* $\mathbf{M}$ *used to span the pencil.*

Proof. This is proved by consideration of the set S as a vector space spanned by $\mathbf{L}$ and $\mathbf{M}$. Any other basis for S is related by a nonsingular change of coordinates $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, which acts on $\mathbf{L}$ and $\mathbf{M}$ as follows:

The vector $\lambda\mathbf{L} + \mu\mathbf{M}$ is transformed to:

$$\lambda(\alpha\mathbf{L} + \gamma\mathbf{M}) + \mu(\beta\mathbf{L} + \delta\mathbf{M}) = (\alpha\lambda + \beta\mu)\mathbf{L} + (\gamma\lambda + \delta\mu)\mathbf{M}.$$

$d_S(\lambda, \mu)$ does not necessarily equal $d_S(\alpha\lambda + \beta\mu, \gamma\lambda + \delta\mu)$, and $[\lambda : \mu]$ does not necessarily equal $[\alpha\lambda + \beta\mu : \gamma\lambda + \delta\mu]$. However, the number of intersections with Ω and V does not change, since S itself is not changed as a set. This should clarify the earlier abuse of language: ρ from Notation 5.3 is defined and computed in terms of d_S from Notation 5.1, which depends on a choice of basis for S, but ρ may be called the number of root ratios of S without regard to any particular choice of $\mathbf{L}$ and $\mathbf{M}$, and similarly v. ∎

The methods in Theorem 4.3 can be used in another theorem:

Theorem 5.8. *Two complex pencils of matrices S and T have the same characteristic (ρ, v) if and only if they are related by congruence (over $\mathbb{C}$).*

Proof. 1. Suppose a nonsingular $\mathbf{A}$ relates S and T by congruence. Then, for $\mathbf{T} \in T$, there is some $\mathbf{S} = \lambda\mathbf{L} + \mu\mathbf{M} \in S$ so that $\mathbf{T} = \mathbf{A}^T\mathbf{S}\mathbf{A} = \mathbf{A}^T(\lambda\mathbf{L} + \mu\mathbf{M})\mathbf{A} = \lambda\mathbf{A}^T\mathbf{L}\mathbf{A} + \mu\mathbf{A}^T\mathbf{M}\mathbf{A}$, and choosing $\mathbf{A}^T\mathbf{L}\mathbf{A}$ and $\mathbf{A}^T\mathbf{M}\mathbf{A}$ as a basis for T, $d_T(\lambda, \mu) = (\det(\mathbf{A}))^2 d_S(\lambda, \mu)$. So, the root ratios are identical and they must occur the same number of times, ρ. Because the operation $\mathbf{S} \mapsto \mathbf{A}^T\mathbf{S}\mathbf{A}$ preserves rank, v is also the same for S and T.

2. The converse must be proven separately for pencils of each characteristic. As an example, the $(\rho, v) = (3, 0)$ case can be proved with an argument adapted from [B].

S is spanned by $\mathbf{L}$ and $\mathbf{M}$; choose $[\lambda_1 : 1]$ a simple root of $d_S(\lambda, \mu)$. (This is possible because S has three distinct root ratios.) Temporarily fixing $\mu = 1$, $\lambda\mathbf{L} + \mathbf{M} = \mathbf{M} + \lambda_1\mathbf{L} + (\lambda - \lambda_1)\mathbf{L}$. $\mathbf{M} + \lambda_1\mathbf{L}$ is singular, so it is congruent to $\mathbf{M}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & y & w \\ 0 & w & z \end{bmatrix}$. It follows that $\lambda\mathbf{L} + \mathbf{M}$ is congruent to an expression of the form

$$\mathbf{M}' + (\lambda - \lambda_1)\mathbf{L}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & y & w \\ 0 & w & z \end{bmatrix} + (\lambda - \lambda_1) \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}. \tag{5.1}$$

By hypothesis, $\det(\lambda\mathbf{L} + \mathbf{M})$ has $(\lambda - \lambda_1)$ as non-repeated factor, so $a \neq 0$, since otherwise $(\lambda - \lambda_1)$ could be factored out of both the first row and the first column of (5.1). The expression (5.1) is then congruent to

$$\mathbf{M}'' + (\lambda - \lambda_1)\mathbf{L}'' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & y & w \\ 0 & w & z \end{bmatrix} + (\lambda - \lambda_1) \begin{bmatrix} a & 0 & 0 \\ 0 & b & f \\ 0 & f & c \end{bmatrix},$$

where in matrix calculations like this, the small latin letters are place-holding entries without fixed value. Let $\mathbf{N} = \mathbf{M}'' - \lambda_1 \mathbf{L}''$, so that $\lambda\mathbf{L} + \mathbf{M}$ is congruent to $\lambda\mathbf{L}'' + \mathbf{N}$. The $1,1$ entry of $\mathbf{N}$ is $-\lambda_1 a$.

Choosing one of the remaining root ratios, $[\lambda_2 : 1]$, and repeating the above argument, it is proved that $\lambda\mathbf{L} + \mathbf{M}$ is congruent to $\lambda \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} q & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix}$, and reinstating μ as a variable, $\lambda\mathbf{L} + \mu\mathbf{M}$ is congruent to $\lambda \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \mu \begin{bmatrix} q & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix}$, with $aq \neq 0$, since otherwise $d_S(\lambda, \mu)$ would be identically zero. Rescaling the parameters, $\lambda\mathbf{L} + \mu\mathbf{M}$ is congruent to $\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \mu \begin{bmatrix} q & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix}$.

By a transformation of S by $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ as in Lemma 5.7, let $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}$, so S is congruent to the subspace of matrices $\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & s \end{bmatrix}$. $s \neq 0$, since otherwise $d_S(\lambda, \mu)$ would have a double root, so again by scaling, we can set $s = 1$ and apply a transformation $\begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}$ to get that S is congruent to the pencil $\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $br \neq 0$, since otherwise $d_S(\lambda, \mu)$ would have a double root, so the subspace is spanned by $\lambda \begin{bmatrix} a & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$. Multiplication on both sides by $\mathbf{A} = \begin{bmatrix} \sqrt{\frac{1}{a}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{-\frac{1}{s}} \end{bmatrix}$ gives the result: S is congruent to the subspace $\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. This subspace has characteristic $(3, 0)$, and the above calculation showed that any pencil of characteristic $(3, 0)$ is congruent to this one, and by transitivity, to any other pencil of characteristic $(3, 0)$. ∎

Remark 5.9. The above Proof did not treat cases 1., 2., or 3. from Proposition 5.4. Case 1., where all the matrices in the pencil are singular ($\rho = \infty$) corresponds, by

Theorem 4.3, to coefficient matrices $\mathbf{P}$ where the parametric equations satisfy an implicit equation of degree 2 — a quadric surface such as a sphere or cone.

6 Congruence over the Reals

If $\mathbf{L}$ and $\mathbf{M}$ are symmetric matrices with real entries and λ and μ are real coefficients, then $S = \{\lambda\mathbf{L} + \mu\mathbf{M}\}$ is called a real pencil of matrices. Two real pencils S and T are related by congruence over $\mathbb{R}$ means:

$$\mathbf{S} \in S \implies \mathbf{A}^T\mathbf{S}\mathbf{A} \in T \text{ and } \mathbf{T} \in T \implies (\mathbf{A}^{-1})^T\mathbf{T}\mathbf{A}^{-1} \in S,$$

where $\mathbf{A}$ is a nonsingular three-by-three matrix with real entries.

If S and T are congruent over $\mathbb{R}$, then they are also congruent over $\mathbb{C}$, so each equivalence class under this new relation is a subset of an equivalence class under congruence over $\mathbb{C}$. An analogue of Lemma 5.7 holds, with a similar proof.

Lemma 6.1. *The number of real root ratios of a real pencil $S = \{\lambda\mathbf{L} + \mu\mathbf{M}\}$ does not depend on the choice of $\mathbf{L}$ and $\mathbf{M}$.* ∎

Considering again (as in part 2. of the Proof of Theorem 5.8) only the "generic" case 4.a. (from Proposition 5.4), there are three equivalence classes under congruence over $\mathbb{R}$. Geometrically, as in Theorem 4.3, there are three "surfaces" given by coefficient matrices whose kernels fall into case 4.a., but which are not related to each other by real transformations of coordinates in the parameter or target spaces.

Theorem 6.2. *Suppose $\mathbf{L}$ and $\mathbf{M}$ are real symmetric matrices and that they are independent, spanning a real pencil S. Suppose also that $d_S(\lambda, \mu)$ has three distinct root ratios, possibly complex. Then $S = \{\lambda\mathbf{L} + \mu\mathbf{M}\}$ is congruent over $\mathbb{R}$ to one of the following:*

i) $\left\{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}$ *and $d_S(\lambda, \mu)$ has three real root ratios,*

ii) $\left\{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}$ *and $d_S(\lambda, \mu)$ has three real root ratios,*

iii) $\left\{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}$ *and $d_S(\lambda, \mu)$ has only one real root ratio.*

Proof. Since $d_S(\lambda,\mu)$ is a real polynomial of degree three in λ and μ, it has at least one real root ratio, $[\lambda_1 : \mu_1]$. Choose a new basis $\mathbf{L}'$ and $\mathbf{M}'$, so that $\mathbf{L}' = \lambda_1\mathbf{L} + \mu_1\mathbf{M}$, and $\mathbf{L}'$ has rank two.

By Sylvester's Law of Inertia, $\mathbf{L}'$ is congruent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or to $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and the entries of $\mathbf{M}'$ are real and labeled $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

If $\mathbf{L}'$ is congruent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then

$$d_S(\lambda,\mu) = f\lambda^2\mu + (fd - e^2 + af - c^2)\lambda\mu^2 + \mu^3 \det(\mathbf{M}'),$$

which by assumption has three distinct roots, so $f \neq 0$. $\lambda\mathbf{L}' + \mu\mathbf{M}'$ is congruent to

$$\lambda\mathbf{L}' + \mu\mathbf{M}'' = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{6.1}$$

and such matrices span the subspace $\left\{ \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & b & 0 \\ b & d-a & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$. Set $c = d - a$. S is congruent as a set to $T = \left\{ \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$, with

$$d_T(\lambda,\mu) = \lambda^2\mu + c\lambda\mu^2 - b^2\mu^3,$$

which has discriminant $c^2 + 4b^2$. There are always two more distinct real roots.

$\begin{bmatrix} 0 & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$ can be diagonalized by an orthogonal matrix, with entries expressed in terms of its eigenvalues:

$$
\begin{aligned}
e_1 &= \frac{c+\sqrt{4b^2+c^2}}{2} > 0 \\
e_2 &= \frac{c-\sqrt{4b^2+c^2}}{2} < 0 \\
z &= \frac{-e_2}{\sqrt{b^2+e_2^2}} \\
\mathbf{A} &= \begin{bmatrix} \frac{bz}{e_1}\sqrt{\frac{-e_1}{e_2}} & \frac{bz}{e_2} & 0 \\ z\sqrt{\frac{-e_1}{e_2}} & z & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
\mathbf{A}^T\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 0 & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\mathbf{A} &= \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Three more steps give result i) of the Theorem. By a transformation $\begin{bmatrix} 1 & 0 \\ -e_1 & 1 \end{bmatrix}$ as in Lemma 5.7, scaling μ by $\frac{1}{e_2-e_1}$, and then congruence by $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{e_1-e_2} \end{bmatrix}$, S is congruent to $\left\{\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}$.

In upcoming computations, simple intermediate steps are sketched or left entirely to the reader.

If $\mathbf{L}'$ is congruent to $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then again (as in (6.1)) $\mathbf{M}'$ can be transformed to $\begin{bmatrix} a & b & 0 \\ b & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and these two matrices also span $T = \left\{\lambda\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$, with $d_T(\lambda,\mu) = \lambda^2 + ad\mu^3$, which has two real roots if and only if $ad > 0$.

If $a > 0$ and $d > 0$ then, for $\mathbf{A} = \begin{bmatrix} \sqrt{d} & \sqrt{d} & 0 \\ -\sqrt{a} & \sqrt{a} & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$
\mathbf{A}^T T\mathbf{A} = \left\{\lambda\begin{bmatrix} -2\sqrt{ad} & 0 & 0 \\ 0 & 2\sqrt{ad} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 2ad & 0 & 0 \\ 0 & 2ad & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\},
$$

which is related by scaling, congruence by a permutation matrix, and exchange of parameters to $\left\{\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}\right\}$, which is the same set as in case i).

If $a<0$ and $d<0$ then, for $\mathbf{A}=\begin{bmatrix}\sqrt{-d}&\sqrt{-d}&0\\-\sqrt{-a}&\sqrt{-a}&0\\0&0&1\end{bmatrix}$,

$$\mathbf{A}^T T\mathbf{A}=\left\{\lambda\begin{bmatrix}-2\sqrt{ad}&0&0\\0&2\sqrt{ad}&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}-2ad&0&0\\0&-2ad&0\\0&0&1\end{bmatrix}\right\},$$

which is related by scaling and congruence to $\left\{\lambda\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&0&0\\0&1&0\\0&0&-1\end{bmatrix}\right\}$. This is not congruent to the case i) set, and gives case ii) of the Theorem.

If $a<0$ and $d>0$ then, for $\mathbf{A}=\begin{bmatrix}\sqrt{d}&\sqrt{d}&0\\-\sqrt{-a}&\sqrt{-a}&0\\0&0&1\end{bmatrix}$,

$$\mathbf{A}^T T\mathbf{A}=\left\{\lambda\begin{bmatrix}-2\sqrt{-ad}&0&0\\0&2\sqrt{-ad}&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&2ad&0\\2ad&0&0\\0&0&1\end{bmatrix}\right\},$$

which is related by scaling and congruence to $\left\{\lambda\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}\right\}$. This is not congruent to any previous cases, and gives case iii) of the Theorem.

If $a>0$ and $d<0$ then, for $\mathbf{A}=\begin{bmatrix}\sqrt{-d}&\sqrt{-d}&0\\-\sqrt{a}&\sqrt{a}&0\\0&0&1\end{bmatrix}$,

$$\mathbf{A}^T T\mathbf{A}=\left\{\lambda\begin{bmatrix}-2\sqrt{-ad}&0&0\\0&2\sqrt{-ad}&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&-2ad&0\\-2ad&0&0\\0&0&1\end{bmatrix}\right\},$$

which is related by scaling and congruence to $\left\{\lambda\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}+\mu\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\right\}$.

This is related to the set in case iii) by congruence using $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and scaling.

∎

Remark 6.3. In this case with three complex root ratios, case iii) is distinguished from the other two by only having one real root ratio. To distinguish the other two cases, i) and ii) with three real root ratios, the above Proof suggests examining the non-zero eigenvalues of the singular elements of the pencil. The signs of eigenvalues of the corresponding singular elements of the pencils are invariants under real congruence (by an argument using Sylvester's Law of Inertia). Any (non-zero) singular matrix corresponding to a root ratio in a pencil from case ii) will have eigenvalues of opposite sign, as seen by inspecting the representative pencil from the statement of the Theorem. In case i), some singular matrices will have eigenvalues of opposite sign, but others will have eigenvalues of the same sign (possibly repeated).

7 Elimination of Parameters

Once a congruence class of a matrix pencil S is established, there are many ways to choose two linearly independent matrices for a basis, and many ways to choose a coefficient matrix $\mathbf{P}$ with a kernel equal to $w(S)$ as in Theorem 4.3.

However, once a set of parametric equations is given that defines a surface (as in (3.2)), an implicit equation of minimal degree is uniquely determined (up to scalar multiplication). Any process that achieves this conversion of parametric equations to an implicit equation is called <u>implicitization</u>.

It turns out that quadratic parametrizations, as in Section 3, always satisfy a polynomial implicit equation of degree at most 4.

We illustrate this for the three cases from Theorem 6.2.

Example 7.1. From case i)

$$\begin{aligned} w\left(\left\{\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}\right) &= \{(\lambda, \lambda+\mu, -\mu, 0, 0, 0)\} \\ &= \ker\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The parametric equations are:

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 - u_2^2 \\ x_1 &= u_1 u_2 \\ x_2 &= u_0 u_2 \\ x_3 &= u_0 u_1 \end{aligned}$$

These are similar to the equations from Example 1.2, and the calculation leading to an implicit equation is similar to the steps leading up to (1.9), only some signs are different in step (1.8). These parametric equations satisfy the implicit equation

$$x_1 x_2 x_3 x_0 + x_1^2 x_2^2 + x_1^2 x_3^2 - x_2^2 x_3^2 = 0.$$

Example 7.2. From case ii)

$$\begin{aligned} w\left(\left\{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}\right) &= \{(\lambda, -\lambda + \mu, -\mu, 0, 0, 0)\} \\ &= \ker \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This **P** matrix is the same as (3.3), and the parametric equations are exactly as in (3.1) from Example 3.1. The implicitization was carried out in Example 1.2, so in the $[x_0 : \ldots : x_3]$ coordinates the implicit equation is

$$x_1 x_2 x_3 x_0 - x_1^2 x_2^2 - x_1^2 x_3^2 - x_2^2 x_3^2 = 0.$$

Example 7.3. From case iii)

$$\begin{aligned} w\left(\left\{\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}\right) &= \{(\lambda, -\lambda, -\mu, 0, 0, \mu)\} \\ &= \ker \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned}$$

The parametric equations are:

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 \\
x_1 &= u_2^2 + u_0 u_1 \\
x_2 &= u_1 u_2 \\
x_3 &= u_0 u_2
\end{aligned}$$

The steps eliminating the parameters are different than in the other cases:

$$\begin{aligned}
u_1 &= \frac{x_2}{u_2} \\
u_0 &= \frac{x_3}{u_2} \\
x_0 &= \frac{x_2^2 + x_3^2}{u_2^2} \\
x_1 &= \frac{x_2^2 + x_3^2}{x_0} + \frac{x_0 x_2 x_3}{x_2^2 + x_3^2} \\
0 &= (x_2^2 + x_3^2)^2 - x_0 x_1 (x_2^2 + x_3^2) + x_0^2 x_2 x_3.
\end{aligned}$$

Example 7.4. For some coefficient matrices, implicitization is not so easily done by hand. Consider, for example,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Its kernel is spanned by $(1, 0, 0, 1, 0, 0)$ and $(0, 0, 1, 0, 0, 1)$. The image of the kernel under w^{-1} is $S = \left\{ \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$. $d_S(\lambda, \mu) = -(\lambda^3 + \mu^3)$, which has three distinct root ratios, only one of which is real. This falls under case iii) from Theorem 6.2.

The parametric equations defined by the coefficient matrix are

$$\begin{aligned}
x_0 &= u_0^2 - u_1 u_2 \\
x_1 &= u_2^2 - u_0 u_1 \\
x_2 &= u_1^2 \\
x_3 &= u_0 u_2.
\end{aligned}$$

What follows is an attempt to eliminate parameters by steps similar to the previous Example:

$$
\begin{aligned}
u_0 &= \frac{x_3}{u_2} \\
x_0 &= \frac{x_3^2}{u_2^2} - u_1 u_2 \\
x_1 &= u_2^2 - \frac{x_3 u_1}{u_2} \\
u_1 &= \frac{x_3^2}{u_2^3} - \frac{x_0}{u_2} \\
x_2 &= \frac{x_3^4}{u_2^6} - \frac{2x_0 x_3^2}{u_2^4} + \frac{x_0^2}{u_2^2} && (7.1) \\
x_1 &= u_2^2 - \frac{x_3^3}{u_2^4} + \frac{x_0 x_3}{u_2^2}. && (7.2)
\end{aligned}
$$

All that remains now is two equations, (7.1) and (7.2), and one variable (u_2) to be eliminated. The Sylvester Resultant can now give a polynomial expression in the x_i variables, but since this method depends on the degree of each equation, v can be substituted for u_2^2 to simplify the calculation:

$$
\begin{aligned}
0 &= v^3 - x_1 v^2 + x_0 x_3 v - x_3^3 \\
0 &= x_2 v^3 - x_0^2 v^2 + 2x_0 x_3^2 v - x_3^4.
\end{aligned}
$$

Subtracting the first equation multiplied by x_2 from the second to lower the degree of the second equation gives:

$$
\begin{aligned}
0 &= v^3 - x_1 v^2 + x_0 x_3 v - x_3^3 \\
0 &= (x_1 x_2 - x_0^2) v^2 + (2x_0 x_3^2 - x_0 x_2 x_3) v - x_3^4 + x_2 x_3^3.
\end{aligned}
$$

The resultant of this system of equations is the determinant of the 5×5 matrix

$$
\begin{bmatrix}
1 & -x_1 & x_0 x_3 & -x_3^3 & 0 \\
0 & 1 & -x_1 & x_0 x_3 & -x_3^3 \\
x_1 x_2 - x_0^2 & 2x_0 x_3^2 - x_0 x_2 x_3 & x_2 x_3^3 - x_3^4 & 0 & 0 \\
0 & x_1 x_2 - x_0^2 & 2x_0 x_3^2 - x_0 x_2 x_3 & x_2 x_3^3 - x_3^4 & 0 \\
0 & 0 & x_1 x_2 - x_0^2 & 2x_0 x_3^2 - x_0 x_2 x_3 & x_2 x_3^3 - x_3^4
\end{bmatrix},
$$

which is

$$
-x_3^8 \left(x_3^4 - 3x_2 x_3^3 - 2x_0 x_1 x_3^2 + 3x_2^2 x_3^2 + x_0 x_1 x_2 x_3 - x_2^3 x_3 - x_0^3 x_3 + x_0^2 x_1^2 + x_0 x_1 x_2^2 - x_1^3 x_2 \right).
$$

This symbolic calculation was done easily on a personal computer using [Derive]. As sometimes occurs with resultants, there is an extraneous factor, in this case $-x_3^8$, and the other factor,

$$x_3^4 - 3x_2x_3^3 - 2x_0x_1x_3^2 + 3x_2^2x_3^2 + x_0x_1x_2x_3 - x_2^3x_3 - x_0^3x_3 + x_0^2x_1^2 + x_0x_1x_2^2 - x_1^3x_2,$$

is an irreducible, homogeneous, fourth degree polynomial which is an implicit equation satisfied by the above parametric equations.

8 Geometric Consequences

In all examples so far, an implicit equation has been derived so that algebraic relations among variables called x_i are satisfied by values of x_i given by quadratic polynomials in three parameters u_j. However, the implicit equations can have some solutions in addition to any presentable in terms of the given parameters.

For quadratically parametrized surfaces with a fourth degree implicitization, these extra solutions always appear along the double lines, as described in Section 2. In a particular affine neighborhood, the extra solutions form either a ray of whisker points starting at a singularity on the surface (there can be one or two such rays), or one segment of whisker points between two singularities.

Example 8.1. Points in the image of the parametric map

$$[u_0 : u_1 : u_2] \mapsto [u_0^2 + u_1^2 + u_2^2 : u_1u_2 : u_0u_2 : u_0u_1]$$

satisfy the implicit equation

$$x_1x_2x_3x_0 - x_1^2x_2^2 - x_1^2x_3^2 - x_2^2x_3^2 = 0.$$

from Examples 1.2, 3.1, and 7.2, in case ii) of Theorem 6.2. The equation is homogeneous, defining a real hypersurface in $\mathbb{R}^4$, and a real surface in $\mathbb{R}P^3$.

Every point in the parameter domain $[u_0 : u_1 : u_2]$ has an image with $x_0 \neq 0$ (there are no points in the domain $\mathbb{R}P^2$ where $x_0 = u_0^2 + u_1^2 + u_2^2 = 0$ holds), so setting $x_0 = 1$ to look at an affine subset $(x, y, z) = [1 : x_1 : x_2 : x_3]$ of the target $\mathbb{R}P^3$, will show every image point of the parametric map. The image shape could also be considered as the image of the restriction of the mapping from Equation (3.1) to the unit sphere $\{u_0^2 + u_1^2 + u_2^2 = 1\}$ in $\mathbb{R}^3$. This restriction is two-to-one on the sphere: antipodal points are mapped to the same image, as described in Section 2.

The image of the above parametric map from the real projective plane to $\mathbb{R}^3$ is called Steiner's Roman Surface. This is the first of the illustrations in Appendix 11. The image itself is not homeomorphic to a real projective plane because it has self-intersections, and the image is contained in, but not equal to, the solution set of the fourth degree implicit equation; both of these issues are visible as the three double lines appearing on the Roman Surface. In the (x, y, z) coordinate system, these double lines are exactly the coordinate axes, meeting at the triple point $(0, 0, 0)$ on the surface.

If we again consider the parametric map as a restriction to the unit sphere, this puts a constraint on the real parameters:

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 = 1 \\ &\implies -1 \le u_j \le 1. \end{aligned} \tag{8.1}$$

In the xyz-space where $x_0 = 1$, $x = u_1 u_2$ is also constrained, $-1 \le u_1 u_2 \le 1$, but the affine point $(x, y, z) = (5, 0, 0)$, and every other point on the x-axis, satisfies the implicit equation $xyz - x^2y^2 - x^2z^2 - y^2z^2 = 0$.

The Roman Surface is the first of the illustrations in Appendix 11; Surface Plotter ([SP]) used polar parameters in Figure 1. as follows:

$$\begin{aligned} u_1 &= \rho \cos\theta \\ u_2 &= \rho \sin\theta \\ u_0^2 &= 1 - u_1^2 - u_2^2 = 1 - \rho^2 \\ x &= u_1 u_2 = \rho^2 \cos\theta \sin\theta \\ y &= u_0 u_2 = \rho \sin\theta \sqrt{1 - \rho^2} \\ z &= u_0 u_1 = \rho \cos\theta \sqrt{1 - \rho^2}. \end{aligned}$$

Example 8.2. [A] uses the equations

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 \\ x_1 &= u_1 u_2 \\ x_2 &= 2u_0 u_1 \\ x_3 &= u_0^2 - u_1^2 \end{aligned}$$

as a parametrization of Steiner's Cross Cap Surface, and they satisfy the implicit equation

$$4x_1^2(x_1^2 + x_2^2 + x_3^2 + x_0x_3) + x_2^2(x_2^2 + x_3^2 - x_0^2) = 0.$$

Setting $x_0 = 1$ does not lose any image points in the parametrization. As in Example 8.1, this can be considered as a map of the sphere. There is only one double line, $\{x_1 = x_2 = 0\}$, and in the xyz-space where $x_0 = 1$, the implicit equation

$$4x^2(x^2 + y^2 + z^2 + z) + y^2(y^2 + z^2 - 1) = 0$$

describes the Cross-Cap Surface with its double line on the z-axis. There are two more algebraic double lines, defined by systems of linear equations with complex conjugate coefficients:

$$\begin{aligned} 0 &= y - i\sqrt{2}x = z + 1, \\ 0 &= y + i\sqrt{2}x = z + 1. \end{aligned}$$

The three complex lines meet at a triple point, which happens to have real coordinates, on the surface at $(0, 0, -1)$. The real surface does not show three real double lines meeting at a point, but $(0, 0, -1)$ is where a whisker intersects the surface. This falls under case iii) of Theorem 6.2.

The Cross-Cap Surface is the second illustration in Appendix 11, with polar coordinates in Figure 2:

$$\begin{aligned} u_1 &= \rho \cos\theta \\ u_2 &= \rho \sin\theta \\ u_0^2 &= 1 - u_1^2 - u_2^2 = 1 - \rho^2 \\ x &= u_1 u_2 = \rho^2 \cos\theta \sin\theta \\ y &= 2u_0 u_1 = 2\rho \cos\theta \sqrt{1 - \rho^2} \\ z &= u_0^2 - u_1^2 = 1 - \rho^2(1 + \cos^2\theta). \end{aligned}$$

Example 8.3. For the implicit equation from Example 7.1,

$$x_1 x_2 x_3 x_0 + x_1^2 x_2^2 + x_1^2 x_3^2 - x_2^2 x_3^2 = 0$$

and the parametric equations, falling in case i) of Theorem 6.2:

$$x_0 = u_0^2 - u_1^2 - u_2^2 \tag{8.2}$$
$$x_1 = u_1 u_2 \tag{8.3}$$
$$x_2 = u_0 u_2 \tag{8.4}$$
$$x_3 = u_0 u_1, \tag{8.5}$$

the surface can no longer be considered as an image of the sphere. There are nonzero values of x_1, x_2, x_3 for some points where $x_0 = 0$, so the image cannot be contained in the $x_0 = 1$ affine space (nor any other). Further, in Examples 8.1 and 8.2, x_0 was positive for any real u_i values, but in this case x_0 can be negative, so the parametric image will appear to have two "pieces" in the $x_0 \neq 0$ neighborhood. In Equation (8.1) from Example 8.1, setting $u_0^2 + u_1^2 + u_2^2 = 1$ captured representatives of every line through the origin in the $u_0u_1u_2$-space, and setting $x_0 = 1$ mapped every point in the domain to the $\{x_0 \neq 0\}$ affine neighborhood in the target. For this example, setting

$u_0^2 - u_1^2 - u_2^2 = 1$ from (8.2) only meets some of the lines through the origin; setting $u_0^2 - u_1^2 - u_2^2 = -1$ gets most of the rest. So to view the image of the parametrization in the $\{x_0 \neq 0\}$ neighborhood, and before converting to polar coordinates for plotting, the two cases will need to be considered separately.

The solution set of the implicit equation has two pinch points, as in Example 8.2, and three real double lines that meet at a real triple point, as in Example 8.1. In xyz-space, the surface

$$xyz + x^2y^2 + x^2z^2 - y^2z^2 = 0$$

has pinch point singularities at $(-0.5, 0, 0)$ and $(0.5, 0, 0)$, and double lines on each of the three coordinate axes, since setting any two of the x, y, z variables equal to zero satisfies the implicit equation. The double lines on the y and z axes do not have pinch points and lie inside the image of the parametric map. Like the previous Examples, some of the points on the x-axis are whiskers that are not images of the parameters, for example, $x = u_1u_2 = 7$ and $y = z = 0$. Only points between the pinch points are in the image of the parametric map: from (8.4), (8.5), if $y = z = 0$, then $u_0 = 0$, and the real ratio $x = \frac{x_1}{x_0} = \frac{u_1u_2}{-u_1^2 - u_2^2}$ is constrained to $-0.5 \leq x \leq 0.5$.

In polar coordinates where $u_1 = \rho\cos\theta$, $u_2 = \rho\sin\theta$, the parametric plot can be done piecewise. The first piece has no pinch points and looks like a smooth saddle surface.

$$\begin{aligned}
x_0 &= u_0^2 - u_1^2 - u_2^2 = 1 \implies u_0^2 = 1 + u_1^2 + u_2^2 = 1 + \rho^2 \\
x &= \frac{x_1}{x_0} = u_1u_2 = \rho^2\cos\theta\sin\theta \\
y &= \frac{x_2}{x_0} = u_0u_2 = \rho\sin\theta\sqrt{1+\rho^2} \\
z &= \frac{x_3}{x_0} = u_0u_1 = \rho\cos\theta\sqrt{1+\rho^2}.
\end{aligned}$$

For the other piece, $\rho \geq 1$, as shown in Figure 3:

$$\begin{aligned}
x_0 &= u_0^2 - u_1^2 - u_2^2 = -1 \implies u_0^2 = -1 + u_1^2 + u_2^2 = \rho^2 - 1 \\
x &= \frac{x_1}{x_0} = -u_1u_2 = -\rho^2\cos\theta\sin\theta \\
y &= \frac{x_2}{x_0} = -u_0u_2 = -\rho\sin\theta\sqrt{\rho^2 - 1} \\
z &= \frac{x_3}{x_0} = -u_0u_1 = -\rho\cos\theta\sqrt{\rho^2 - 1}.
\end{aligned}$$

Figure 4. shows both pieces plotted together.

Example 8.4. Another set of parametric equations falling in case i) of Theorem 6.2:

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 \\ x_1 &= u_1^2 - u_2^2 \\ x_2 &= u_1 u_2 \\ x_3 &= u_0 u_2 \end{aligned} \tag{8.6}$$

with homogeneous implicit equation

$$(x_3^2 - x_2^2)^2 + x_0 x_1 (x_3^2 - x_2^2) - x_0^2 x_2^2 = 0. \tag{8.7}$$

This surface also has two pinch points, but only one double line visible in the $\{x_0 \neq 0\}$ affine neighborhood with coordinates (x, y, z), where the implicit equation is:

$$(z^2 - y^2)^2 + x(z^2 - y^2) - y^2 = 0. \tag{8.8}$$

The solution set of (8.8) contains the x-axis, but if $y = z = 0$ and $x_0 \neq 0$ then $u_2 = 0$ and $x = \frac{x_1}{x_0} = \frac{u_1^2}{u_0^2 - u_1^2}$ has values in $(\infty, -1] \cup [0, \infty)$. So, the whisker is the segment from $(-1, 0, 0)$ to $(0, 0, 0)$, unlike in Example 8.3 where the double line met the parametrized surface in a segment and the whiskers formed two rays.

The other two double lines are real, but contained in the plane at infinity, $\{x_0 = 0\}$, where they meet the first double line at a real triple point.

The two pieces of the parametric map formed by positive or negative values of $x_0 = u_0^2 - u_1^2$ do not intersect in xyz-space, and each contains one pinch point.

In polar coordinates where $u_1 = \rho \cos\theta$, $u_2 = \rho \sin\theta$, the parametric map can be plotted piecewise:

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 = 1 \implies u_0^2 = 1 + u_1^2 = 1 + \rho^2 \cos^2\theta \\ x &= \frac{x_1}{x_0} = u_1^2 - u_2^2 = \rho^2(\cos^2\theta - \sin^2\theta) \\ y &= \frac{x_2}{x_0} = u_1 u_2 = \rho^2 \cos\theta \sin\theta \\ z &= \frac{x_3}{x_0} = u_0 u_2 = \rho \sin\theta \sqrt{1 + \rho^2 \cos^2\theta}. \end{aligned}$$

This piece is shown by itself in Figure 5. For the other piece, the polar coordinates in the domain are changed to $u_0 = \rho \cos\theta$, $u_2 = \rho \sin\theta$:

$$\begin{aligned} x_0 &= u_0^2 - u_1^2 = -1 \implies u_1^2 = 1 + u_0^2 = 1 + \rho^2 \cos^2\theta \\ x &= \frac{x_1}{x_0} = -(u_1^2 - u_2^2) = -1 - \rho^2(\cos^2\theta - \sin^2\theta) \\ y &= \frac{x_2}{x_0} = -u_1 u_2 = -\rho \sin\theta \sqrt{1 + \rho^2 \cos^2\theta} \\ z &= \frac{x_3}{x_0} = -u_0 u_2 = -\rho^2 \cos\theta \sin\theta. \end{aligned}$$

See Figure 6. in Appendix 11, where both pieces are shown, the first in front of the second, and they are congruent as geometric objects in xyz-space.

Example 8.5. Another set of parametric equations falling in case i) of Theorem 6.2:

$$\begin{aligned} x_0 &= u_1^2 - u_2^2 \\ x_1 &= u_1 u_2 - u_0^2 \\ x_2 &= u_0 u_2 \\ x_3 &= u_0 u_1 \end{aligned}$$

has homogeneous implicit equation

$$(x_3^2 - x_2^2)^2 + x_0 x_1 (x_3^2 - x_2^2) - x_0^2 x_2 x_3 = 0. \tag{8.9}$$

This implicit equation differs from (8.7) only in the last term.

Looking at the $\{x_3 \neq 0\}$ affine neighborhood with coordinates $(x, y, z, 1)$ (instead of the $\{x_0 \neq 0\}$ neighborhood as in previous Examples), the implicit equation where $x_3 = 1$ becomes

$$(1 - z^2)^2 + xy(1 - z^2) - x^2 z = 0.$$

This surface in $\mathbb{R}^3$ has two double lines, but the third, and the triple point where they meet, are at the $\{x_3 = 0\}$ plane at infinity. The two visible double lines are parallel to each other and to the y-axis; one is $\{x = 0, z = 1\}$ which is entirely in the image of the parametric map, the other is $\{x = 0, z = -1\}$, which has two pinch points bounding a segment of whisker points.

Figure 7. in Appendix 11 is a view in a small box around the whisker: $-4 < x, y < 4$, $-2 < z < 0$, so only one double line is visible. Figure 7. was generated by plotting points on curves on the surface where z is constant. Figure 8. shows a wider view, with both double lines.

Example 8.6. This example shows how the surface from Example 8.4 can undergo a transformation so that it looks like the surface in Example 8.3. Both are in the same equivalence class of parametrizations, from case i) of Theorem 6.2, so this will give an example of an equivalence by a specific real matrix $\mathbf{B}$ as in Theorem 4.3.

The plane $\{x = -0.5\}$ is the perpendicular bisector of the whisker segment in the surface (8.8) in xyz-space. This plane can be mapped to the plane at infinity, corresponding to $y_0 = 0$ in a new $[y_0 : y_1 : y_2 : y_3]$ target space. There is a new coordinate system (x', y', z') for the affine neighborhood $\{y_0 \neq 0\}$. The transformation matrix is $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; let $(y_0, y_1, y_2, y_3) = (x_0, x_1, x_2, x_3)\mathbf{B}$.

Then, for any point where $x_0 = 1$ and $x_1 = -0.5$, we have $y_0 = x_0 + 2x_1 = 1 - 1 = 0$.

The parametric equations (8.6) are transformed to new parametric equations:

$$\begin{aligned}
y_0 &= u_0^2 - u_1^2 + 2(u_1^2 - u_2^2) = u_0^2 + u_1^2 - 2u_2^2 \\
y_1 &= u_1^2 - u_2^2 \\
y_2 &= u_1 u_2 \\
y_3 &= u_0 u_2,
\end{aligned}$$

and the implicit equation (8.7) is transformed to:

$$(y_3^2 - y_2^2)^2 + (y_0 - 2y_1)y_1(y_3^2 - y_2^2) - (y_0 - 2y_1)^2 y_2^2 = 0.$$

One double line is $\{y' = z' = 0\}$, where $x' = \frac{y_1}{y_0} = \frac{u_1^2}{u_0^2 + u_1^2}$, so the parametric image meets the double line only in the segment $0 \le x' \le 1$ between two pinch points. The other two double lines, which were at infinity in Example 8.4, are now visible at $\{x' = 0.5, y' = \pm z'\}$, meeting the first double line at a triple point, as in Example 8.3 and Figure 4.

Example 8.7. The following parametric equations:

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 - u_2^2 && (8.10) \\
x_1 &= u_1^2 - u_2^2 && (8.11) \\
x_2 &= u_1 u_2 \\
x_3 &= u_0 u_2,
\end{aligned}$$

differ from those in Examples 8.4 and 8.6 only in the x_0 expression (8.10). They fall in case 3.a. of Proposition 5.4: the kernel of the coefficient matrix for these equations looks like $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$ and has determinant $-\mu\lambda^2$.

The parametric image has a singularity that is different from a pinch point; there is a point that looks like a union of a pinch point and a tangent plane, as in Example 2.2. To demonstrate the expected instability of such a point, consider the following perturbation of the parametric equations: change Equation (8.11) to $x_1 = u_1^2 - (1+\varepsilon)u_2^2$ for small real ε. The kernel of the coefficient matrix changes to $\begin{bmatrix} -\varepsilon\mu & \lambda & 0 \\ \lambda & (1+\varepsilon)\mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$, and has determinant $-\mu(\lambda^2 + \varepsilon(1+\varepsilon)\mu^2)$, which, for values of ε other than 0 or -1, is in case 4.a. of Proposition 5.4 and in case i) or iii) from Theorem 6.2. The singularities in the 4.a. cases are two pinch points; the 3.a. case has one standard pinch point and one point with the singularity from Example 2.2.

The polar coordinatization has two pieces, both shown in Figure 9., with $u_0 = \rho\cos\theta$, $u_1 = \rho\sin\theta$:

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 - u_2^2 = 1 \implies u_2^2 = u_0^2 + u_1^2 - 1 = \rho^2 - 1 \\
x &= \frac{x_1}{x_0} = u_1^2 - u_2^2 = 1 - \rho^2\cos^2\theta \\
y &= \frac{x_2}{x_0} = u_1 u_2 = \rho\sin\theta\sqrt{\rho^2 - 1} \\
z &= \frac{x_3}{x_0} = u_0 u_2 = \rho\cos\theta\sqrt{\rho^2 - 1}.
\end{aligned}$$

Similarly for the second piece,

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 - u_2^2 = -1 \implies u_2^2 = u_0^2 + u_1^2 + 1 = \rho^2 + 1 \\
x &= \frac{x_1}{x_0} = -\left(u_1^2 - u_2^2\right) = -1 - \rho^2\cos^2\theta \\
y &= \frac{x_2}{x_0} = -u_1 u_2 = -\rho\sin\theta\sqrt{\rho^2 + 1} \\
z &= \frac{x_3}{x_0} = -u_0 u_2 = -\rho\cos\theta\sqrt{\rho^2 + 1}.
\end{aligned}$$

These can be easily modified to get piecewise polar expressions for the perturbed equations (with ε).

Example 8.8. The following parametric equations:

$$\begin{aligned}
x_0 &= u_0^2 + u_1^2 + u_2^2 \\
x_1 &= 2u_0 u_2 \\
x_2 &= 2u_0 u_1 \\
x_3 &= u_0^2 - u_1^2 - u_2^2,
\end{aligned}$$

satisfy the implicit equation $x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0$, which is quadratic (unlike the previous Examples in Section 8, which were all of degree 4). In the $x_0 = 1$ affine neighborhood, this is an equation for the unit sphere in 3-space, as mentioned in Section 1. The parametric equations fall into case 1.c. from Proposition 5.4. A polar coordinatization uses $u_1 = \rho\cos\theta$, $u_2 = \rho\sin\theta$, $0 \le \rho \le 1$:

$$\begin{aligned}
x &= \frac{2u_0 u_2}{u_0^2 + u_1^2 + u_2^2} = 2\rho\sin\theta\sqrt{1 - \rho^2} \qquad (8.12) \\
y &= \frac{2u_0 u_1}{u_0^2 + u_1^2 + u_2^2} = 2\rho\cos\theta\sqrt{1 - \rho^2} \\
z &= \frac{u_0^2 - u_1^2 - u_2^2}{u_0^2 + u_1^2 + u_2^2} = 1 - 2\rho^2.
\end{aligned}$$

Some perturbations of the parametric equations continue to describe quadric surfaces, for example, if $x_1 = 2u_0u_2 + \varepsilon u_0u_1$, then the parametrization is still in case 1.c. and the equations with $x = 2\rho(\sin\theta + \varepsilon\cos\theta)\sqrt{1-\rho^2}$ describe an ellipsoid. This perturbation results just from a linear transformation of the xyz-space, so the coefficient matrices are equivalent. The curve where $x = 0$ is always the circle $y^2 + z^2 = 1$.

Example 8.9. Another perturbation of the parametrization of the sphere from Example 8.8 is:

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 \\ x_1 &= 2u_0u_2 + \varepsilon u_1u_2 \\ x_2 &= 2u_0u_1 \\ x_3 &= u_0^2 - u_1^2 - u_2^2, \end{aligned}$$

which changes the polar formula (8.12) to

$$x = 2\rho\sin\theta\sqrt{1-\rho^2} + \varepsilon\rho^2\sin\theta\cos\theta.$$

The homogeneous implicit equation is

$$4x_1^2(x_3+x_0)^2 + (x_2^2 + x_3^2 - x_0^2)(2(x_3+x_0) + \varepsilon x_2)^2 = 0.$$

So for $\varepsilon = 0$, $x_0 = 1$, this is the sphere from Example 8.8, but for any other value of ϵ, the kernel of the coefficient matrix looks like $\begin{bmatrix} 0 & 0 & -\varepsilon\mu \\ 0 & \lambda & 2\mu \\ -\varepsilon\mu & 2\mu & -\lambda \end{bmatrix}$, with determinant $-\varepsilon^2\lambda\mu^2$. This is case 3.a. from Proposition 5.4, so the coefficient matrix is not equivalent to the coefficient matrix for the sphere.

This is a notable example in that it is a family of surfaces that always, except at $\varepsilon = 0$, retains an unstable singularity at $(0,0,-1)$ (locally, the same pinch point meeting a plane shape as in Example 8.7). There are more pinch points, $(\pm\frac{\varepsilon}{2}, 0, -1)$ that move toward each other along a double line as $\varepsilon \to 0$. Another double line is given by the equations $\{x = 0, z + \frac{\varepsilon}{2}y + 1 = 0\}$, which intersects the circle $\{x = 0, y^2 + z^2 = 1\}$ at two points, $(0,0,-1)$ and another pinch point, $\left(0, \frac{-4\varepsilon}{\varepsilon^2+4}, \frac{\varepsilon^2-4}{\varepsilon^2+4}\right)$. See Figure 10.

Remark 8.10. The real surfaces from Examples 8.7 and 8.9 falling in case 3.a. from Proposition 5.4 are related by complex equivalence as in Theorem 4.3, but are not related by real equivalence. Their corresponding real pencils are not congruent over $\mathbb{R}$. There is an analogue of Theorem 6.2, not proved here, stating that case 3.a. admits exactly two real equivalence classes. The surfaces are different geometrically, in their number of pinch points and whiskers, and because the parametric equations from Example 8.9 have an image in $\mathbb{R}P^3$ which is contained in an affine neighborhood, while the image from Example 8.7 cannot be contained in any affine neighborhood.

Example 8.11. One more perturbation of the parametrization of the sphere from Example 8.8 is given by:

$$\begin{aligned} x_0 &= u_0^2 + u_1^2 + u_2^2 \\ x_1 &= 2u_0u_2 \\ x_2 &= 2u_0u_1 \\ x_3 &= u_0^2 - u_1^2 - (1+\varepsilon)u_2^2. \end{aligned}$$

The homogeneous implicit equation is:

$$\left(x_2^2 + \left(1+\frac{\varepsilon}{2}\right)x_1^2\right)^2 + (x_1^2+x_2^2)(x_3-x_0)^2 + 2x_0(x_3-x_0)\left(x_2^2+\left(1+\frac{\varepsilon}{2}\right)x_1^2\right) = 0.$$

At $\varepsilon = 0$, the polynomial is not irreducible, it factors as

$$(x_1^2+x_2^2)(x_1^2+x_2^2+x_3^2-x_0^2),$$

so the zero set is a union of a sphere and some extraneous solutions, although only the sphere is in the image of the parameters, and the second quadratic factor is the lowest degree implicitization of the parametric equations at $\varepsilon = 0$.

For $\varepsilon \neq 0$, the kernel of the coefficient matrix looks like $\begin{bmatrix} \frac{\varepsilon\lambda}{2} & 0 & 0 \\ 0 & -\left(1+\frac{\varepsilon}{2}\right)\lambda & \mu \\ 0 & \mu & \lambda \end{bmatrix}$, with determinant $-\frac{\varepsilon\lambda}{2}\left(\left(1+\frac{\varepsilon}{2}\right)\lambda^2+\mu^2\right)$.

For $\varepsilon = -2$, the coefficients fall in case 3.a. from Proposition 5.4 and the surface is similar in appearance to the surfaces from Example 8.9, with a T-shaped self-intersection set along two double lines: the z-axis and $\{(x,0,1)\}$. It has the implicit equation

$$x_1^2(x_3-x_0)^2 + x_2^2(x_2^2+x_3^2-x_0^2) = 0.$$

Considering the family of surfaces for real values of ε in the $x_0 = 1$ neighborhood with coordinates (x,y,z), the z-axis is always a double line (except at $\varepsilon = 0$). For non-zero $\varepsilon > -2$, the parametric equations are in case 4.a. from Proposition 5.4 and case iii) from Theorem 6.2, the Cross-Cap Surface. For $\varepsilon < -2$ the equations fall in case ii) from Theorem 6.2, in the same class as the Roman Surface.

The Cross-Cap Surface has one real double line and two singularities. The Roman Surface has three real double lines and six singularities. The intermediate surface at $\varepsilon = -2$ has two real double lines and four singularities. The animation feature of Surface Plotter ([SP]) can be used with ε as a time parameter, to observe the transitions from Cross-Cap to sphere to Cross-Cap to the intermediate surface to the Roman Surface, as ε decreases. The double lines appearing in the Roman Surface, depending on ε, are given by the equations $\{z = 1, y^2 + \left(1+\frac{\varepsilon}{2}\right)x^2 = 0\}$.

9 Combinatorial Models

The surfaces described above, even with the pictures in Appendix 11, can be hard to understand and visualize. As a guide for interpretation, geometric constructions call "combinatorial surfaces" are useful. Triangles and squares can be joined along common edges or corners to create a surface "ambient isotopic" to the actual image of the parameters in xyz-space. Two surfaces M_0 and M_1 in $\mathbb{R}^3$ are ambient isotopic means: there is a continuous map $F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ such that for every $t \in [0,1]$, $F(.,t)$ is a homeomorphism, $F(.,0) = id_{\mathbb{R}^3}$, and $F(M_0, 1) = M_1$.

Imagine a solid cube, centered at the origin of $\mathbb{R}^3$, so that each face of the cube is perpendicular to one of the coordinate axes, and each edge is of length 2. Now remove two smaller cubes from this cube: one where the larger cube intersects the first octant $\{x > 0, y > 0, z > 0\}$, and another from $\{x < 0, y < 0, z > 0\}$. The Cross-Cap Surface is ambient isotopic to the boundary of the remainder of this solid. Pinch points are represented where corners meet, at $(0,0,0)$, and $(0,0,1)$. The segment between the pinch points on the z-axis represents the self-intersection set.

Imagine this solid again, but remove two additional cubes, at $\{x < 0, y > 0, z < 0\}$ and $\{x > 0, y < 0, z < 0\}$. What is left is four solid cubes, each touching two others along an edge, and each meeting the other three at the origin point. The solid's boundary has six pinch points: $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$, with the three self-intersection segments meeting in a triple point at the center. This surface is a model of the Roman Surface.

10 Computer Software and Graphic Media

[Derive] was used in the calculation of determinants when using resultants to eliminate parameters. On a personal computer, reasonable time periods (on the order of seconds) were spent on matrices of size up to 10×10.

Surface Plotter ([SP]) was used to plot surfaces during the research for this project and for the final draft of this paper. In most cases, converting to polar coordinates was more efficient than the quadratic parametrizations, in terms of visualizing the whole surface rather than just a local patch.

Level curves of surfaces were also observed using plots generated by a BASIC program on an Amiga personal computer.

Paper models were of use in the visualization of, and communication about, the pinch points, self-intersections, and symmetry of some surfaces; the square pages of a desk calendar (appropriately, for non-orientable surfaces, [L_1], [L_2]) were suitable as an artistic medium.

11 Appendix — Illustrations

[Some of the original illustrations from 1991 were not available, and were reconstructed in 2018 using [Maple*], as indicated.]

- Figure 1. Steiner's Roman Surface, plotted using polar parameters, as in Example 8.1, using Surface Plotter [SP].
- Figure 2. Steiner's Cross-Cap Surface, plotted using polar parameters, as in Example 8.2, using Surface Plotter [SP].
- Figure 3. One piece of the polar parametrization from Example 8.3, using Surface Plotter [SP].
- Figure 4. Both pieces of the surface from Example 8.3, using [Maple*].
- Figure 5. One piece of the polar parametrization from Example 8.4, using Surface Plotter [SP].
- Figure 6. Both pieces of the polar parametrization from Example 8.4, using Surface Plotter [SP].
- Figure 7. A local view of the surface from Example 8.5 near a whisker segment. The plot was generated by BASIC code on an Amiga personal computer.
- Figure 8. A wider view of the surface from Example 8.5, using [Maple*].
- Figure 9. Both pieces of the polar parametrization from Example 8.7, using Surface Plotter [SP].
- Figure 10. The surface from Example 8.9, with $\varepsilon = 6$, using [Maple*].

12 Postscript — added in 2018

This Bachelor's thesis was submitted to the Department of Mathematics to fulfill the requirements to graduate in May 1991 with High Honors in mathematics. It was supervised by Art Schwartz, and my research was supported in the Summer of 1990 by a Research Experiences for Undergraduates grant from the National Science Foundation.

This thesis remained unpublished (and unavailable on the internet until 2018), but Prof. Schwartz and I continued to work on this topic, leading to a paper published in 1996: [CSS*], and also a web page with graphics and animation, currently at this address: [C*]. Some of my subsequent research papers (which I hope have better writing, proofs, and graphics than here…) have been on related topics, in real and complex projective geometry, linear algebra and matrix pencils, analogues of Theorem 4.3 and Theorem 6.2, and the mathematics of surfaces in computer graphics.

Theorem 6.2 stated the classification of 2-dimensional real pencils of 3×3 symmetric matrices up to real congruence, but only for the case of distinct root ratios. The classification for the other cases was essentially known to me and Schwartz by 1991, but not included in the thesis I submitted in early 1991. In the following "Addendum" Section, I am copying some of my notes from later in 1991, in a format similar to Proposition 5.4 and Example 5.5. We ended up using different notation in the published paper [CSS*], but this classification of real pencils is the calculation behind the main geometric result of [CSS*], the classification of quadratically parametrized maps into the nine real types of Steiner Surfaces (three non-degenerate types in case 4, three degenerate quartics with $v = 0$, three ruled cubics with $v = 1$) and quadric surfaces (case 1).

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13 Addendum — notes from 1991

In the following Proposition, the (ρ, v) pair from Notation 5.3 is modified by replacing ρ with symbols indicating whether each of the ρ root ratios corresponds to a matrix with eigenvalues of the same sign $(+)$, opposite signs $(-)$, or which are non-real (0).

Proposition 13.1. *Any two-dimensional real pencil of symmetric three-by-three matrices falls into exactly one of the following cases, which determines its congruence class over $\mathbb{R}$:*

1. $d_S(\lambda, \mu)$ *is identically zero.*
    - (a) $(\rho, v) = (\infty, 0)$.
    - (b) $(\rho, v) = (\infty, 1)$.
    - (c) $(\rho, v) = (\infty, 2)$.
        - i. *Two distinct, real root ratios each correspond to a matrix of rank* 1.
        - ii. *No real root ratio* $[\lambda : \mu]$ *will give an intersection with* V*, but there are two complex intersections.*
2. $d_S(\lambda, \mu)$ *has one root ratio (of multiplicity three).*
    - (a) $(\rho, v) = (1, 0) = (-, 0)$.
    - (b) $(\rho, v) = (1, 1) = (+, 1)$.
3. $d_S(\lambda, \mu)$ *has two distinct root ratios.*
    - (a) $(\rho, v) = (2, 0)$.
        - i. $(+-, 0)$.
        - ii. $(--, 0)$.
    - (b) $(\rho, v) = (2, 1)$.
        - i. $(++, 1)$.
        - ii. $(+-, 1)$.
4. $d_S(\lambda, \mu)$ *has three distinct root ratios.*
    - (a) $(\rho, v) = (3, 0)$.
        - i. $(+ + -, 0)$.
        - ii. $(- - -, 0)$.
        - iii. $(-\ 0\ 0, 0)$.

$\blacksquare$

Example 13.2.

- 1.a. $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix}$ 1.b. $\begin{bmatrix} \lambda & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- 1.c.i. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 1.c.ii. $\begin{bmatrix} \lambda & \mu & 0 \\ \mu & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- 2.a. $\begin{bmatrix} 0 & \lambda & \mu \\ \lambda & \mu & 0 \\ \mu & 0 & 0 \end{bmatrix}$ 2.b. $\begin{bmatrix} \lambda & 0 & \mu \\ 0 & \mu & 0 \\ \mu & 0 & 0 \end{bmatrix}$
- 3.a.i. $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}$ 3.a.ii. $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & \mu & 0 \\ 0 & 0 & -\mu \end{bmatrix}$
- 3.b.i. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$ 3.b.ii. $\begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \mu \end{bmatrix}$
- 4.a.i. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu+\lambda & 0 \\ 0 & 0 & -\mu \end{bmatrix}$ 4.a.ii. $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu-\lambda & 0 \\ 0 & 0 & -\mu \end{bmatrix}$
- 4.a.iii. $\begin{bmatrix} \lambda & \mu & 0 \\ \mu & -\lambda & 0 \\ 0 & 0 & -\mu \end{bmatrix}$

Remark 13.3. The 4.a. cases were established in Theorem 6.2. The two 3.a. cases appeared in Examples 8.7 and 8.9, as mentioned in Remark 8.10. Note that even with the refined $+/-/0$ invariant for singular matrices in a real pencil, the information about rank 1 matrices (the number v) is still needed to distinguish case 3.a.i. with $(+-,0)$ from 3.b.ii. with $(+-,1)$, and the two cases from 1.c.

Remark 13.4. The quadratically parametrized maps corresponding to case 1. of Propositions 5.4 and 13.1 have images contained in quadric surfaces. In cases 1.a., 1.b., and 1.c.i., the real parametric image is contained in (but not necessarily equal to) a real ruled quadric surface (cone, cylinder, hyperboloid of one sheet, etc., depending on the affine neighborhood). In case 1.c.ii., the real parametric image is a sphere, as in Example 8.8, or something projectively equivalent to a sphere (ellipsoid, paraboloid, hyperboloid of two sheets).

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[* = added to the list in 2018]

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[CSS*] A. Coffman, A. J. Schwartz, and C. M. Stanton, *The algebra and geometry of Steiner and other quadratically parametrizable surfaces*, Computer Aided Geometric Design (3) **13** (1996), 257–286. MR 1390390 (97g:53012), Zbl 0875.68860. Errata and updates available at `http://users.pfw.edu/CoffmanA/`

[Derive] *Derive*, computer algebra software, Soft Warehouse, 1988.

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[L_1] G. Larson, *The Far Side* desk calendar, 1990.

[L_2] G. Larson, *The Far Side* desk calendar, 1991.

[Maple*] *Maple 2018.1*, Waterloo Maple Inc., 2018.

[SP] R. Myers, *Surface Plotter*, computer graphics software, Brooks/Cole, 1990.

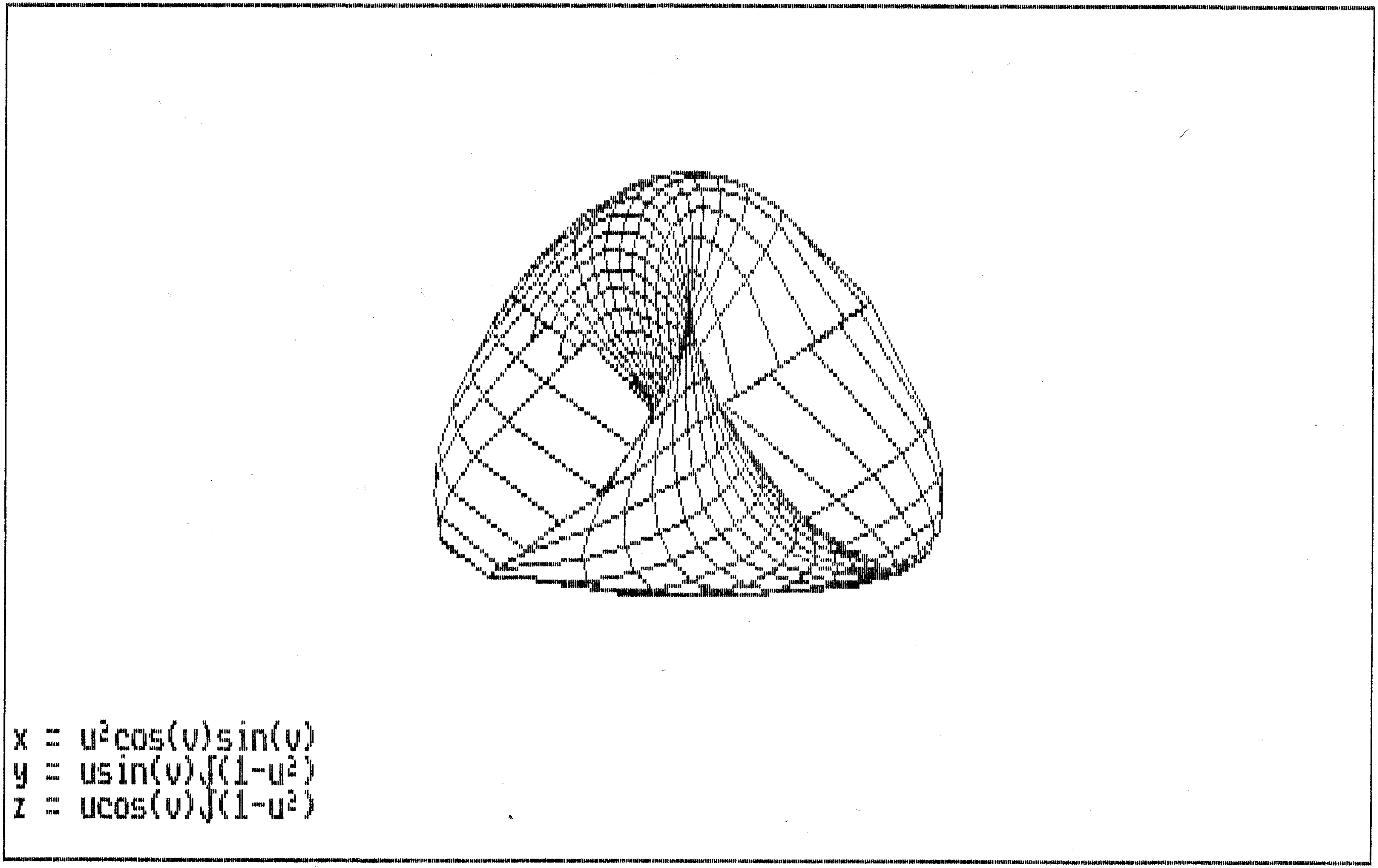


Figure 1

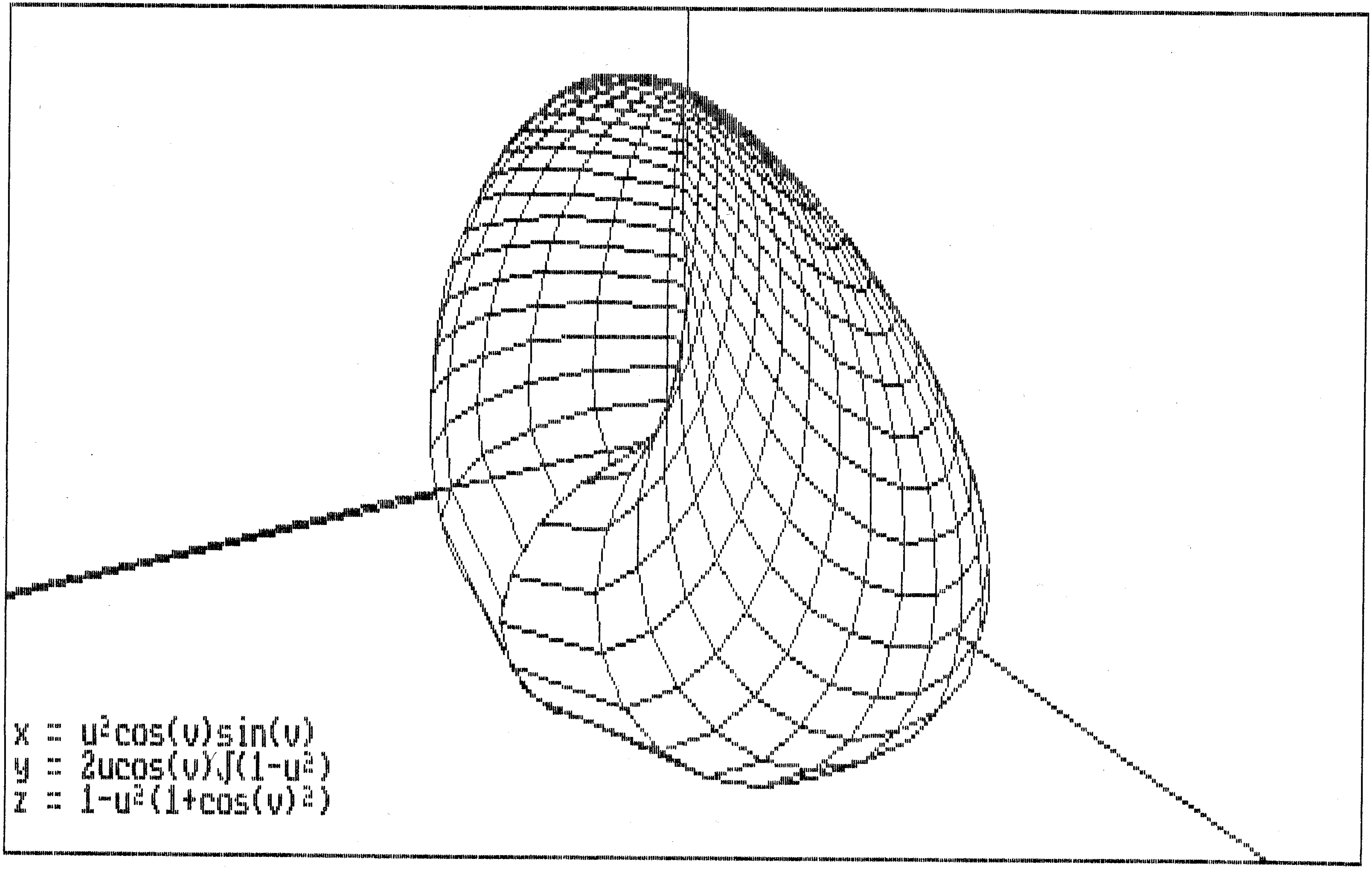


Figure 2

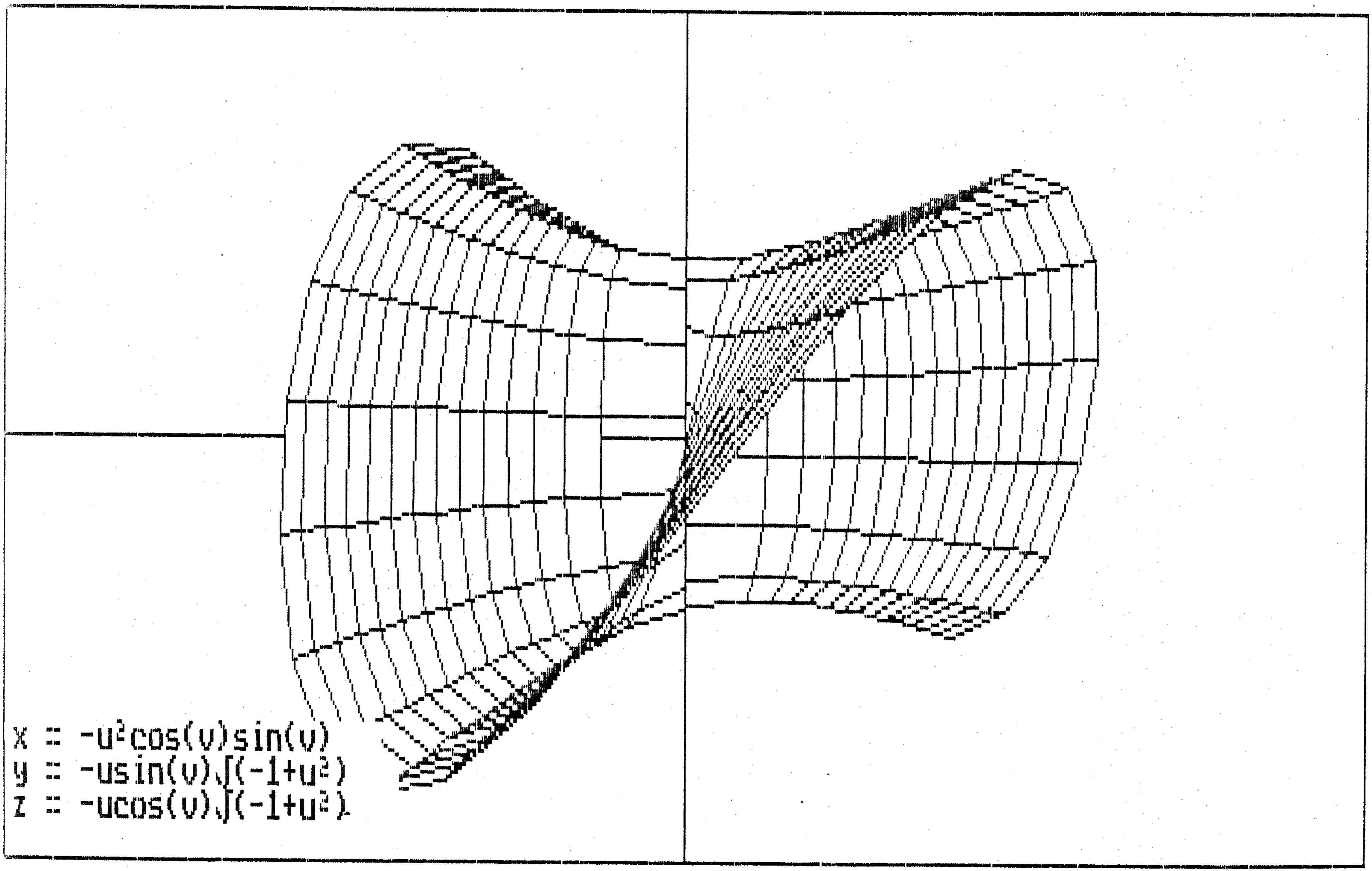
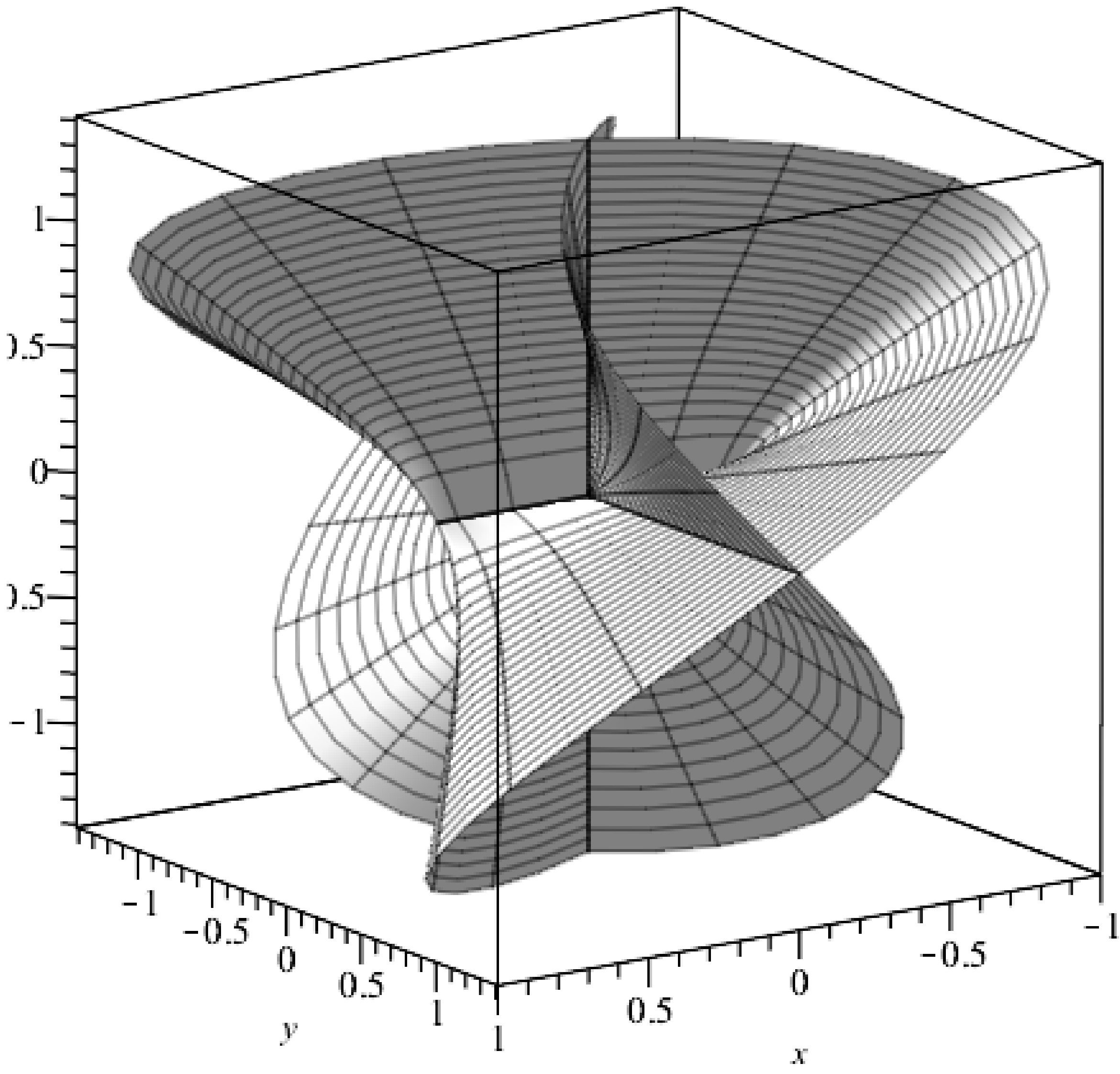


Figure 3

```
> restart; with(plots) :  #The surface from Example 8.3
> Piece1 := plot3d([r^2·cos(t)·sin(t), r·sin(t)·sqrt(1 + r^2), r·cos(t)·sqrt(1 + r^2)], r = 0 ..1, t = 0
      ..2·Pi) :
> Piece2 := plot3d([-r^2·cos(t)·sin(t), -r·sin(t)·sqrt(-1 + r^2), -r·cos(t)·sqrt(-1 + r^2)], r = 1
      ..sqrt(2), t = 0 ..2·Pi) :
> display({Piece1, Piece2});
```




```
>
```

Figure 4

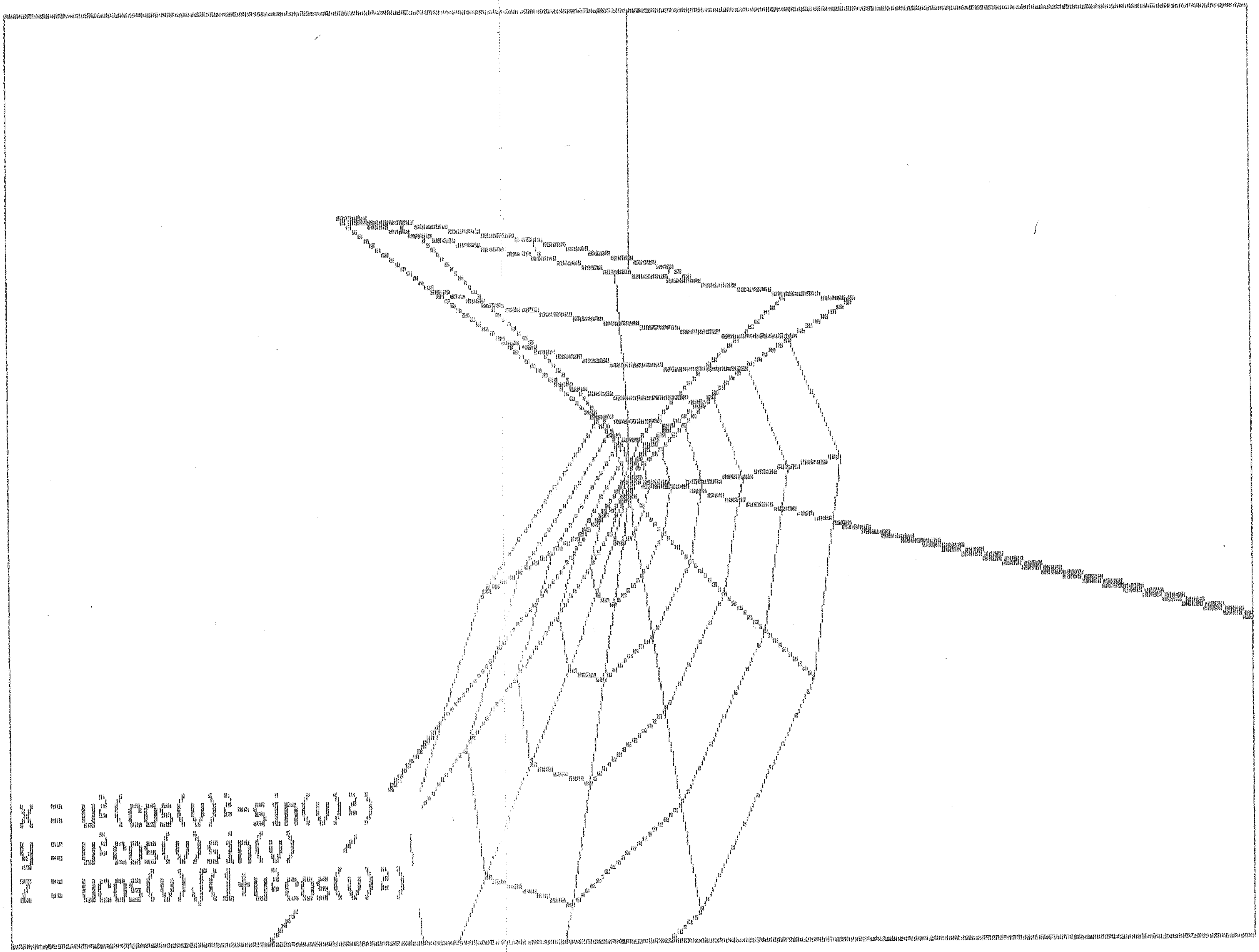


Figure 5

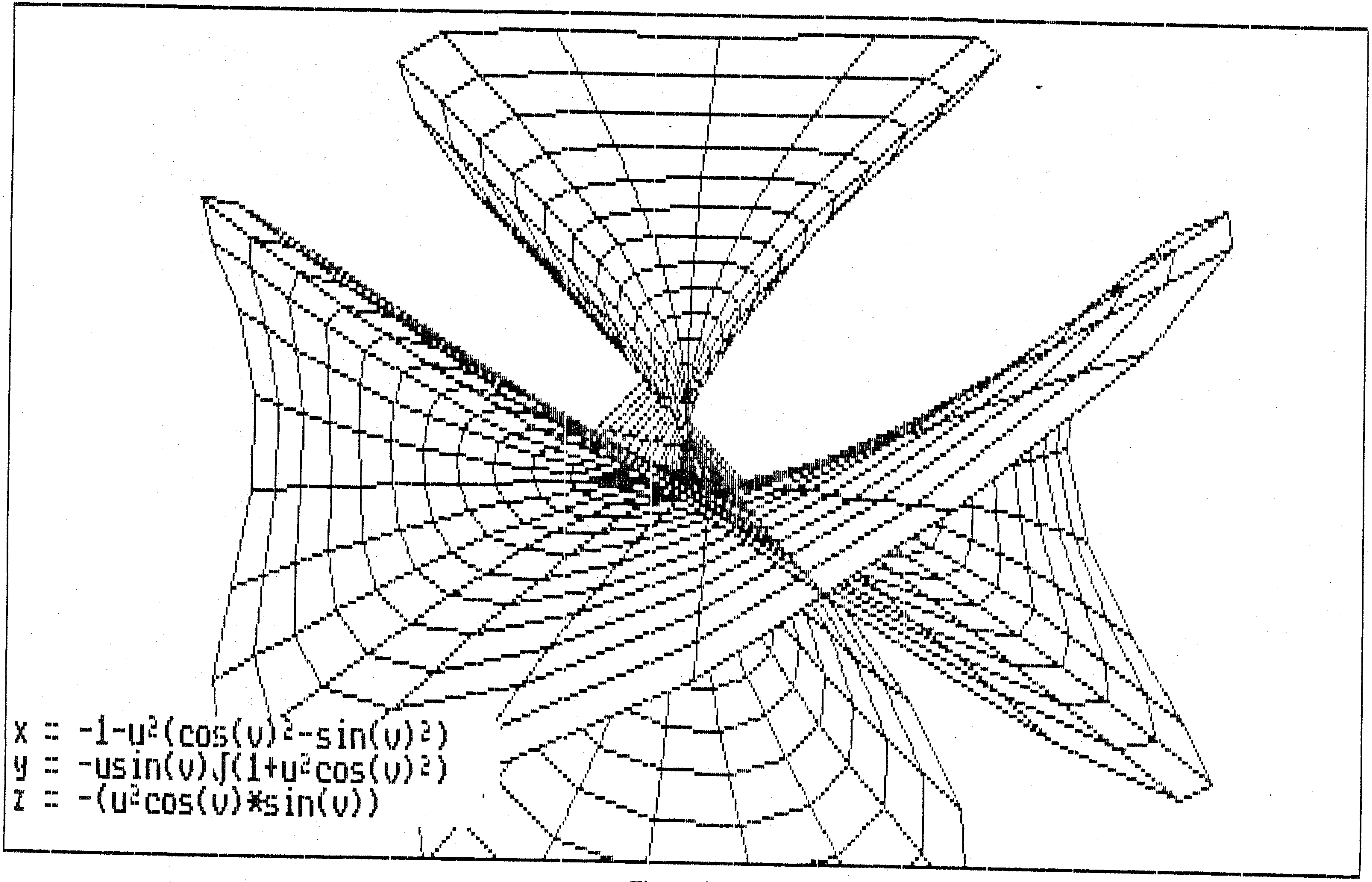
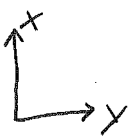


Figure 6

$$\begin{bmatrix} \mu & & \\ & \lambda & \mu \\ & \mu & \lambda \end{bmatrix}$$

$$(z^2-1)^2 - xy\,(z^2-1) - x^2 z = 0$$



$-4 < x, y < 4$

$-2 < z < 0$

Double line at $x=0$, $z=-1$, $y<-2$ or $y>2$

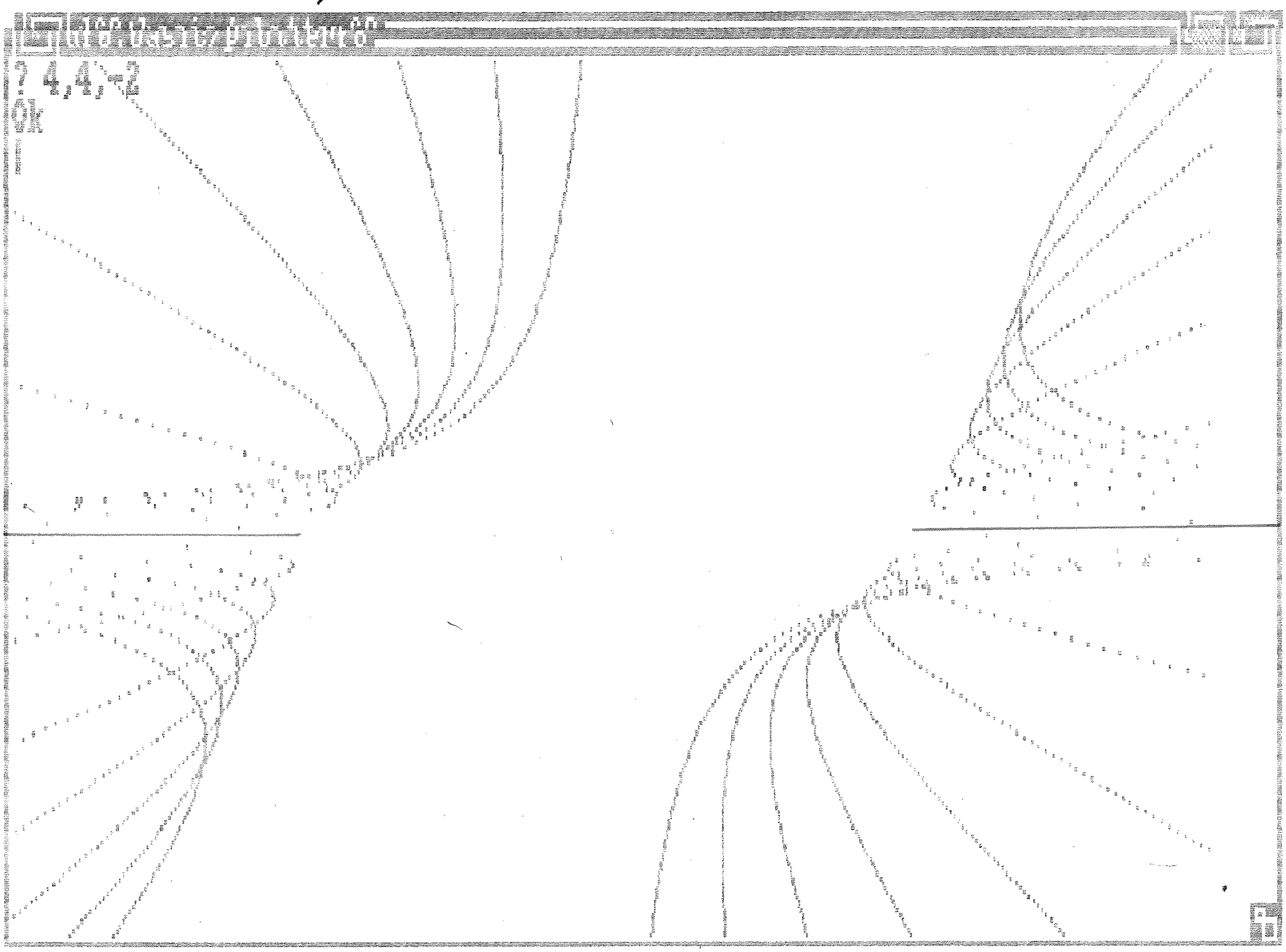


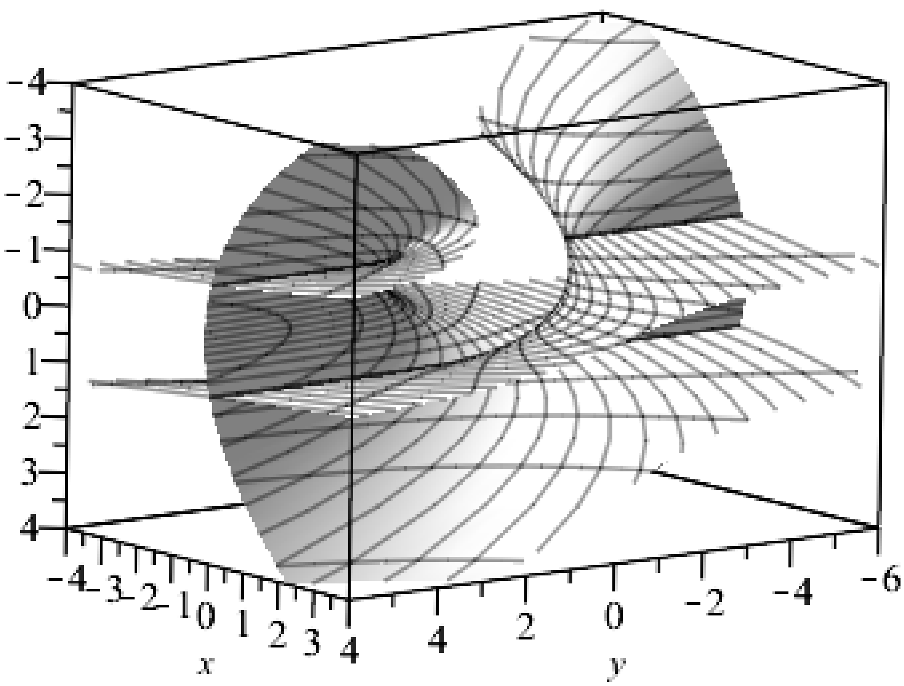
Figure 7

> $restart;\ with(plots):$ #The surface from Example 8.5

> $piece1 := plot3d\left(\left[r^2\cdot\left((\cos(t))^2-(\sin(t))^2\right),\ r^2\cdot\cos(t)\cdot\sin(t)-\frac{1}{r^2\cdot(\cos(t))^2},\ \tan(t)\right], r=0..4,\ t=-1.57..1.57,\ view=[-4..4,-6..6,-4..4]\right):$

> $piece2 := plot3d\left(\left[-r^2\cdot\left((\cos(t))^2-(\sin(t))^2\right), -\left(r^2\cdot\cos(t)\cdot\sin(t)-\frac{1}{r^2\cdot(\cos(t))^2}\right),\ \tan(t)\right], r=0..4,\ t=-1.57..1.57,\ view=[-4..4,-6..6,-4..4]\right):$

> $display([piece1, piece2]);$



>

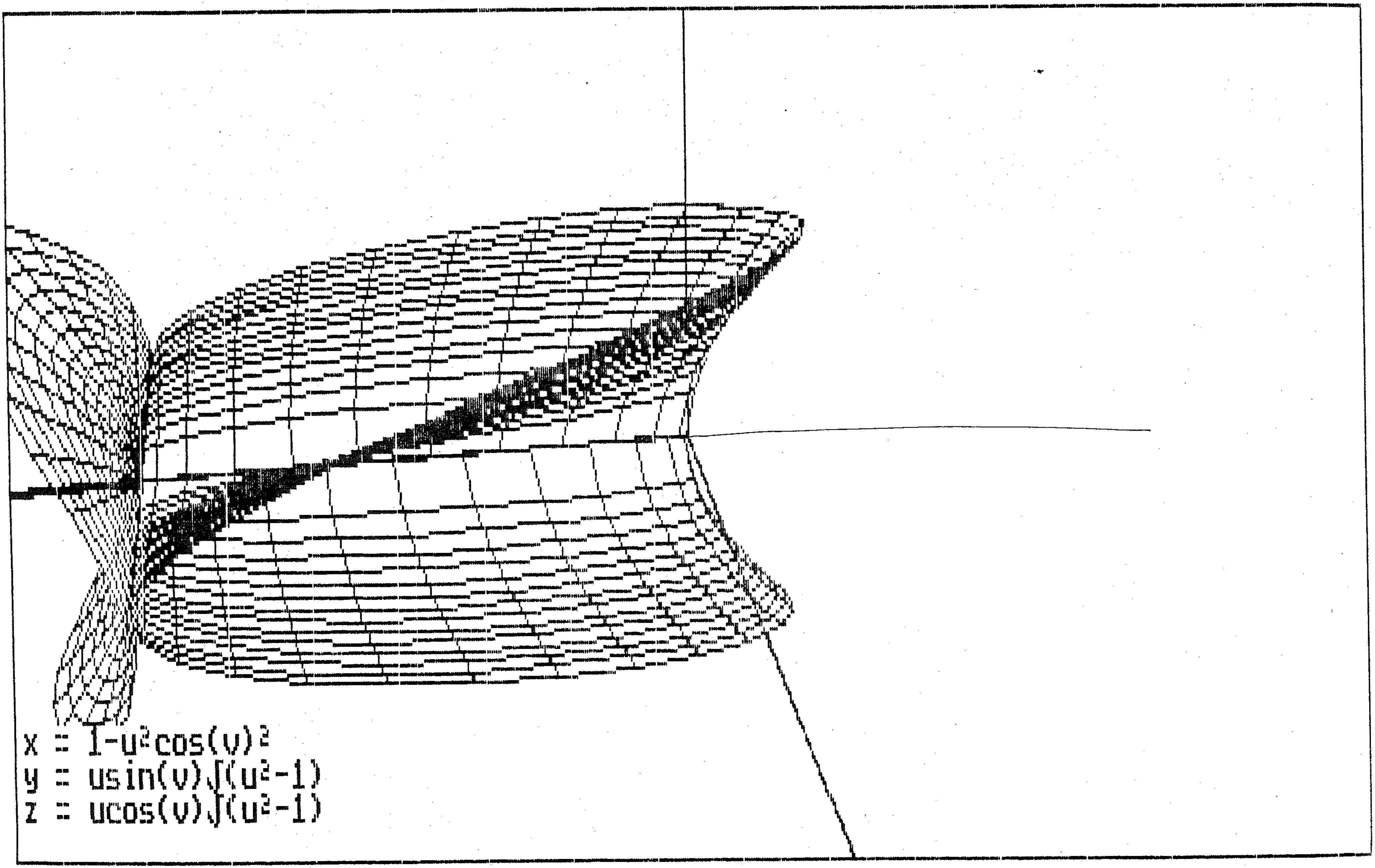
>

>

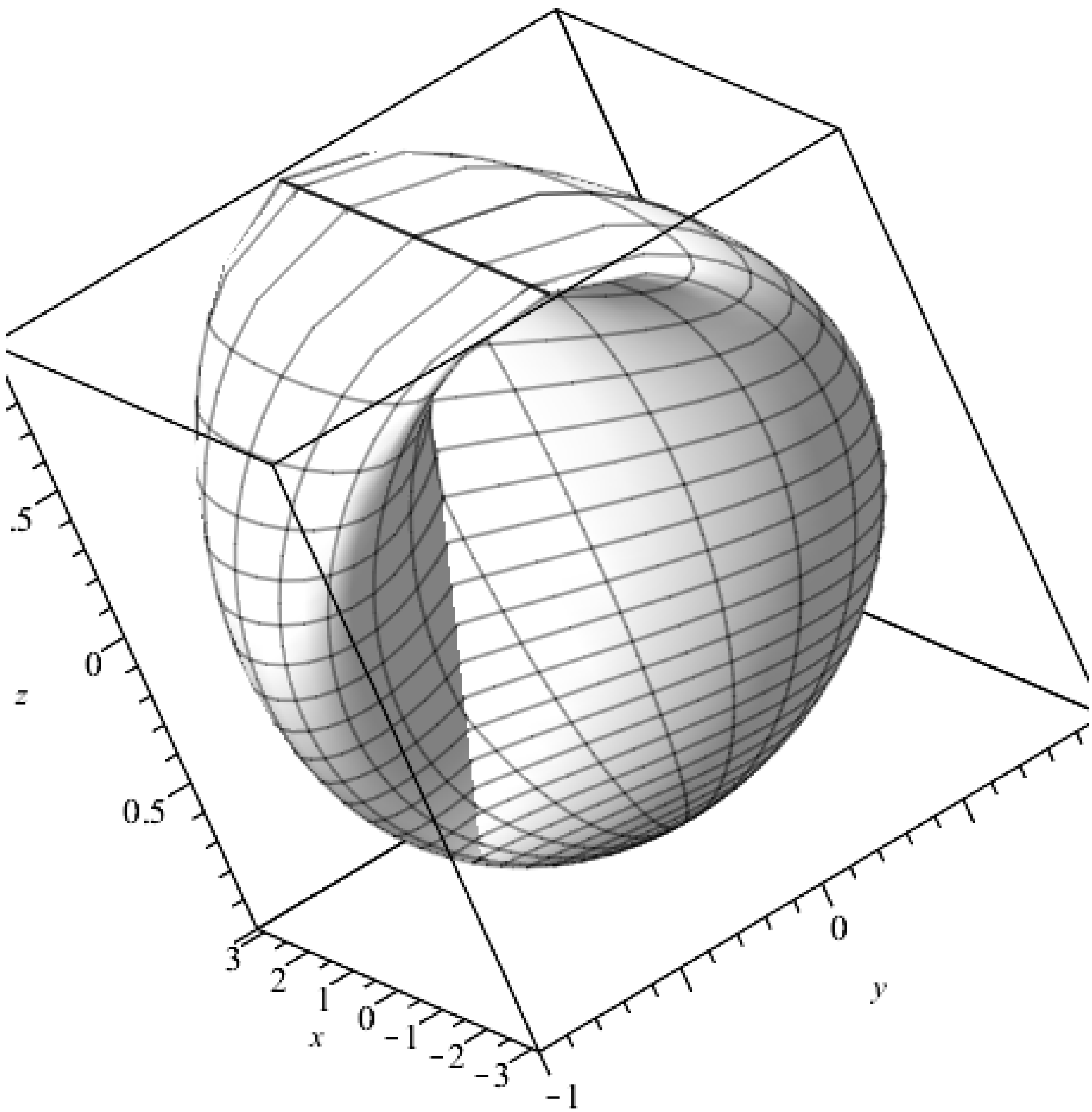
>

Figure 8

Figure 9



```
> restart; with(plots) :  #The surface from Example 8.9, with epsilon=6
> Surface := plot3d([2·r·sin(t)·sqrt(1 − r^2) + 6·r^2·sin(t)·cos(t), 2·r·cos(t)·sqrt(1 − r^2), 1
      − 2·r^2], r = 0 ..1, t = 0 ..2·Pi) :
> display(Surface);
```




```
>
```

Figure 10