# Isolated CR singularities of real threefolds in $\mathbb{C}^3$

Adam Coffman

Indiana University - Purdue University Fort Wayne

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# Real submanifolds of $\mathbb{C}^3$

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What happens to an isolated point under perturbations of $M$?

# Related question in real analytic geometry

## Exercise (easy)

Can a real analytic space curve have an acnode? More specifically, is there a real analytic variety in $\mathbb{R}^3$ ($xyz$ space), with two defining functions and which contains (or is) an isolated point?

$$\begin{aligned} P_1(x,y,z) &= 0 \\ P_2(x,y,z) &= 0 \end{aligned}$$

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## Exercise (not as easy)

Is there a real analytic variety $\{P_1 = P_2 = 0\}$ in $\mathbb{R}^3$ with an isolated point $\vec{a}$ that doesn't disappear under some small perturbations of the defining equations? For all $t_1$, $t_2$ close to 0, the solution set is non-empty near $\vec{a}$:

$$\begin{aligned} P_1(x,y,z) &= t_1 \\ P_2(x,y,z) &= t_2 \end{aligned}$$

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After a translation $\vec{z}_0 \mapsto \vec{0}$ and complex linear transformation, $M$ is in standard position, where the tangent 3-plane is the subspace $\{(z_1, x_2 + i0, 0)\}$, and near the origin, $M$ is described as a graph of real analytic functions:

$$\begin{aligned} y_2 &= H_2(z_1, \bar{z}_1, x_2), \\ &= \alpha_2 z_1^2 + \beta_2 z_1 \bar{z}_1 + \gamma_2 \bar{z}_1^2 + \delta_2 z_1 x_2 + \epsilon_2 \bar{z}_1 x_2 + \theta_2 x_2^2 + O(3) \\ z_3 &= h_3(z_1, \bar{z}_1, x_2) \\ &= \alpha_3 z_1^2 + \beta_3 z_1 \bar{z}_1 + \gamma_3 \bar{z}_1^2 + \delta_3 z_1 x_2 + \epsilon_3 \bar{z}_1 x_2 + \theta_3 x_2^2 + O(3). \end{aligned}$$

$H_2, h_3$ are real analytic (defined by convergent power series with complex coefficients, centered at the origin) functions of $x_1$, $y_1$, $x_2$, or equivalently, $z_1$, $\bar{z}_1 = x_1 - iy_1$, $x_2$.

# Holomorphic coordinate changes

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$$\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)^T = \mathbf{A}_{3\times 3}\vec{z} + \vec{p}(\vec{z}),$$

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where $\vec{p}(\vec{z}) = (p_1, p_2, p_3)^T$ is a column vector of 3 functions of 3 variables, each of which is holomorphic in a neighborhood of $\vec{0}$ and has no constant or linear terms, and where $\mathbf{A}$, the invertible linear part of the transformation, has matrix representation of the form

$$\mathbf{A}_{3\times 3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & r_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}.$$

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The defining equations in the new $\tilde{z}$ coordinate system will still be in standard position but the goal is to find normal forms that expose the geometry of the equivalence classes.

# Other dimensions

Generalizing from 3 to $n$ — For $n$-dimensional $M$ in $\mathbb{C}^n$, we will still only consider points where the tangent space contains exactly one line (not a higher-dimensional complex subspace).

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Local defining equations:

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$x = (x_2, \ldots, x_{n-1}), \qquad \sigma = 2, \ldots, n-1.$

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Coordinate changes: $\tilde{z} = \mathbf{A}_{n\times n}\vec{z} + \vec{p}(\vec{z})$

# Quadratic normal forms

## Proposition

*Given $M$ with a CR singularity in standard position, there exists a holomorphic change of coordinates where:*

$$\begin{aligned} y_\sigma &= H_\sigma(z_1, \bar{z}_1, x) = b_\sigma z_1 \bar{z}_1 + O(3), \\ z_n &= h_n(z_1, \bar{z}_1, x) = a(z_1^2 + \bar{z}_1^2) + b z_1 \bar{z}_1 + i c_\beta x_\beta (z_1 - \bar{z}_1) + O(3). \end{aligned}$$

$a \geq 0$, $b \in \{0, 1\}$, $b_\sigma \in \{0, 1\}$. ∎

## Remark

Summation convention for $\beta = 2, \ldots, n-1$.
Also dropping the tilde notation, $\tilde{z}$, after changing coordinates.

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## Remark

If $b \neq 0$, then there is a further change of coordinates, with:

$$\begin{aligned} y_\sigma &= H_\sigma = O(3), \\ z_n &= h_n = z_1 \bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + i c_\beta x_\beta (z_1 - \bar{z}_1) + O(3). \end{aligned}$$

where $\gamma \geq 0$ is the well-known Bishop invariant, and $c_\beta \in \mathbb{R}$.

# Graphics by POV-Ray

Surfaces in $\mathbb{C}^2$:

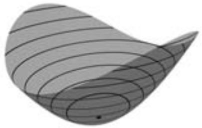

Figure : Elliptic point $0 \leq \gamma < \frac{1}{2}$

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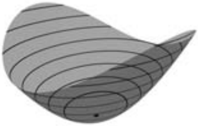

Figure : Elliptic point $0 \leq \gamma < \frac{1}{2}$

Figure : Hyperbolic point $\frac{1}{2} < \gamma \leq \infty$

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The borderline case: Parabolic points with $\gamma = \frac{1}{2}$

Figure : $z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) = \frac{1}{2}(z_1 + \bar{z}_1)^2$

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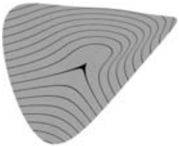

Figure : with some cubic terms: $z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(\bar{z}_1 - z_1)z_1\bar{z}_1$

Parabolic points with quartic terms:

Figure : $z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) - (z_1^2 + \bar{z}_1^2)z_1\bar{z}_1$

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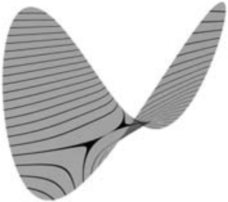

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The $\pm$ sign is a holomorphic invariant.

# Back to $n$-manifolds in $\mathbb{C}^n$

For $n > 2$, and with the goal of finding examples of $M$ with isolated complex tangents, consider parabolic points, and the cubic (& higher) terms of the defining equations:

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For $n > 2$, and with the goal of finding examples of $M$ with isolated complex tangents, consider parabolic points, and the cubic (& higher) terms of the defining equations:

$$\begin{aligned} y_\sigma &= H_\sigma \\ &= B^\sigma_{300} z_1^3 + B^\sigma_{210} z_1^2 \bar{z}_1 + B^\sigma_{120} z_1 \bar{z}_1^2 + B^\sigma_{030} \bar{z}_1^3 \\ &\quad + B^\sigma_{20\alpha} z_1^2 x_\alpha + B^\sigma_{11\alpha} z_1 \bar{z}_1 x_\alpha + B^\sigma_{02\alpha} \bar{z}_1^2 x_\alpha \\ &\quad + B^\sigma_{10\alpha\beta} z_1 x_\alpha x_\beta + B^\sigma_{01\alpha\beta} \bar{z}_1 x_\alpha x_\beta + B^\sigma_{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma + O(4). \\ z_n &= h_n = \frac{1}{2}(z_1 + \bar{z}_1)^2 + i c_\beta x_\beta (z_1 - \bar{z}_1) \\ &\quad + b_{300} z_1^3 + b_{210} z_1^2 \bar{z}_1 + b_{120} z_1 \bar{z}_1^2 + b_{030} \bar{z}_1^3 \\ &\quad + b_{20\alpha} z_1^2 x_\alpha + b_{11\alpha} z_1 \bar{z}_1 x_\alpha + b_{02\alpha} \bar{z}_1^2 x_\alpha \\ &\quad + b_{10\alpha\beta} z_1 x_\alpha x_\beta + b_{01\alpha\beta} \bar{z}_1 x_\alpha x_\beta + b_{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma + O(4). \end{aligned}$$

# Cubic normal form for $n$-manifolds in $\mathbb{C}^n$

## Proposition (Webster 1985)

*Let $n \geq 3$. Given $M$ with a parabolic CR singularity, there exists a coordinate system in which the defining equations are of the form:*

$$\begin{aligned} y_\sigma &= O(4), \quad \sigma = 2, \ldots, n-1 \\ z_n &= z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + ic_2x_2(z_1 - \bar{z}_1) \\ &\quad +(-i\eta(z_1 + \bar{z}_1) + \eta_\beta x_\beta)z_1\bar{z}_1 \\ &\quad +iK_{\alpha\beta 1}\bar{z}_1 x_\alpha x_\beta + O(4). \end{aligned}$$

*The quadratic coefficient $c_2$ is either $0$ or $1$, and is a biholomorphic invariant of $M$. Similarly, the cubic coefficient $\eta$ is either $0$ or $1$, and is also a biholomorphic invariant. The coefficients $\eta_\beta$, $K_{\alpha\beta 1}$ are real.*

# Sketch of Proof of Webster's Classification

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\begin{aligned}
y_\sigma &= H_\sigma \\
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z_n &= h_n = \frac{1}{2}(z_1 + \bar{z}_1)^2 + i c_\beta x_\beta (z_1 - \bar{z}_1) \\
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z_n &= h_n = \frac{1}{2}(z_1+\bar{z}_1)^2 + ic_2x_2(z_1-\bar{z}_1) \\
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\end{aligned}
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# Sketch of Proof of Webster's Classification

$$
\begin{aligned}
y_\sigma &= H_\sigma \\
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&\quad +0z_1x_\alpha x_\beta + iK_{\alpha\beta1}\bar{z}_1x_\alpha x_\beta + 0x_\alpha x_\beta x_\gamma + O(4).
\end{aligned}
$$

■

# Cubic normal form for $n$-manifolds in $\mathbb{C}^n$

## Proposition (Webster 1985)

*Let $n \geq 3$. Given $M$ with a parabolic CR singularity, there exists a coordinate system in which the defining equations are of the form:*

$$\begin{aligned} y_\sigma &= O(4), \\ z_n &= z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + ic_2x_2(z_1 - \bar{z}_1) \\ &\quad +(-i\eta(z_1 + \bar{z}_1) + \eta_\beta x_\beta)z_1\bar{z}_1 \\ &\quad +iK_{\alpha\beta 1}\bar{z}_1 x_\alpha x_\beta + O(4). \end{aligned}$$

$c_2 \in \{0, 1\}$, $\eta \in \{0, 1\}$, $\eta_\beta \in \mathbb{R}$, $K_{\alpha\beta 1} \in \mathbb{R}$.

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$c_2 \in \{0,1\}$, $\eta \in \{0,1\}$, $\eta_\beta \in \mathbb{R}$, $K_{\alpha\beta 1} \in \mathbb{R}$.

## Remark

The normalization can be continued, depending on $c_2$, $\eta$, in the following classification into 6 types of normal forms.

# Cubic normal form for parabolic points in $\mathbb{C}^n$

| | Case | normal form for $h_n$ | comment |
|---|---|---|---|
| P1 | $c_2 = 1$<br>$\eta = 1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)x_2$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + O(4)$ | $\eta_\beta \equiv 0$<br>$K_{\alpha\beta 1} \equiv 0$ |
| P2 | $c_2 = 0$<br>$\eta = 1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + iK_\alpha(z_1 - \bar{z}_1)x_\alpha^2 + O(4)$ | $\eta_\beta \equiv 0$<br>$K_\alpha \in \{\pm 1, 0\}$<br>$\alpha = 2, \ldots, n-1$ |
| P3 | $c_2 = 1$<br>$\eta = 0$<br>$\eta_\beta \not\equiv 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)x_2$<br>$+z_1\bar{z}_1x_3 + O(4)$ | $n \geq 4$<br>$K_{\alpha\beta 1} \equiv 0$ |
| P4 | $c_2 = 1$<br>$\eta = 0$<br>$\eta_\beta \equiv 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)x_2 + O(4)$ | $K_{\alpha\beta 1} \equiv 0$ |
| P5 | $c_2 = 0$<br>$\eta = 0$<br>$\eta_\beta \not\equiv 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$+z_1\bar{z}_1x_2 + iK_\alpha(z_1 - \bar{z}_1)x_\alpha^2 + O(4)$ | $K_\alpha \in \{\pm 1, 0\}$<br>$\alpha = 3, \ldots, n-1$ |
| P6 | $c_2 = 0$<br>$\eta = 0$<br>$\eta_\beta \equiv 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$+iK_\alpha(z_1 - \bar{z}_1)x_\alpha^2 + O(4)$ | $K_\alpha \in \{\pm 1, 0\}$<br>$\alpha = 2, \ldots, n-1$ |

# Cubic normal form for parabolic points in $\mathbb{C}^3$ — first 4 of 8

For $n = 3$, get a complete list of 8 inequivalent cubic normal forms:

| | Case | normal form for $h_3$ |
|---|---|---|
| P1 | $c_2 = 1$<br>$\eta = 1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)x_2$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + O(4)$ |
| P2a | $c_2 = 0$<br>$\eta = 1$<br>$K_2 = +1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + i(z_1 - \bar{z}_1)x_2^2 + O(4)$ |
| P2b | $c_2 = 0$<br>$\eta = 1$<br>$K_2 = -1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 - i(z_1 - \bar{z}_1)x_2^2 + O(4)$ |
| P2c | $c_2 = 0$<br>$\eta = 1$<br>$K_2 = 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + O(4)$ |

**Remark**

In the P2 case, the signature $K_2$ is a holomorphic invariant.

# Cubic normal form for parabolic points in $\mathbb{C}^3$ — last 4 of 8

| | | |
|---|---|---|
| P4 | $c_2 = 1$<br>$\eta = 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)x_2 + O(4)$ |
| P5 | $c_2 = 0$<br>$\eta = 0$<br>$\eta_2 = 1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$+z_1\bar{z}_1x_2 + O(4)$ |
| P6a | $c_2 = 0$<br>$\eta = 0$<br>$\eta_2 = 0$<br>$K_2 = 1$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$<br>$+i(z_1 - \bar{z}_1)x_2^2 + O(4)$ |
| P6b | $c_2 = 0$<br>$\eta = 0$<br>$\eta_2 = 0$<br>$K_2 = 0$ | $z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + O(4)$ |

## Remark

In the P6 case, the $K_2 = \pm 1$ signatures are equivalent. $(\tilde{z}_1 = -z_1)$

# Where are the CR singular points?

## Lemma

*For a real analytic n-manifold M in* $\mathbb{C}^n$ *of the form*

$$\begin{aligned} y_\sigma &\equiv 0, \\ z_n &= h_n(z_1, \bar{z}_1, x), \end{aligned}$$

*a point* $(z_1, x, h_n(z_1, \bar{z}_1, x)) \in M$ *is a CR singular point* $\iff$ $(z_1, \bar{z}_1, x)$ *satisfies*

$$\frac{\partial h_n}{\partial \bar{z}_1}(z_1, \bar{z}_1, x) = 0.$$

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$\blacksquare$

So the CR singular locus $N \subseteq M$ is usually a subset of codimension 2.

# Finally, an example!

Recall the P2a normal form; consider the real algebraic variety $M^3 \subseteq \mathbb{C}^3$:

$$\begin{aligned} y_2 &\equiv 0, \\ z_3 &= z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) - i(z_1 + \bar{z}_1)z_1\bar{z}_1 + i(z_1 - \bar{z}_1)x_2^2. \end{aligned}$$

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$$

Then

$$
\begin{aligned}
\frac{\partial h_n}{\partial \bar{z}_1}(z_1, \bar{z}_1, x) &= z_1 + \bar{z}_1 - iz_1^2 - 2iz_1\bar{z}_1 - ix_2^2 \\
&= 2x_1 + 2x_1y_1 \\
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So $N$ is an isolated point.

# A real perturbation

For a small real parameter $t$ and complex constant $\mu$, consider the family of real algebraic varieties in $\mathbb{C}^3$, so $M$ is the variety at $t = 0$:

$$\begin{aligned} y_2 &\equiv 0, \\ z_3 &= z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) - i(z_1 + \bar{z}_1)z_1\bar{z}_1 + i(z_1 - \bar{z}_1)x_2^2 + \mu t(z_1 + \bar{z}_1). \end{aligned}$$

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\end{aligned}
$$

Then (for each fixed $t$):

$$
\begin{aligned}
\frac{\partial h_n}{\partial \bar{z}_1}(z_1, \bar{z}_1, x) &= z_1 + \bar{z}_1 - iz_1^2 - 2iz_1\bar{z}_1 - ix_2^2 + \mu t \\
&= 2x_1 + 2x_1y_1 + \mathrm{Re}(\mu)t \\
&\quad -i(3x_1^2 + y_1^2 + x_2^2 - \mathrm{Im}(\mu)t)
\end{aligned}
$$

so $N$ is empty for some $\mu$ and $t$.

# Solution for Exercise

## Exercise

Is there a real analytic variety $\{P_1 = P_2 = 0\}$ in $\mathbb{R}^3$ with an isolated point $\vec{a}$ that doesn't disappear under some small perturbations of the defining equations? For all $t_1$, $t_2$ close to 0, the solution set is non-empty near $\vec{a}$:

$$\begin{aligned} P_1(x, y, z) &= t_1 \\ P_2(x, y, z) &= t_2 \end{aligned}$$

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Answer: Yes

$$\begin{aligned} P_1(x,y,z) &= 8x^2 + 8y^2 - z^2 \\ P_2(x,y,z) &= z(x^2 + y^2) - x^3 \end{aligned}$$

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Cone

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Answer: Yes

| | | |
|---|---|---|
| $P_1(x, y, z) = 8x^2 + 8y^2 - z^2$ | | Cone |
| $P_2(x, y, z) = z(x^2 + y^2) - x^3$ | | Cartan Umbrella |

# Graphics by Maple

Figure : left: $z(x^2 + y^2) - x^3 = 0$. middle: $z = \frac{x^3 - 0.01}{x^2 + y^2}$. right: $z = \frac{x^3 + 0.01}{x^2 + y^2}$.

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