Enumeration and Normal Forms for Singularities in Cauchy-Riemann Structures

CHAPTER I:

Introduction

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Starting on the next page is the 3-page **Introduction** (Chapter I), excerpted from my dissertation, "Enumeration and Normal Forms for Singularities in Cauchy-Riemann Structures," 1997, University of Chicago.

For more details about, and excerpts from, my thesis, see my web site.

1 Citations

The dissertation has Math Reviews number MR 2716702, and is cited in this paper: [G], as well as several of my own later publications.

References

[G] T. GARRITY, *Global structures on CR manifolds via Nash blow-ups*, Fulton Birthday Volume, Michigan Math. J. **48** (2000), 281–294, MR 1786491 (2001h:32058), Zbl 0995.32023.

INTRODUCTION

The objects of study are real m-submanifolds M of complex n-manifolds (A, J). The goal is to describe some of the topological and complex-analytic behavior of submanifolds with complex tangents— points where the tangent space and its rotation by the complex structure operator do not meet transversely. Such points x where the intersection $T_x \cap JT_x$ is a j-dimensional complex subspace H^j of the real tangent space T_x form a locus N_j in M. Complex tangents when $m = n$ and $j = 1$ have been studied by many authors, but the local geometry of the $m < n$ case is analyzed for the first time in this thesis.

An immersion of M is defined to be "generic" with respect to the complex structure in terms of a transversality condition on the Gauss map. N_j can be interpreted as a degeneracy locus of a bundle map over M, and a construction of Thom and Porteous is used in a theorem relating the fundamental class of N_j to determinantal formulas in chern and pontrjagin classes. Recent theorems of Fulton and Pragacz are applied to describe chern classes of the bundle $H^j \to N_j$, complex tangents of multifoliations, and complex tangents in the graph of a smooth map between almost complex manifolds. The global theory is considered only in Chapter I, but the notation used there continues in the later chapters, and the "expected codimension" calculation sheds light on the local geometry of isolated and non-isolated complex tangent loci.

The local geometry near a complex tangent is considered in the $m < n$ case where M is real-analytic, generically embedded in a neighborhood of the origin $0 \in \mathbb{C}^n = A$, and totally real except for a submanifold N_1 of points where the tangent plane contains a complex line H^1. Assuming that $0 \in N_1$ and that T_0M has coordinates $z_1, x_2, \ldots, x_{m-1}$, M is locally a graph of $2n - m$ real functions of these variables. Holomorphic coordinate changes in $\mathbb{C}^n$ put the quadratic parts of these functions into a normal form. The simplest case is when M is a 4-manifold in $\mathbb{C}^5$, so that $N_1 = \{0\}$ and the "non-degenerate" quadratic normal form is

$$\begin{aligned} y_2 &= 0 + O(3) \\ y_3 &= 0 + O(3) \\ z_4 &= (\bar{z}_1 + x_2 + ix_3)^2 + O(3) \\ z_5 &= z_1(\bar{z}_1 + x_2 + ix_3) + O(3). \end{aligned}$$

The $O(3)$ abbreviates real-analytic functions vanishing to third order, with complex coefficients on monomials of the form $z_1^a \bar{z}_1^b x_2^c x_3^d$. The non-degenerate quadratic normal form for general m is similar. The main theorem is that in the non-degenerate case, there exists a polynomial coordinate change eliminating the higher-order terms up to arbitrary degree, improving $O(3)$ to $O(N)$, and so there is a *formal* coordinate change transforming M into the real variety Q defined by the quadratic terms only. The calculations leading to the formulation of this "stability theorem" and its proof were facilitated by *Maple V* software.

Both the global and the local holomorphic geometry resemble classical theorems of Whitney. For $m < \frac{2}{3}(n+1)$, M is generically totally real, in analogy with the embedding theorem for smooth manifolds. For $\frac{2}{3}(n+1) \leq m < n$, complex tangents are expected to occur, having a non-degenerate normal form as above, which resembles Whitney's cross-cap singularity. The 4-manifold Q is contained in the singular complex hypersurface $z_1^2 z_4 - z_5^2 = 0$ in $\mathbb{C}^5$, a variety resembling Whitney's umbrella surface. Whitney also showed there is a smooth coordinate system in a neighborhood of a non-degenerate singularity so that the cross-cap has the polynomial normal form. Constructing holomorphic coordinates normalizing the complex tangent is complicated by the appearance of antiholomorphic factors $\bar{z}_1$.

Finding a holomorphic coordinate change bringing arbitrary real-analytic functions on a polydisc in T_0M to the quadratic normal form in a neighborhood of 0 in $\mathbb{C}^n$ is equivalent to solving a nonlinear functional equation. One of the difficulties here is that formal power series solutions are not unique. A phenomenon contributing to this is that M is not a uniqueness set for holomorphic functions of n variables; for example, $z_1^2 z_4 - z_5^2 = 0$ on Q. A normalized solution p of a related linear equation is constructed, which is only shown to converge on a polydisc a fraction of the size of the starting disc. Considering p as an approximate solution of the nonlinear equation and iterating the construction provides a formal transformation, but the shrinking of the radius means that the question of the existence of a convergent coordinate change remains open.

Chapter III is a real algebraic interlude. With the help of the computer algebra program *Macaulay*, the geometry of some real varieties with complex tangents is considered. The defining polynomials are the normal forms of the previous chapter, but the methods apply to find and analyze complex tangents of any real variety.