CR SINGULARITIES OF REAL THREEFOLDS IN $\mathbb{C}^4$

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Abstract. CR singularities of real threefolds in $\mathbb{C}^4$ are classified by using holomorphic coordinate changes to transform the quadratic part of the real defining equations into one of a list of normal forms. In the non-degenerate case, it is shown that a real analytic manifold near a CR singular point is formally equivalent to a real algebraic model. Some degenerate cases also have this property.

1. Introduction

If a real 3-manifold M is embedded in $\mathbb{C}^4$, then for each point x on M there are two possibilities: the tangent 3-plane at x may contain a complex line, so M is said to be "CR singular" at x, or it may not, so M is said to be "totally real" at x. This article will consider the local extrinsic geometry of a real analytically embedded M near a CR singular point, by finding invariants under biholomorphic coordinate changes. The first main result is a classification of quadratic normal forms for the defining equations. The next result concerns the higher-order terms of the normal forms in the non-degenerate case. It will be shown that in this case, M is formally equivalent to a fixed real algebraic variety, in the sense that there are holomorphic coordinate systems in which the defining equations for M agree with the polynomial normal form to an arbitrarily high degree. The final result is a similar formal stability property for some of the degenerate CR singularities.

The analysis of normal forms near CR singular points is part of the program studying the local equivalence problem for real m-submanifolds of $\mathbb{C}^n$, as described in the survey paper, [BER]. Normal forms for CR singular real n-manifolds in $\mathbb{C}^n$, $n \geq 2$, have been considered by [Bishop], [H_2], [M], [W], and others, and for real surfaces in $\mathbb{C}^n$ ($m = 2$, $n \geq 3$) by [H_1] and [C_2]. A formal normal form for a CR singular real 4-manifold in $\mathbb{C}^5$ was found by [Beloshapka] and [C_1]. The case $m = 3$, $n = 4$ to be considered here was motivated in part by [C_3], which considers a certain family of maps $\mathbb{R}P^2 \times \mathbb{R}P^1 \to \mathbb{C}P^4$. Most of those maps are totally real embeddings, but some of the exceptional cases exhibit complex tangents.

2. A quadratic classification of CR singularities

Although we could consider real threefolds in any complex 4-manifold, we will only be considering a small neighborhood, so it will be convenient to regard the ambient complex space to be $\mathbb{C}^4$, with coordinates (z_1, z_2, z_3, z_4). The real and imaginary parts of the coordinate functions are labeled $z_j = x_j + iy_j$.

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2.1. **A general quadratic normal form.**

We begin by assuming M is a real analytic three-dimensional submanifold in $\mathbb{C}^4$ with a complex tangent at some point. By a translation that moves that point to the origin $\vec{0}$, and then a complex linear transformation of $\mathbb{C}^4$, the tangent 3-plane $T = T_{\vec{0}}M$ can be assumed to be the (x_1, y_1, x_2)-space, which contains the z_1-axis. Then there is some neighborhood Δ of the origin in $\mathbb{C}^4$ so that the defining equations of M in Δ are in the form of a graph over a neighborhood of the origin in T:

$$\begin{aligned} y_2 &= H_2(z_1, \bar{z}_1, x_2) \\ z_3 &= h_3(z_1, \bar{z}_1, x_2), \\ z_4 &= h_4(z_1, \bar{z}_1, x_2), \end{aligned} \tag{2.1}$$

where H_2, h_3 and h_4 are real analytic functions defined in a neighborhood of the origin in T, and vanishing to second order at $(x_1, y_1, x_2) = (0, 0, 0)$, with H_2 real-valued and h_3, h_4 complex-valued. So, they are of the following form:

$$\begin{aligned} H_2(z_1, \bar{z}_1, x_2) &= \alpha_2 z_1^2 + \beta_2 z_1\bar{z}_1 + \gamma_2 \bar{z}_1^2 + \delta_2 z_1 x_2 + \epsilon_2 \bar{z}_1 x_2 + \theta_2 x_2^2 + E_2(z_1, \bar{z}_1, x_2), \\ h_3(z_1, \bar{z}_1, x_2) &= \alpha_3 z_1^2 + \beta_3 z_1\bar{z}_1 + \gamma_3 \bar{z}_1^2 + \delta_3 z_1 x_2 + \epsilon_3 \bar{z}_1 x_2 + \theta_3 x_2^2 + e_3(z_1, \bar{z}_1, x_2), \\ h_4(z_1, \bar{z}_1, x_2) &= \alpha_4 z_1^2 + \beta_4 z_1\bar{z}_1 + \gamma_4 \bar{z}_1^2 + \delta_4 z_1 x_2 + \epsilon_4 \bar{z}_1 x_2 + \theta_4 x_2^2 + e_4(z_1, \bar{z}_1, x_2), \end{aligned}$$

with E_2, e_3, e_4 having terms of degree three or higher. These functions can be expressed as the restriction to $\{(z, \zeta, x) \in \mathbb{C}^3 : \zeta = \bar{z}, x = \bar{x}\}$ of the three-variable series:

$$\begin{aligned} H_2(z, \zeta, x) &= \alpha_2 z^2 + \beta_2 z\zeta + \gamma_2 \zeta^2 + \delta_2 zx + \epsilon_2 \zeta x \\ &\quad + \theta_2 x^2 + \sum_{a+b+c \geq 3} E_2^{a,b,c} z^a \zeta^b x^c, \\ h_3(z, \zeta, x) &= \alpha_3 z^2 + \beta_3 z\zeta + \gamma_3 \zeta^2 + \delta_3 zx + \epsilon_3 \zeta x \\ &\quad + \theta_3 x^2 + \sum_{a+b+c \geq 3} e_3^{a,b,c} z^a \zeta^b x^c, \\ h_4(z, \zeta, x) &= \alpha_4 z^2 + \beta_4 z\zeta + \gamma_4 \zeta^2 + \delta_4 zx + \epsilon_4 \zeta x \\ &\quad + \theta_4 x^2 + \sum_{a+b+c \geq 3} e_4^{a,b,c} z^a \zeta^b x^c, \end{aligned} \tag{2.2}$$

each of which converges on the set $\{(z, \zeta, x) : |z| < R_1, |\zeta| < R_1, |x| < R_2\}$ to a complex analytic function, but with $\gamma_2 = \overline{\alpha_2}$, $\epsilon_2 = \overline{\delta_2}$, $\beta_2 = \overline{\beta_2}$, $\theta_2 = \overline{\theta_2}$, and with similar restrictions on the coefficients of E_2, so that $E_2(z_1, \bar{z}_1, x_2)$ is real-valued.

Definition 2.1. A (formal, with complex coefficient C) monomial of the form $Cz^a\zeta^b x^c$ has "degree" $a + b + c$. A power series (convergent or formal) in three variables $e(z, \zeta, x) = \sum e^{a,b,c} z^a \zeta^b x^c$, is said to have "degree" n if $e^{a,b,c} = 0$ for all (a, b, c) such that $a + b + c < n$. Sometimes a series of degree n will be abbreviated $O(n)$. An ordered triple of series (e, f, g) has degree n if all its components have degree n.

Definition 2.2. Similarly for four variables, a monomial of the form $Cz_1^a z_2^b z_3^c z_4^d$ has degree $a+b+c+d$, but we will more often work with the "weight," $a+b+2c+2d$. A series $p(z_1, z_2, z_3, z_4) = \sum p^{abcd} z_1^a z_2^b z_3^c z_4^d$ has "weight" n if $p^{abcd} = 0$ when $a + b + 2c + 2d < n$.

We consider the effect of a coordinate change of the following form:

$$\begin{aligned} \tilde{z}_1 &= z_1 + p_1(z_1, z_2, z_3, z_4) \\ \tilde{z}_2 &= z_2 + p_2(z_1, z_2, z_3, z_4) \\ \tilde{z}_3 &= z_3 + p_3(z_1, z_2, z_3, z_4) \\ \tilde{z}_4 &= z_4 + p_4(z_1, z_2, z_3, z_4), \end{aligned} \tag{2.3}$$

where p_1, p_2, p_3, p_4 are holomorphic functions with no linear or constant terms (so they have weight 2). Since this transformation of $\mathbb{C}^4$ has its linear part equal to the identity map, it is invertible on some neighborhood of the origin. In the following calculations, we will neglect considering the size of that neighborhood, and consider only points close enough to the origin. We denote the real and imaginary parts of the new coordinates $\tilde{z}_j = \tilde{x}_j + \tilde{y}_j$.

As the first special case of a transformation of the form (2.3) to be used, let p_1 be identically zero and let p_2, p_3, p_4 be homogeneous quadratic polynomials. Given a point on M near $\vec{0}$, its coordinates $\vec{z} = (z_1, \ldots, z_4)$ satisfy $\text{Im}(z_2) - H_2 = z_3 - h_3 = z_4 - h_4 = 0$. The new coordinates satisfy:

$$\begin{aligned} &\tilde{z}_3 - (\beta_3 \tilde{z}_1 \bar{\tilde{z}}_1 + \gamma_3 \bar{\tilde{z}}_1^2 + \epsilon_3 \bar{\tilde{z}}_1 \text{Re}(\tilde{z}_2)) \\ = \ & z_3 + p_3(\vec{z}) - (\beta_3 z_1 \bar{z}_1 + \gamma_3 \bar{z}_1^2 + \epsilon_3 \bar{z}_1 \text{Re}(z_2 + p_2(\vec{z}))) \\ = \ & \alpha_3 z_1^2 + \delta_3 z_1 x_2 + \theta_3 x_2^2 \\ & \quad + p_3(z_1, x_2 + iH_2, h_3, h_4) - \epsilon_3 \bar{z}_1 \text{Re}(p_2(z_1, x_2 + iH_2, h_3, h_4)) + e_3(z_1, \bar{z}_1, x_2) \\ = \ & \alpha_3 z_1^2 + \delta_3 z_1 x_2 + \theta_3 x_2^2 + p_3(z_1, x_2, 0, 0) + O(3) \\ = \ & \alpha_3 \tilde{z}_1^2 + \delta_3 \tilde{z}_1 \tilde{x}_2 + \theta_3 \tilde{x}_2^2 + p_3(\tilde{z}_1, \tilde{x}_2, 0, 0) + \tilde{O}(3), \end{aligned} \tag{2.4}$$

where $O(3)$ denotes a convergent series of degree three or higher in $z_1, \bar{z}_1, x_2$, and $\tilde{O}(3)$ denotes another series of degree three or higher in $\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}_2$ (also convergent, possibly on a different neighborhood). The last step of the above process converts back to the $\tilde{z}$ coordinates, using the fact that for points on M sufficiently close to the origin, z_1 and x_2 can be expressed as real analytic functions of $\tilde{z}_1$ and $\tilde{x}_2$; the details are given by Lemma 2.3, below.

This calculation shows that the coefficients α_3, δ_3, θ_3 can be altered to attain any complex values, by a suitable choice of p_3 — for example, the transformation $\tilde{z}_3 = z_3 - \alpha_3 z_1^2 - \delta_3 z_1 z_2 - \theta_3 z_2^2$ eliminates these terms, so that the defining equations in the $\tilde{z}$ coordinate system will be of the form $\tilde{z}_3 = \beta_3 \tilde{z}_1 \bar{\tilde{z}}_1 + \gamma_3 \bar{\tilde{z}}_1^2 + \epsilon_3 \bar{\tilde{z}}_1 \tilde{x}_2 + \tilde{O}(3)$. We note that such a quadratic transformation does not affect the quadratic coefficients of H_2 or h_4, but may change their higher-degree terms.

A similar calculation shows that the coefficients α_4, δ_4, θ_4 can be altered arbitrarily by choice of p_4, without changing the quadratic terms of H_2 or h_3.

The quantity p_2 can be used to transform H_2:

$$\begin{aligned} & \text{Im}(\tilde{z}_2 - i\beta_2 \tilde{z}_1 \bar{\tilde{z}}_1) \\ = \ & \text{Im}(z_2 + p_2(\vec{z}) - i\beta_2 z_1 \bar{z}_1) \\ = \ & H_2(z_1, \bar{z}_1, x_2) + \text{Im}(p_2(z_1, x_2 + iH_2, h_3, h_4)) - \beta_2 z_1 \bar{z}_1 \\ = \ & \alpha_2 z_1^2 + \overline{\alpha_2} \bar{z}_1^2 + \delta_2 z_1 x_2 + \overline{\delta_2} \bar{z}_1 x_2 + \theta_2 x_2^2 \\ & \quad + \text{Im}(p_2(z_1, x_2, 0, 0)) + O(3) \\ = \ & \text{Re}(2\alpha_2 z_1^2 + 2\delta_2 z_1 x_2 + \theta_2 x_2^2 - ip_2(z_1, x_2, 0, 0) + O(3)) \\ = \ & \text{Re}(2\alpha_2 \tilde{z}_1^2 + 2\delta_2 \tilde{z}_1 \tilde{x}_2 + \theta_2 \tilde{x}_2^2 - ip_2(\tilde{z}_1, \tilde{x}_2, 0, 0) + \tilde{O}(3)), \end{aligned} \tag{2.5}$$

the last step using Lemma 2.3 again. So, the complex coefficients α_2, δ_2 and the real coefficient θ_2 can be altered arbitrarily by choice of p_2 — for example, the transformation $\tilde{z}_2 = z_2 - 2i\alpha_2 z_1^2 - 2i\delta_2 z_1 z_2 - i\theta_2 z_2^2$ eliminates these terms so that the defining equations in the $\tilde{z}$ coordinate system will be of the form $\mathrm{Im}(\tilde{z}_2) = \beta_2 \tilde{z}_1 \bar{\tilde{z}}_1 + \tilde{O}(3)$. Such a quadratic transformation does not change β_2, and does not affect the quadratic coefficients of h_3 or h_4, but may change their higher-degree terms.

So, the conclusion is that for any CR singular submanifold M, there exists a quadratic coordinate transformation of the form (2.3) with $p_1 = 0$, so that M has the following general normal form. In a local coordinate system (z_1, z_2, z_3, z_4) in some neighborhood of the CR singularity, the defining equations of M are of the form (2.1), with

$$\begin{aligned} y_2 = H_2(z_1, \bar{z}_1, x_2) &= \beta_2 z_1 \bar{z}_1 + O(3) \\ z_3 = h_3(z_1, \bar{z}_1, x_2) &= \beta_3 z_1 \bar{z}_1 + \gamma_3 \bar{z}_1^2 + \epsilon_3 \bar{z}_1 x_2 + O(3) \\ z_4 = h_4(z_1, \bar{z}_1, x_2) &= \beta_4 z_1 \bar{z}_1 + \gamma_4 \bar{z}_1^2 + \epsilon_4 \bar{z}_1 x_2 + O(3). \end{aligned} \tag{2.6}$$

The following Lemma states that for points on M, the coordinates z_1, x_2 can be expressed as real analytic functions of $\tilde{z}_1, \tilde{x}_2$ in some neighborhood of $(0, 0)$ in $\mathbb{C} \times \mathbb{R}$, and that these functions depend on the defining equations of M and the p_1, p_2 components of the coordinate transformation (2.3). We are interested mostly in the existence of the neighborhood and not its size, about which something could be said, given more information about p_1, p_2, H_2, h_3, h_4, in analogy with the results of [C_2].

Lemma 2.3. *Given any functions $p_1(z_1, z_2, z_3, z_4)$, $p_2(z_1, z_2, z_3, z_4)$ which are holomorphic in a neighborhood of the origin of $\mathbb{C}^4$ and have weight 2, and given H_2, h_3, h_4 as in (2.2) which define M in a neighborhood of the origin in $\mathbb{C}^4$, there exist functions ϕ_1, $\phi_3 : \mathbb{C}^3 \to \mathbb{C}$ which are holomorphic on some neighborhood of the origin and which vanish to at least second order there, such that for points (z_1, z_2, z_3, z_4) on M and sufficiently close to the origin, if $\tilde{z}_1 = z_1 + p_1(z_1, z_2, z_3, z_4)$ and $\tilde{z}_2 = z_2 + p_2(z_1, z_2, z_3, z_4)$, then*

$$\begin{aligned} z_1 &= \tilde{z}_1 + \phi_1(\tilde{z}_1, \bar{\tilde{z}}_1, \mathrm{Re}(\tilde{z}_2)) \\ \mathrm{Re}(z_2) &= \mathrm{Re}(\tilde{z}_2) + \phi_3(\tilde{z}_1, \bar{\tilde{z}}_1, \mathrm{Re}(\tilde{z}_2)). \end{aligned}$$

Proof. Consider the map τ that takes (z, ζ, x) to the ordered triple:

$$\begin{aligned} &(z + p_1(z, x + iH_2(z, \zeta, x), h_3(z, \zeta, x), h_4(z, \zeta, x)), \\ &\zeta + \overline{p_1(\bar{\zeta}, \bar{x} + iH_2(\bar{\zeta}, \bar{z}, \bar{x}), h_3(\bar{\zeta}, \bar{z}, \bar{x}), h_4(\bar{\zeta}, \bar{z}, \bar{x}))}, \\ &x + \frac{1}{2} p_2(z, x + iH_2(z, \zeta, x), h_3(z, \zeta, x), h_4(z, \zeta, x)) \\ &\quad + \frac{1}{2} \overline{p_2(\bar{\zeta}, \bar{x} + iH_2(\bar{\zeta}, \bar{z}, \bar{x}), h_3(\bar{\zeta}, \bar{z}, \bar{x}), h_4(\bar{\zeta}, \bar{z}, \bar{x}))}). \end{aligned}$$

Note that τ is holomorphic and invertible in a neighborhood of the origin of $\mathbb{C}^3$, with identity linear part. Its inverse is given by a function ϕ such that $\tau(\phi(z', \zeta', x')) = (z', \zeta', x')$, with

$$\phi(z', \zeta', x') = (z' + \phi_1(z', \zeta', x'), \zeta' + \phi_2(z', \zeta', x'), x' + \phi_3(z', \zeta', x')).$$

Also note that if $\tau(z,\zeta,x)=(z',\zeta',x')$, then $\tau(\bar\zeta,\bar z,\bar x)=(\bar\zeta',\bar z',\bar x')$. Denoting by C the antiholomorphic involution of $\mathbb{C}^3$ where $C(z,\zeta,x)=(\bar\zeta,\bar z,\bar x)$, it follows from $C\circ\tau\circ C=\tau$ that $C\circ\phi\circ C=\phi$ on some neighborhood of the origin of $\mathbb{C}^3$.

By construction, for $x_2=\mathrm{Re}(z_2)$,

$$\begin{aligned}\tau(z_1,\bar z_1,x_2) \;=\;& (z_1+p_1(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2)),\\ & \overline{z_1+p_1(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2))},\\ & x_2+\mathrm{Re}(p_2(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2)))).\end{aligned}$$

If a point (z_1,z_2,z_3,z_4) is on M and close enough to the origin, then it is of the form $(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2))$, and

$$\begin{aligned}\tilde z_1 &= z_1+p_1(z_1,z_2,z_3,z_4)\\ &= z_1+p_1(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2)),\\ \tilde z_2 &= z_2+p_2(z_1,z_2,z_3,z_4)\\ &= z_2+p_2(z_1,x_2+iH_2(z_1,\bar z_1,x_2),h_3(z_1,\bar z_1,x_2),h_4(z_1,\bar z_1,x_2)),\end{aligned}$$

so $\tau(z_1,\bar z_1,x_2)=(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))=\tau(\phi(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2)))$. The conclusion is that

$$\begin{aligned}z_1 &= \tilde z_1+\phi_1(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))\\ \bar z_1 &= \bar{\tilde z}_1+\phi_2(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))=\bar{\tilde z}_1+\overline{\phi_1(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))}\\ x_2 &= \mathrm{Re}(\tilde z_2)+\phi_3(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))=\mathrm{Re}(\tilde z_2)+\overline{\phi_3(\tilde z_1,\bar{\tilde z}_1,\mathrm{Re}(\tilde z_2))}.\end{aligned}$$

$\blacksquare$

2.2. **The non-degenerate case.**

Next, we consider some linear transformations of $\mathbb{C}^4$, but only those which fix the tangent 3-plane T, so they are of the form

$$\tilde z_{4\times 1}=\begin{pmatrix} c_1 & * & * & *\\ 0 & r_2 & * & *\\ 0 & 0 & c_3 & c_4\\ 0 & 0 & c_5 & c_6\end{pmatrix}\vec z_{4\times 1}, \tag{2.7}$$

with complex entries such that $c_1r_2\neq 0$, $r_2=\overline{r_2}$, $c_3c_6-c_4c_5\neq 0$.

First, consider the linear coordinate change $\tilde z_3=c_3z_3+c_4z_4$, $\tilde z_4=c_5z_3+c_6z_6$. In this coordinate system, the coefficients of H_2 are not changed, and we get

$$\begin{aligned}\tilde z_3 &= c_3(\beta_3 z_1\bar z_1+\gamma_3\bar z_1^2+\epsilon_3\bar z_1x_2+O(3))+c_4(\beta_4 z_1\bar z_1+\gamma_4\bar z_1^2+\epsilon_4\bar z_1x_2+O(3))\\ &= (c_3\beta_3+c_4\beta_4)\tilde z_1\bar{\tilde z}_1+(c_3\gamma_3+c_4\gamma_4)\bar{\tilde z}_1^2+(c_3\epsilon_3+c_4\epsilon_4)\bar{\tilde z}_1\tilde x_2+\tilde O(3),\end{aligned}$$

and similarly

$$\tilde z_4=(c_5\beta_3+c_6\beta_4)\tilde z_1\bar{\tilde z}_1+(c_5\gamma_3+c_6\gamma_4)\bar{\tilde z}_1^2+(c_5\epsilon_3+c_6\epsilon_4)\bar{\tilde z}_1\tilde x_2+\tilde O(3).$$

At this point we introduce the "first non-degeneracy condition," which is satisfied if:

$$\det\begin{pmatrix}\beta_3 & \gamma_3\\ \beta_4 & \gamma_4\end{pmatrix}\;\neq\;0, \tag{2.8}$$

so that there is a complex linear coordinate change transforming the general normal form (2.6) to:

$$\begin{aligned} y_2 &= \beta_2 z_1\bar{z}_1 + E_2(z_1,\bar{z}_1,x_2) \\ z_3 &= \bar{z}_1^2 + \epsilon_3\bar{z}_1 x_2 + e_3(z_1,\bar{z}_1,x_2) \\ z_4 &= z_1\bar{z}_1 + \epsilon_4\bar{z}_1 x_2 + e_4(z_1,\bar{z}_1,x_2). \end{aligned} \tag{2.9}$$

Next, consider a linear transformation of the form $\tilde{z}_2 = z_2 + irz_4$, for some real r, so that for (z_1,z_2,z_3,z_4) on M:

$$\begin{aligned} &\operatorname{Im}(\tilde{z}_2) \\ = &\operatorname{Im}(z_2 + irz_4) \\ = &H_2(z_1,\bar{z}_1,x_2) + \operatorname{Im}(irh_4(z_1,\bar{z}_1,x_2)) \\ = &\beta_2 z_1\bar{z}_1 + E_3(z_1,\bar{z}_1,x_2) + rz_1\bar{z}_1 + \operatorname{Re}(r\epsilon_4\bar{z}_1x_2) + \operatorname{Re}(re_4(z_1,\bar{z}_1,x_2)) \\ = &(\beta_2 + r)z_1\bar{z}_1 + \frac{1}{2}r\epsilon_4\bar{z}_1x_2 + \frac{1}{2}r\overline{\epsilon_4}z_1x_2 + O(3) \\ = &(\beta_2 + r)\tilde{z}_1\bar{\tilde{z}}_1 + \frac{1}{2}r\epsilon_4\bar{\tilde{z}}_1\tilde{x}_2 + \frac{1}{2}r\overline{\epsilon_4}\tilde{z}_1\tilde{x}_2 + \tilde{O}(3). \end{aligned} \tag{2.10}$$

Even though $\tilde{z}_2 = z_2 + irz_4$ is a linear transformation, the hypotheses of Lemma 2.3 are satisfied, since in this case $p_2(\vec{z}) = irz_4$ has weight 2, so $\tau(z,\zeta,x)$ is defined by:

$$(z,\zeta,x + \frac{1}{2}(irz\zeta + ir\epsilon_4\zeta x + ire_4(z,\zeta,x)) + \frac{1}{2}\overline{(ir\bar{\zeta}\bar{z} + ir\epsilon_4\bar{z}\bar{x} + ire_4(\bar{\zeta},\bar{z},\bar{x}))}),$$

which has identity linear part. By inspection of (2.4) with $p_2(\vec{z}) = irz_4$, the transformation does not affect the quadratic coefficients of h_3 or h_4 but may change their higher-degree terms. Clearly, choosing $r = -\beta_2$ will eliminate the $\tilde{z}_1\bar{\tilde{z}}_1$ term, leaving only the $\bar{\tilde{z}}_1\tilde{x}_2$ and $\tilde{z}_1\tilde{x}_2$ terms which can be eliminated by another quadratic transformation as in (2.5) without re-introducing a $\tilde{z}_1\bar{\tilde{z}}_1$ term.

Another linear transformation is $\tilde{z}_1 = z_1 + cz_2$, for some complex coefficient c.

$$\begin{aligned} \tilde{z}_4 - \tilde{z}_1\bar{\tilde{z}}_1 &= z_4 - (z_1 + cz_2)(\bar{z}_1 + \bar{c}\bar{z}_2) \\ &= \epsilon_4\bar{z}_1x_2 + e_4(z_1,\bar{z}_1,x_2) \\ &\quad -c\bar{z}_1(x_2 + iH_2(z_1,\bar{z}_1,x_2)) - \bar{c}z_1(x_2 - i\overline{H_2(z_1,\bar{z}_1,x_2)}) \\ &\quad -c\bar{c}(x_2 + iH_2(z_1,\bar{z}_1,x_2))(x_2 - i\overline{H_2(z_1,\bar{z}_1,x_2)}) \\ &= (\epsilon_4 - c)\bar{z}_1x_2 - \bar{c}z_1x_2 - c\bar{c}x_2^2 + O(3). \end{aligned}$$

This time, the hypotheses of Lemma 2.3 are not satisfied, since in this case $p_1(\vec{z}) = cz_2$ has weight 1. The map $\tau(z,\zeta,x)$ defined by

$$(z + c(x + iH_2(z,\zeta,x)), \zeta + \overline{c(\bar{x} + iH_2(\bar{\zeta},\bar{z},\bar{x}))}, x)$$

has an invertible linear part which is not the identity, but the inverse transformation is holomorphic on some neighborhood, and given by

$$\phi(z',\zeta',x') = (z' - cx' + \phi_1, \zeta' - \bar{c}x' + \phi_2, x' + \phi_3).$$

The conclusion analogous to Lemma 2.3 is then that $z_1 = \tilde{z}_1 - c\tilde{x}_2 + \phi_1(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}_2)$, $x_2 = \tilde{x}_2 + \phi_3(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}_2)$ is real analytic, so the new $\tilde{z}_4$ equation is

$$\begin{aligned} \tilde{z}_4 &= \tilde{z}_1\bar{\tilde{z}}_1 + (\epsilon_4 - c)(\bar{\tilde{z}}_1 - \bar{c}\tilde{x}_2)\tilde{x}_2 - \bar{c}(\tilde{z}_1 - c\tilde{x}_2)\tilde{x}_2 - c\bar{c}\tilde{x}_2^2 + \tilde{O}(3) \\ &= \tilde{z}_1\bar{\tilde{z}}_1 + (\epsilon_4 - c)\bar{\tilde{z}}_1\tilde{x}_2 - \bar{c}\tilde{z}_1\tilde{x}_2 + (-\epsilon_4\bar{c} + c\bar{c})\tilde{x}_2^2 + \tilde{O}(3). \end{aligned} \tag{2.11}$$

This transformation does not introduce any quadratic terms in H_2, but it does change h_3:

$$\begin{aligned}
\tilde{z}_3 - \bar{\tilde{z}}_1^2 &= z_3 - (\bar{z}_1 + \bar{c}\bar{z}_2)^2 \\
&= \epsilon_3 \bar{z}_1 x_2 + e_3(z_1, \bar{z}_1, x_2) \\
&\quad -2\bar{c}\bar{z}_1(x_2 - i\overline{H_2(z_1, \bar{z}_1, x_2)}) - \bar{c}^2(x_2 - i\overline{H_2(z_1, \bar{z}_1, x_2)})^2 \\
&= (\epsilon_3 - 2\bar{c})\bar{z}_1 x_2 - \bar{c}^2 x_2^2 + O(3) \\
&= (\epsilon_3 - 2\bar{c})(\bar{\tilde{z}}_1 - \bar{c}\tilde{x}_2)\tilde{x}_2 - \bar{c}^2 \tilde{x}_2^2 + \tilde{O}(3) \\
&= (\epsilon_3 - 2\bar{c})\bar{\tilde{z}}_1 \tilde{x}_2 + (-\epsilon_3 \bar{c} + \bar{c}^2)\tilde{x}_2^2 + \tilde{O}(3). \qquad (2.12)
\end{aligned}$$

From (2.11) and (2.12), we see c can be chosen to eliminate the $\bar{\tilde{z}}_1\tilde{x}_2$ term from either the h_3 or the h_4 series, but it is unlikely that the transformation will simultaneously eliminate the $\bar{\tilde{z}}_1\tilde{x}_2$ term from both. At this point, we choose to eliminate $\bar{\tilde{z}}_1\tilde{x}_2$ term from h_3 by selecting $c = \overline{\epsilon_3}/2$, and then another transformation as in (2.4) with suitable p_3, p_4 will clean up the quadratic terms introduced above, giving the following quadratic normal form for any M satisfying the first non-degeneracy condition:

$$\begin{aligned}
y_2 &= E_2(z_1, \bar{z}_1, x_2) = O(3) \\
z_3 &= \bar{z}_1^2 + e_3(z_1, \bar{z}_1, x_2) \\
z_4 &= (z_1 + \epsilon_4 x_2)\bar{z}_1 + e_4(z_1, \bar{z}_1, x_2).
\end{aligned} \qquad (2.13)$$

The "second non-degeneracy condition" is that $\epsilon_4 \neq 0$, after M is put into the above normal form (so the first non-degeneracy condition is assumed). In this case, an invertible linear coordinate change of the form $\tilde{z}_1 = c_1 z_1$, $\tilde{z}_3 = c_3 z_3$, $\tilde{z}_4 = c_6 z_4$ gives:

$$\begin{aligned}
\tilde{y}_2 &= E_2(\frac{\tilde{z}_1}{c_1}, \frac{\bar{\tilde{z}}_1}{\overline{c_1}}, \tilde{x}_2) = \tilde{O}(3) \\
\tilde{z}_3 &= c_3((\frac{\bar{\tilde{z}}_1}{\overline{c_1}})^2 + e_3(\frac{\tilde{z}_1}{c_1}, \frac{\bar{\tilde{z}}_1}{\overline{c_1}}, \tilde{x}_2)) \\
\tilde{z}_4 &= c_6(\frac{\tilde{z}_1}{c_1}\frac{\bar{\tilde{z}}_1}{\overline{c_1}} + \epsilon_4 \frac{\bar{\tilde{z}}_1}{\overline{c_1}}\tilde{x}_2 + e_4(\frac{\tilde{z}_1}{c_1}, \frac{\bar{\tilde{z}}_1}{\overline{c_1}}, \tilde{x}_2)),
\end{aligned} \qquad (2.14)$$

and choosing $(c_1, c_3, c_6) = (1/\epsilon_4, 1/\overline{\epsilon_4}^2, 1/|\epsilon_4|^2)$ normalizes the coefficient of $\bar{\tilde{z}}_1\tilde{x}_2$ in h_4 to 1.

2.3. **Classifying the degenerate cases.** Returning to (2.6), suppose that the determinant from (2.8) is zero. Then a linear transformation (2.7) could put the rank 1 coefficient matrix $\begin{pmatrix} \beta_3 & \gamma_3 \\ \beta_4 & \gamma_4 \end{pmatrix}$ into one of three normal forms, taking the defining equations (2.6) to one of the following cases:

$$\begin{aligned}
y_2 &= \beta_2 z_1 \bar{z}_1 + E_2(z_1, \bar{z}_1, x_2) \\
z_3 &= z_1\bar{z}_1 + \gamma_3 \bar{z}_1^2 + \epsilon_3 \bar{z}_1 x_2 + e_3(z_1, \bar{z}_1, x_2) \\
z_4 &= \epsilon_4 \bar{z}_1 x_2 + e_4(z_1, \bar{z}_1, x_2).
\end{aligned} \qquad (2.15)$$

$$\begin{aligned}
y_2 &= \beta_2 z_1 \bar{z}_1 + E_2(z_1, \bar{z}_1, x_2) \\
z_3 &= \bar{z}_1^2 + \epsilon_3 \bar{z}_1 x_2 + e_3(z_1, \bar{z}_1, x_2) \\
z_4 &= \epsilon_4 \bar{z}_1 x_2 + e_4(z_1, \bar{z}_1, x_2).
\end{aligned} \qquad (2.16)$$

$$(2.17)\qquad \begin{aligned} y_2 &= \beta_2 z_1\bar{z}_1 + E_2(z_1,\bar{z}_1,x_2) \\ z_3 &= \epsilon_3\bar{z}_1x_2 + e_3(z_1,\bar{z}_1,x_2) \\ z_4 &= \epsilon_4\bar{z}_1x_2 + e_4(z_1,\bar{z}_1,x_2). \end{aligned}$$

In (2.15), a re-scaling as in (2.14) can transform the coefficient γ_3 to a non-negative real number, resembling Bishop's invariant of surfaces in $\mathbb{C}^2$, ([Bishop]). The z_3 expression in (2.16) could be considered the "$\gamma_3 = \infty$" case of (2.15). If $\epsilon_4 \neq 0$, another linear transformation of the z_3, z_4 coordinates could re-scale $\epsilon_4 = 1$ and eliminate ϵ_3 from (2.15), (2.16), or (2.17). Otherwise, if $\epsilon_4 = 0$, ϵ_3 could be eliminated from (2.15) or (2.16) by a linear transformation of the form $\tilde{z}_1 = z_1 + cz_2$ as in (2.12), unless $\gamma_3 = \frac{1}{2}$ in (2.15), where only the real part of ϵ_3 can be eliminated, and $\mathrm{Im}(\epsilon_3)$ can be re-scaled to 1 or 0. The $\gamma_3 = \frac{1}{2}$, $\epsilon_3 = i$, $\epsilon_4 = 0$ case (type (IVp) in the table below) corresponds to the parabolic normal form of [W]. In (2.17), if $\epsilon_4 = 0$, ϵ_3 could be re-scaled to be either 1 or 0. In (2.15), a linear transformation of the form $\tilde{z}_2 = z_2 + irz_3$ as in (2.10) will eliminate β_2. In the other cases, β_2 can only be re-scaled to be either 1 or 0.

To summarize, for any real threefold M in $\mathbb{C}^4$ with a CR singular point, there is some coordinate system in a neighborhood of that point so that the defining equations of M in that neighborhood are of the form

$$\begin{aligned} y_2 &= Q_2 + E_2(z_1,\bar{z}_1,x_2) \\ z_3 &= q_3 + e_3(z_1,\bar{z}_1,x_2) \\ z_4 &= q_4 + e_4(z_1,\bar{z}_1,x_2), \end{aligned}$$

where E_2, e_3, e_4 vanish to third order, E_2 is real-valued, and Q_2, q_3, q_4 are quadratic quantities falling into exactly one form from the following list.

(I)	$Q_2 = 0$ $q_3 = \bar{z}_1^2$ $q_4 = z_1\bar{z}_1 + x_2\bar{z}_1$	(V)	$Q_2 = z_1\bar{z}_1$ $q_3 = \bar{z}_1^2$ $q_4 = \bar{z}_1x_2$	(IX)	$Q_2 = z_1\bar{z}_1$ $q_3 = 0$ $q_4 = \bar{z}_1x_2$
(II)	$Q_2 = 0$ $q_3 = \bar{z}_1^2$ $q_4 = z_1\bar{z}_1$	(VI)	$Q_2 = z_1\bar{z}_1$ $q_3 = \bar{z}_1^2$ $q_4 = 0$	(X)	$Q_2 = z_1\bar{z}_1$ $q_3 = 0$ $q_4 = 0$
(III)	$Q_2 = 0$ $q_3 = z_1\bar{z}_1 + \gamma\bar{z}_1^2,\ \gamma \geq 0$ $q_4 = \bar{z}_1x_2$	(VII)	$Q_2 = 0$ $q_3 = \bar{z}_1^2$ $q_4 = \bar{z}_1x_2$	(XI)	$Q_2 = 0$ $q_3 = 0$ $q_4 = \bar{z}_1x_2$
(IV)	$Q_2 = 0$ $q_3 = z_1\bar{z}_1 + \gamma\bar{z}_1^2,\ \gamma \geq 0$ $q_4 = 0$	(VIII)	$Q_2 = 0$ $q_3 = \bar{z}_1^2$ $q_4 = 0$	(XII)	$Q_2 = 0$ $q_3 = 0$ $q_4 = 0$
(IVp)	$Q_2 = 0$ $q_3 = z_1\bar{z}_1 + \frac{1}{2}\bar{z}_1^2 + i\bar{z}_1x_2$ $q_4 = 0$				

Type (I) is the non-degenerate quadratic normal form, (2.13) with ϵ_4 scaled to 1 as in (2.14). Type (II) is the quadratic normal form for a manifold satisfying the first non-degeneracy condition but not the second, so $\epsilon_4 = 0$ in (2.13). The middle column has the $\gamma = \infty$ cases.

2.4. **Examples.** Most real analytic threefolds in $\mathbb{C}^4$ are totally real at every point — a CR singularity is topologically unstable, in the sense that a real manifold with a CR singular point can be perturbed by a small amount to become totally real. However, simple examples of CR singular submanifolds in complex Euclidean 4-space are not hard to construct and we consider a few here.

Example 2.4. Singularities of type (I), (III) with $\gamma > \frac{1}{2}$, (VII), and (XI) can occur in projections of the real Segre threefold from $\mathbb{C}P^5$ to $\mathbb{C}P^4$, as shown in [C_3].

Example 2.5. Among the three-dimensional real affine subspaces of $\mathbb{C}^4$, most are totally real. The rest, those that contain a complex line, are related to the z_1, x_2-subspace by some complex affine transformation, and have a type (XII) singularity at every point.

Example 2.6. Considering the Euclidean 3-sphere contained inside some four-dimensional real affine subspace of $\mathbb{C}^4$, again such a subspace in general position is totally real, and so is the sphere it contains. One exceptional case is that the real affine subspace is contained in a complex 3-subspace but is not itself a complex subspace. In this case, the real subspace necessarily contains a complex line, and the standard hypersphere is CR singular along a real circle. A complex affine coordinate change takes a such a sphere to $\{x_1^2+y_1^2+x_2^2+(x_3-1)^2=1\}$ inside the z_1, x_2, x_3-subspace, and the CR singular locus is the circle of points on the sphere with $z_1 = 0$. In a neighborhood of the origin, the defining equations of the sphere become $y_2 = y_3 = z_4 = 0$ and

$$\begin{aligned} x_3 &= 1-\sqrt{1-x_1^2-y_1^2-x_2^2} \\ \implies z_3 &= \frac{1}{2}(z_1\bar{z}_1 + x_2^2) + O(4). \end{aligned}$$

This is a type (IV) singularity with $\gamma = 0$. More generally, the quadric $\{ax_1^2+y_1^2+x_2^2+(x_3-1)^2=1\}$, for $a \in \mathbb{R}$, has a CR singularity of type (IV) with $\gamma = \left|\frac{a-1}{2(a+1)}\right|$ for $a \neq -1$, or type (VIII) for $a = -1$. So, each γ in the range $[0, \infty]$ has a representative of this form, and it is exactly the well-known Bishop invariant for n-manifolds in $\mathbb{C}^n$, with $n = 3$.

Example 2.7. The other exceptional case of a four-dimensional real affine subspace of $\mathbb{C}^4$ is that it is a two-dimensional complex affine subspace. Then, a standard 3-sphere that it contains is a real hypersurface in a complex 2-manifold, so it has a complex tangent at every point, and is CR singular at every point when considered as a real submanifold of $\mathbb{C}^4$. A complex affine coordinate change takes such a sphere to $\{x_1^2+y_1^2+x_2^2+(y_2-1)^2=1\}$ in the z_1, z_2-subspace, and in a neighborhood of the origin, the defining equations of the sphere become $z_3 = z_4 = 0$ and

$$\begin{aligned} y_2 &= 1-\sqrt{1-x_1^2-y_1^2-x_2^2} \\ &= \frac{1}{2}(z_1\bar{z}_1 + x_2^2) + O(4). \end{aligned}$$

This is a type (X) singularity. Again generalizing to the quadric $\{ax_1^2+y_1^2+x_2^2+(y_2-1)^2=1\}$, for $a \in \mathbb{R}$, the CR singularity is of type (X) for all a except $a = -1$, where the $z_1\bar{z}_1$ term has coefficient 0 and the quadratic normal form is of type (XII).

3. Higher-order terms in the non-degenerate case

After a transformation as in the previous Section, the equations near a non-degenerate (type (I)) CR singularity are in the following form:

$$\begin{aligned} y_2 &= E_2(z_1, \bar{z}_1, x_2) \\ z_3 &= \bar{z}_1^2 + e_3(z_1, \bar{z}_1, x_2) \\ z_4 &= (z_1 + x_2)\bar{z}_1 + e_4(z_1, \bar{z}_1, x_2), \end{aligned} \tag{3.1}$$

where E_2, e_3, e_4 have degree 3 in z_1, $\bar{z}_1$, x_2, and are real analytic, converging for z_1, x_2 in some neighborhood of the origin in the z_1, x_2 3-space. It will be shown in this Section that there exists $\vec{p} = (p_1, p_2, p_3, p_4)$ defining a coordinate transformation as in (2.3) such that the defining equations in the new coordinates are as in (3.1), but with E_2, e_3, e_4 vanishing to any degree n, $n \geq 3$.

We will consider formal coordinate transformations $\vec{p}$ as in (2.3) where p_1 has weight 2 and p_2, p_3, p_4 have weight 3. We will think of these transformations as having identity linear part, although a weight 2 transformation of z_1 could have linear terms of the form $\tilde{z}_1 = z_1 + c_3 z_3 + c_4 z_4$. The effect of the transformation is that for points on M, of the form $\vec{z} = (z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)$,

$$\begin{aligned} \operatorname{Im}(\tilde{z}_2) &= \operatorname{Im}(z_2 + p_2(\vec{z})) \\ &= \operatorname{Im}(x_2 + iE_2(z_1, \bar{z}_1, x_2) + p_2(\vec{z})) \\ &= E_2(z_1, \bar{z}_1, x_2) + \operatorname{Im}(p_2), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \tilde{z}_3 - \bar{\tilde{z}}_1^2 &= z_3 + p_3(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2 \\ &= e_3 + p_3 - 2\bar{z}_1\overline{p_1} - \overline{p_1}^2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\tilde{z}_4 - (\tilde{z}_1 + \tilde{x}_2)\bar{\tilde{z}}_1 \\ = {}& z_4 + p_4(\vec{z}) - (z_1 + p_1(\vec{z}) + x_2 + \operatorname{Re}(p_2(\vec{z})))\overline{(z_1 + p_1(\vec{z}))} \\ = {}& e_4 + p_4 - (z_1 + x_2)\overline{p_1} - \bar{z}_1(p_1 + \operatorname{Re}(p_2)) - p_1\overline{p_1} - \operatorname{Re}(p_2)\overline{p_1}. \end{aligned} \tag{3.4}$$

The "normal form" problem is then the following: given convergent series E_2, e_3, e_4 as in (3.1), with degree ≥ 3, find series p_1, p_2, p_3, p_4 as in (2.3) which are solutions of these non-linear equations:

$$\begin{aligned} 0 &= E_2(z_1, \bar{z}_1, x_2) + \operatorname{Im}(p_2(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)) \\ 0 &= e_3(z_1, \bar{z}_1, x_2) + p_3(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4) \\ &\quad -2\bar{z}_1\overline{p_1(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)} \\ &\quad -\overline{p_1(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)}^2 \\ 0 &= e_4(z_1, \bar{z}_1, x_2) + p_4(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4) \\ &\quad -(z_1 + x_2)\overline{p_1(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)} \\ &\quad -\bar{z}_1 p_1(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4) \\ &\quad -\bar{z}_1\operatorname{Re}(p_2(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)) \\ &\quad -|p_1(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2)\bar{z}_1 + e_4)|^2 - \operatorname{Re}(p_2(\vec{z}))\overline{p_1(\vec{z})}. \end{aligned} \tag{3.5}$$

If, given E_2, e_3, e_4, we can find series solutions p_1, p_2, p_3, p_4 of the above equations, which converge on some neighborhood of $\vec{0} \in \mathbb{C}^4$, then M is "analytically equivalent" to the "polynomial model" $\{\tilde{y}_2 = 0, \tilde{z}_3 = \bar{\tilde{z}}_1^2, \tilde{z}_4 = (\tilde{z}_1 + \tilde{x}_2)\bar{\tilde{z}}_1\}$. More generally, if there exists any formal series solution, then we can say M is "formally equivalent" to the polynomial model.

Theorem 3.1. *Given M with the non-degenerate quadratic normal form (type* (I)*, as in* (3.1)*), there exist formal series p_1, p_2, p_3, p_4 which are solutions of the system of equations* (3.5)*, so that M is formally equivalent to the polynomial model.*

Proof. The idea is to iterate the following step: given series $\vec{e} = (E_2, e_3, e_4)$ of degree $n \geq 3$, we will find a transformation (2.3) using series $\vec{p} = (p_1, p_2, p_3, p_4)$, called an "approximate solution," so that the defining expressions in the new coordinates have degree $\geq n+1$. The existence of a formal transformation follows, since its terms up to any degree can be determined by composing sufficiently many of the approximate solutions.

We begin by replacing (3.5) by a related system of linear equations:

$$\begin{aligned}
(3.6)\quad 0 &= E_2(z_1, \bar{z}_1, x_2) + \mathrm{Im}(p_2(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)) \\
0 &= e_3(z_1, \bar{z}_1, x_2) + p_3(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) \\
&\quad -2\bar{z}_1\overline{p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)} \\
0 &= e_4(z_1, \bar{z}_1, x_2) + p_4(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) \\
&\quad -(z_1 + x_2)\overline{p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)} \\
&\quad -\bar{z}_1(p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) + \mathrm{Re}(p_2(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1))).
\end{aligned}$$

By comparing (3.5) with (3.6), we see that if $\vec{e}$ has degree n, and $\vec{p}$ is an exact solution of (3.6), with the weight of p_1 equal to $n-1$ and the weight of p_2, p_3, and p_4 equal to n, then evaluating the RHS of (3.5) with this $\vec{p}$ will result in an expression with degree $2n-2$ (which is $\geq n+1$ for $n \geq 3$). So, a solution of (3.6) is an approximate solution of (3.5). The degree $2n-2$ quantity, if converted back to the $\tilde{z}$ coordinates using Lemma 2.3, will still have degree $\geq 2n-2$, and, returning to Equations (3.2), (3.3), (3.4), will be the higher-order part of the defining equations in the new coordinate system.

The remainder of the Proof will be the construction of such an approximate solution, assuming $n \geq 3$.

We start with the first equation from the system (3.6):

$$(3.7)\qquad 0 = E_2(z_1, \bar{z}_1, x_2) + \mathrm{Im}(p_2(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)).$$

The real-valued real analytic function E_2 can be written in the following form:

$$\begin{aligned}
E_2(z_1, \bar{z}_1, x_2) &= \sum e_2^{abc} z_1^a \bar{z}_1^b x_2^c \\
&= e_{2A} + \overline{e_{2A}} + e_{2B} + e_{2C} + \overline{e_{2C}} + e_{2D} + \overline{e_{2D}} + e_{2E},
\end{aligned}$$

where E_2 has degree n (as in Definition 2.1), $e_2^{bac} = \overline{e_2^{abc}}$ and

$$\begin{aligned}
e_{2A} &= \sum_{a>b,\ b \text{ even}} e_2^{abc} z_1^a \bar{z}_1^b x_2^c \\
e_{2B} &= \sum_{a \text{ even}} e_2^{aac} z_1^a \bar{z}_1^a x_2^c \\
e_{2C} &= \sum_{a>b,\ a \text{ even},\ b \text{ odd}} e_2^{abc} z_1^a \bar{z}_1^b x_2^c \\
e_{2D} &= \sum_{a>b,\ a,\ b \text{ odd}} e_2^{abc} z_1^a \bar{z}_1^b x_2^c \\
e_{2E} &= \sum_{a \text{ odd}} e_2^{aac} z_1^a \bar{z}_1^a x_2^c.
\end{aligned}$$

We further rearrange e_{2D} and e_{2E}, to get:

$$\begin{aligned}
e_{2D} &= (1+\frac{x_2}{z_1})e_{2D} - \frac{x_2}{z_1}e_{2D} = e_{2F} + e_{2G}, \\
e_{2F} &= \sum_{a>b,\ a,b \text{ odd}} e_2^{abc} z_1^{a-1}(z_1+x_2)\bar{z}_1\bar{z}_1^{b-1}x_2^c \\
e_{2G} &= -\sum_{a>b,\ a,b \text{ odd}} e_2^{abc} z_1^{a-1}\bar{z}_1^b x_2^{c+1}, \\
e_{2E} &= (\frac{1}{2}+\frac{x_2}{2z_1}+\frac{1}{2}+\frac{x_2}{2\bar{z}_1})e_{2E} - (\frac{x_2}{2z_1}+\frac{x_2}{2\bar{z}_1})e_{2E} = e_{2H} + \overline{e_{2H}} + e_{2I} + \overline{e_{2I}}, \\
e_{2H} &= \frac{1}{2}\sum_{a \text{ odd}} e_2^{aac} z_1^{a-1}\bar{z}_1^{a-1}(z_1+x_2)\bar{z}_1 x_2^c, \\
e_{2I} &= -\frac{1}{2}\sum_{a \text{ odd}} e_2^{aac} z_1^{a-1}\bar{z}_1^a x_2^{c+1},
\end{aligned}$$

so

$$E_2 = e_{2A} + \overline{e_{2A}} + e_{2B} + e_{2C} + \overline{e_{2C}} + e_{2F} + \overline{e_{2F}} + e_{2G} + \overline{e_{2G}} + e_{2H} + \overline{e_{2H}} + e_{2I} + \overline{e_{2I}}.$$

To find a function p_2 satisfying (3.7), it is enough to consider a weight n expression of the form:

$$\begin{aligned}
p_2(z_1,z_2,z_3,z_4) &= p_{2A} + p_{2B} + p_{2C} + p_{2D} + p_{2E}, \\
p_{2A} &= \sum_{\alpha>2\gamma,\ \alpha \text{ even}} p_{2A}^{\alpha\beta\gamma} z_1^\alpha z_2^\beta z_3^\gamma z_4 \\
p_{2B} &= \sum p_{2B}^{\beta\gamma} z_1^{2\gamma} z_2^\beta z_3^\gamma z_4 \\
p_{2C} &= \sum_{\alpha>2\gamma} p_{2C}^{\alpha\beta\gamma} z_1^\alpha z_2^\beta z_3^\gamma \\
p_{2D} &= \sum p_{2D}^{\beta\gamma} z_1^{2\gamma} z_2^\beta z_3^\gamma \\
p_{2E} &= \sum_{\alpha<2\gamma,\ \alpha \text{ odd}} p_{2E}^{\alpha\beta\gamma} z_1^\alpha z_2^\beta z_3^\gamma.
\end{aligned} \tag{3.8}$$

Then Equation (3.7) becomes

$$\begin{aligned}
0 &= e_{2A}+\overline{e_{2A}}+e_{2B}+e_{2C}+\overline{e_{2C}}\\
&\quad +e_{2F}+\overline{e_{2F}}+e_{2G}+\overline{e_{2G}}+e_{2H}+\overline{e_{2H}}+e_{2I}+\overline{e_{2I}}\\
&\quad +\frac{1}{2i}\Big(\sum_{\alpha>2\gamma} p_{2A}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma}(z_1+x_2)\bar z_1-\sum_{\alpha>2\gamma}\overline{p_{2A}^{\alpha\beta\gamma}}\bar z_1^\alpha x_2^\beta z_1^{2\gamma}(\bar z_1+x_2)z_1\Big)\\
&\quad +\frac{1}{2i}\Big(\sum p_{2B}^{\beta\gamma} z_1^{2\gamma} x_2^\beta \bar z_1^{2\gamma}(z_1+x_2)\bar z_1-\sum\overline{p_{2B}^{\beta\gamma}}\bar z_1^{2\gamma} x_2^\beta z_1^{2\gamma}(\bar z_1+x_2)z_1\Big)\\
&\quad +\frac{1}{2i}\Big(\sum_{\alpha>2\gamma} p_{2C}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma}-\sum_{\alpha>2\gamma}\overline{p_{2C}^{\alpha\beta\gamma}}\bar z_1^\alpha x_2^\beta z_1^{2\gamma}\Big)\\
&\quad +\frac{1}{2i}\Big(\sum p_{2D}^{\beta\gamma} z_1^{2\gamma} x_2^\beta \bar z_1^{2\gamma}-\sum\overline{p_{2D}^{\beta\gamma}}\bar z_1^{2\gamma} x_2^\beta z_1^{2\gamma}\Big)\\
&\quad +\frac{1}{2i}\Big(\sum_{\alpha<2\gamma} p_{2E}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma}-\sum_{\alpha<2\gamma}\overline{p_{2E}^{\alpha\beta\gamma}}\bar z_1^\alpha x_2^\beta z_1^{2\gamma}\Big).
\end{aligned}$$

By construction, the existence of a (formal) solution p_2 of this equation follows from a straightforward comparison of coefficients.

$$\begin{aligned}
\sum_{\alpha>2\gamma} p_{2A}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma}(z_1+x_2)\bar z_1 &= -2ie_{2F}\\
&= -2i\Big(1+\frac{x_2}{z_1}\Big)e_{2D}\\
\sum p_{2B}^{\beta\gamma} z_1^{2\gamma} x_2^\beta \bar z_1^{2\gamma}(z_1+x_2)\bar z_1 &= -2ie_{2H}\\
&= -i\Big(1+\frac{x_2}{z_1}\Big)e_{2E}\\
\sum_{\alpha>2\gamma} p_{2C}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma} &= -2i(e_{2A}+\overline{e_{2I}})\\
&= -2ie_{2A}+i\frac{x_2}{\bar z_1}e_{2E}\\
\sum p_{2D}^{\beta\gamma} z_1^{2\gamma} x_2^\beta \bar z_1^{2\gamma}-\sum\overline{p_{2D}^{\beta\gamma}}\bar z_1^{2\gamma} x_2^\beta z_1^{2\gamma} &= -2ie_{2B}\\
\sum_{\alpha<2\gamma} p_{2E}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar z_1^{2\gamma} &= -2i(\overline{e_{2C}}+\overline{e_{2G}})\\
&= -2i\overline{e_{2C}}+2i\frac{x_2}{\bar z_1}\overline{e_{2D}}.
\end{aligned}$$

This determines p_2 by giving a formula for each coefficient, using a convention that $e_2^{abc}=0$ if $c<0$, and recalling the normalization (3.8), so that the terms that do not appear in (3.8) have coefficient 0:

$$\begin{aligned}
p_{2A}^{\alpha\beta\gamma} &= -2ie_2^{\alpha+1,2\gamma+1,\beta}\\
p_{2B}^{\beta\gamma} &= -ie_2^{2\gamma+1,2\gamma+1,\beta}\\
p_{2C}^{\alpha\beta\gamma} &= -2ie_2^{\alpha,2\gamma,\beta},\text{ if }\alpha>2\gamma+1\\
&= -2ie_2^{\alpha,2\gamma,\beta}+ie_2^{2\gamma+1,2\gamma+1,\beta-1},\text{ if }\alpha=2\gamma+1\\
p_{2D}^{\beta\gamma} &= -ie_2^{2\gamma,2\gamma,\beta}\\
p_{2E}^{\alpha\beta\gamma} &= -2i\overline{e_2^{2\gamma,\alpha,\beta}}+2i\overline{e_2^{2\gamma+1,\alpha,\beta-1}}.
\end{aligned}$$

As mentioned earlier, only the terms of weight less than $2n-2$ are of interest, and we will not consider the higher order terms or issues of convergence, and instead only try to find formal series solutions of formal series equations. The remaining unknowns, p_1, p_3, p_4, will depend on all three functions E_2, e_3, e_4. We can assume that the solution we want will be of the form:

$$\begin{aligned}
p_1 &= p_1(z_1,z_2,z_3) = p_{1A}+p_{1B}+p_{1C}+p_{1D},\\
p_{1A} &= \sum p_{1A}^{\gamma} z_3^{\gamma},\\
p_{1B} &= \sum_{\alpha>0,\ \alpha\text{ even}} p_{1B}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma},\\
p_{1C} &= \sum_{\beta>0} p_{1C}^{\beta\gamma} z_2^{\beta} z_3^{\gamma},\\
p_{1D} &= \sum_{\alpha\text{ odd}} p_{1D}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma},\\
p_3 &= p_{3A}+p_{3B},\\
p_{3A} &= \sum p_{3A}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma},\\
p_{3B} &= \sum p_{3B}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma} z_4,\\
p_4 &= p_{4A}+p_{4B},\\
p_{4A} &= \sum p_{4A}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma},\\
p_{4B} &= \sum p_{4B}^{\alpha\beta\gamma} z_1^{\alpha} z_2^{\beta} z_3^{\gamma} z_4,
\end{aligned}\tag{3.9}$$

where p_1 has weight $n-1$, and does not depend on z_4, and p_3, p_4 have weight n.

We next turn to the third equation from the system (3.6),

$$\begin{aligned}
0 &= e_4(z_1,\bar z_1,x_2)+p_4(z_1,x_2,\bar z_1^2,(z_1+x_2)\bar z_1)\\
&\quad -(z_1+x_2)\overline{p_1(z_1,x_2,\bar z_1^2)}\\
&\quad -\bar z_1(p_1(z_1,x_2,\bar z_1^2)+\operatorname{Re}(p_2(z_1,x_2,\bar z_1^2,(z_1+x_2)\bar z_1))).
\end{aligned}\tag{3.10}$$

Since e_4 is given and p_2 has already been found, we combine these to get a quantity f_4, and then break the series into subseries:

$$\begin{aligned}
f_4 &= e_4(z_1,\bar z_1,x_2)-\frac{\bar z_1}{2}(p_2(z_1,x_2,\bar z_1^2,(z_1+x_2)\bar z_1)+\overline{p_2(z_1,x_2,\bar z_1^2,(z_1+x_2)\bar z_1)})\\
&= \sum f_4^{abc} z_1^a \bar z_1^b x_2^c\\
&= f_{4A}+f_{4B}+f_{4C}+f_{4D}+f_{4E},\\
f_{4A} &= \sum_{b\text{ even}} f_4^{abc} z_1^a \bar z_1^b x_2^c,\\
f_{4B} &= \sum_{a\text{ odd},\ b\text{ odd}} f_4^{abc} z_1^a \bar z_1^b x_2^c,\\
f_{4C} &= \sum_{a>0,\ a\text{ even},\ b\text{ odd}} f_4^{abc} z_1^a \bar z_1^b x_2^c,\\
f_{4D} &= \sum_{c>0,\ b\text{ odd}} f_4^{0bc} \bar z_1^b x_2^c,\\
f_{4E} &= \sum_{b\text{ odd}} f_4^{0b0} \bar z_1^b.
\end{aligned}$$

Note that e_4 has degree n and the quantity $\frac{\bar{z}_1}{2}\mathrm{Re}(p_2(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1))$ has degree $n + 1$. Equation (3.10) then becomes

$$0 = f_{4A} + f_{4B} + f_{4C} + f_{4D} + f_{4E} + p_4 - (z_1 + x_2)\overline{p_1} - \bar{z}_1(p_{1A} + p_{1B} + p_{1C} + p_{1D}),$$

and by inspection, there are only two parts of this expression with $\bar{z}_1^b$ terms, so we can conclude

$$0 = f_{4E} - \bar{z}_1 p_{1A} = \sum_{b \text{ odd}} f_4^{0b0}\bar{z}_1^b - \bar{z}_1 \sum p_{1A}^{\gamma}\bar{z}_1^{2\gamma},$$

and this determines the coefficients of p_{1A}: $p_{1A}^{\gamma} = f_4^{0,2\gamma+1,0}$. The remaining linear equation from the system (3.6) is

$$0 = e_3(z_1, \bar{z}_1, x_2) + p_3(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) - 2\bar{z}_1\overline{p_1(z_1, x_2, \bar{z}_1^2)} \tag{3.11}$$

and we break e_3 into subseries,

$$\begin{aligned}
e_3 &= e_{3A} + e_{3B} + e_{3C} + e_{3D} + e_{3E}, \\
e_{3A} &= \sum_{b \text{ even}} e_3^{abc} z_1^a \bar{z}_1^b x_2^c, \\
e_{3B} &= \sum_{a \text{ odd},\, b \text{ odd}} e_3^{abc} z_1^a \bar{z}_1^b x_2^c, \\
e_{3C} &= \sum_{a \text{ even},\, b \text{ odd},\, b>1} e_3^{abc} z_1^a \bar{z}_1^b x_2^c, \\
e_{3D} &= \sum_{a \text{ even},\, c>0} e_3^{a1c} z_1^a \bar{z}_1 x_2^c, \\
e_{3E} &= \sum_{a \text{ even},\, a>0} e_3^{a10} z_1^a \bar{z}_1,
\end{aligned}$$

and we further rearrange the e_{3B} quantity:

$$\begin{aligned}
e_{3B} &= e_{3F} + e_{3G} + e_{3H}, \\
e_{3F} &= \sum_{a \text{ odd},\, b \text{ odd}} e_3^{abc} z_1^{a-1}(z_1 + x_2)\bar{z}_1^b x_2^c, \\
e_{3G} &= -\sum_{a \text{ odd},\, b \text{ odd},\, b>1} e_3^{abc} z_1^{a-1} \bar{z}_1^b x_2^{c+1}, \\
e_{3H} &= -\sum_{a \text{ odd}} e_3^{a1c} z_1^{a-1} \bar{z}_1 x_2^{c+1}.
\end{aligned}$$

Equation (3.11) becomes

$$0 = e_{3A} + e_{3F} + e_{3G} + e_{3H} + e_{3C} + e_{3D} + e_{3E} + p_3 - 2\bar{z}_1(\overline{p_{1A}} + \overline{p_{1B}} + \overline{p_{1C}} + \overline{p_{1D}}),$$

and the only terms with monomials of the form $z_1^a\bar{z}_1$, with a even, are e_{3E} and the known quantity $\bar{z}_1\overline{p_{1A}}$, so we collect these together and get three new expressions:

$$\begin{aligned}
e_{3E}-2\bar{z}_1\overline{p_{1A}} &= \sum_{a \text{ even},\ a>0} f_3^a z_1^a \bar{z}_1 \\
&= \left((1+\frac{x_2}{z_1})-(1+\frac{x_2}{z_1})\frac{x_2}{z_1}+\left(\frac{x_2}{z_1}\right)^2\right)(e_{3E}-2\bar{z}_1\overline{p_{1A}}) \\
&= f_{3A}+f_{3B}+f_{3C}, \\
f_{3A} &= \sum_{a \text{ even},\ a>0} f_3^a z_1^{a-1}(z_1+x_2)\bar{z}_1 \\
f_{3B} &= -\sum_{a \text{ even},\ a>0} f_3^a z_1^{a-2}(z_1+x_2)\bar{z}_1 x_2 \\
f_{3C} &= \sum_{a \text{ even},\ a>0} f_3^a z_1^{a-2}\bar{z}_1 x_2^2.
\end{aligned}$$

Equation (3.11) becomes

$$0=e_{3A}+e_{3F}+e_{3G}+e_{3H}+e_{3C}+e_{3D}+f_{3A}+f_{3B}+f_{3C}+p_3-2\bar{z}_1(\overline{p_{1B}}+\overline{p_{1C}}+\overline{p_{1D}}).$$

The terms with monomials of the form $z_1^a\bar{z}_1^b x_2^c$ with a even, b odd, and $b>1$ are e_{3G}, e_{3C}, and $\bar{z}_1\overline{p_{1B}}$, and the terms with monomials of the form $z_1^a\bar{z}_1x_2^c$ with a even and $c>0$ are e_{3H}, e_{3D}, f_{3C}, and $z_1\overline{p_{1C}}$, giving

$$\begin{aligned}
0 &= e_{3G}+e_{3C}-2\bar{z}_1\overline{p_{1B}} \\
&= -\sum_{a \text{ odd},\ b \text{ odd},\ b>1} e_3^{abc} z_1^{a-1}\bar{z}_1^b x_2^{c+1} + \sum_{a \text{ even},\ b \text{ odd},\ b>1} e_3^{abc} z_1^a \bar{z}_1^b x_2^c \\
&\quad -2\bar{z}_1\overline{\sum_{\alpha>0,\ \alpha \text{ even}} p_{1B}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}}, \\
0 &= e_{3H}+e_{3D}+f_{3C}-2\bar{z}_1\overline{p_{1C}} \\
&= -\sum_{a \text{ odd}} e_3^{a1c} z_1^{a-1}\bar{z}_1 x_2^{c+1} + \sum_{a \text{ even},\ c>0} e_3^{a1c} z_1^a \bar{z}_1 x_2^c \\
&\quad + \sum_{a \text{ even},\ a>0} f_3^a z_1^{a-2}\bar{z}_1 x_2^2 - 2\bar{z}_1\overline{\sum_{\beta>0} p_{1C}^{\beta\gamma} x_2^\beta \bar{z}_1^{2\gamma}}.
\end{aligned}$$

This determines the coefficients of p_{1B} and p_{1C}, again using the convention that $e_3^{abc}=0$ if $c<0$, and that the following formulas apply only to terms that appear in the normalization (3.9):

$$\begin{aligned}
p_{1B}^{\alpha\beta\gamma} &= \frac{1}{2}\overline{e_3^{2\gamma,\alpha+1,\beta}}-\frac{1}{2}\overline{e_3^{2\gamma+1,\alpha+1,\beta-1}}, \\
p_{1C}^{\beta\gamma} &= \begin{cases} \frac{1}{2}\overline{e_3^{2\gamma,1,\beta}}-\frac{1}{2}\overline{e_3^{2\gamma+1,1,\beta-1}} & \text{if } \beta\neq 2, \\ \frac{1}{2}\overline{e_3^{2\gamma,1,2}}-\frac{1}{2}\overline{e_3^{2\gamma+1,1,1}}+\frac{1}{2}\overline{f_3^{2\gamma+2}} & \text{if } \beta=2. \end{cases}
\end{aligned}$$

Returning to Equation (3.10),

$$0=f_{4A}+f_{4B}+f_{4C}+f_{4D}+f_{4E}+p_4-(z_1+x_2)\overline{p_1}-\bar{z}_1(p_{1A}+p_{1B}+p_{1C}+p_{1D}),$$

the only terms with monomials of the form $z_1^a\bar{z}_1^b x_2^c$, with a even and positive, and b odd, are f_{4C} and $\bar{z}_1 p_{1B}$, and the only terms with monomials of the form $\bar{z}_1^b x_2^c$, with b odd and c positive, are f_{4D} and $\bar{z}_1 p_{1C}$. So, we collect these together and rearrange:

$$\begin{aligned}
f_{4C} - \bar{z}_1 p_{1B} &= \sum_{a \text{ even},\ a>0,\ b \text{ odd}} g_4^{abc} z_1^a \bar{z}_1^b x_2^c \\
&= (1+\frac{x_2}{z_1})(f_{4C} - \bar{z}_1 p_{1B}) - \frac{x_2}{z_1}(f_{4C} - \bar{z}_1 p_{1B}) = g_{4A} + g_{4B}, \\
g_{4A} &= \sum_{a \text{ even},\ a>0,\ b \text{ odd}} g_4^{abc} z_1^{a-1}(z_1+x_2)\bar{z}_1^b x_2^c, \\
g_{4B} &= -\sum_{a \text{ even},\ a>0,\ b \text{ odd}} g_4^{abc} z_1^{a-1}\bar{z}_1^b x_2^{c+1}, \\
f_{4D} - \bar{z}_1 p_{1C} &= \sum_{c>0,\ b \text{ odd}} g_4^{0bc} \bar{z}_1^b x_2^c \\
&= (1+\frac{z_1}{x_2})(f_{4D} - \bar{z}_1 p_{1C}) - \frac{z_1}{x_2}(f_{4D} - \bar{z}_1 p_{1C}) = g_{4C} + g_{4D}, \\
g_{4C} &= \sum_{c>0,\ b \text{ odd}} g_4^{0bc} \bar{z}_1^b (z_1+x_2) x_2^{c-1}, \\
g_{4D} &= -\sum_{c>0,\ b \text{ odd}} g_4^{0bc} z_1 \bar{z}_1^b x_2^{c-1}.
\end{aligned}$$

Note that the last rearrangement is different from the previous ones. Equation (3.10) then becomes

$$\begin{aligned}
0 &= f_{4A} + f_{4B} + g_{4A} + g_{4B} + g_{4C} + g_{4D} + f_{4E} + p_{4A} + p_{4B} \\
&\quad -(z_1+x_2)(\overline{p_{1A}} + \overline{p_{1B}} + \overline{p_{1C}} + \overline{p_{1D}}) - \bar{z}_1(p_{1A} + p_{1D}).
\end{aligned}$$

The only terms with monomials of the form $z_1^a \bar{z}_1^b x_2^c$, with a and b both odd, are f_{4B}, g_{4B}, g_{4D}, and $\bar{z}_1 p_{1D}$, and collecting these terms gives

$$\begin{aligned}
0 &= f_{4B} + g_{4B} + g_{4D} - \bar{z}_1 p_{1D} \\
&= \sum_{a \text{ odd},\ b \text{ odd}} f_4^{abc} z_1^a \bar{z}_1^b x_2^c - \sum_{a \text{ even},\ a>0,\ b \text{ odd}} g_4^{abc} z_1^{a-1} \bar{z}_1^b x_2^{c+1} \\
&\quad - \sum_{c>0,\ b \text{ odd}} g_4^{0bc} z_1 \bar{z}_1^b x_2^{c-1} - \sum_{\alpha \text{ odd}} p_{1D}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma+1},
\end{aligned}$$

which determines p_{1D}:

$$p_{1D}^{\alpha\beta\gamma} = \begin{cases} f_4^{\alpha,2\gamma+1,\beta} - g_4^{\alpha+1,2\gamma+1,\beta-1} & \text{if } \alpha > 1, \\ f_4^{1,2\gamma+1,\beta} - g_4^{2,2\gamma+1,\beta-1} - g_4^{0,2\gamma+1,\beta+1} & \text{if } \alpha = 1. \end{cases}$$

The only remaining unknowns in this equation are p_{4A} and p_{4B}. The terms involving monomials of the form $z_1^a x_2^b (z_1+x_2)\bar{z}_1^b$, with b odd, are g_{4A}, g_{4C}, the known quantity $(z_1+x)\overline{p_{1D}}$, and p_{4B}, and collecting these terms gives

$$\begin{aligned}
0 &= g_{4A} + g_{4C} - (z_1+x_2)\overline{p_{1D}} + p_{4B} \\
&= \sum_{a \text{ even},\ a>0,\ b \text{ odd}} g_4^{abc} z_1^{a-1}(z_1+x_2)\bar{z}_1^b x_2^c + \sum_{c>0,\ b \text{ odd}} g_4^{0bc} \bar{z}_1^b (z_1+x_2) x_2^{c-1} \\
&\quad -(z_1+x_2) \sum_{\alpha \text{ odd}} \overline{p_{1D}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}} + \sum p_{4B}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}(z_1+x_2)\bar{z}_1,
\end{aligned}$$

which determines p_{4B}. The only terms left are those with even powers of $\bar{z}_1$:

$$\begin{aligned} 0 &= f_{4A} + p_{4A} - (z_1 + x_2)(\overline{p_{1A}} + \overline{p_{1B}} + \overline{p_{1C}}) \\ &= \sum_{b \text{ even}} f_4^{abc} z_1^a \bar{z}_1^b x_2^c + \sum p_{4A}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma} \\ &\quad -(z_1 + x_2)\left(\overline{\sum p_{1A}^\gamma \bar{z}_1^{2\gamma}} + \overline{\sum_{\alpha>0,\ \alpha \text{ even}} p_{1B}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}} + \overline{\sum_{\beta>0} p_{1C}^{\beta\gamma} x_2^\beta \bar{z}_1^{2\gamma}}\right), \end{aligned}$$

which determines p_{4A}.

Similarly for (3.11), the only remaining unknowns are p_{3A} and p_{3B}. The terms involving monomials of the form $z_1^a x_2^b (z_1 + x_2)\bar{z}_1^b$, with b odd, are e_{3F}, f_{3A}, f_{3B}, and p_{3B}, and collecting these terms gives

$$\begin{aligned} 0 &= e_{3F} + f_{3A} + f_{3B} + p_{3B} \\ &= \sum_{a \text{ odd},\ b \text{ odd}} e_3^{abc} z_1^{a-1}(z_1 + x_2)\bar{z}_1^b x_2^c + \sum_{a \text{ even},\ a>0} f_3^a z_1^{a-1}(z_1 + x_2)\bar{z}_1 \\ &\quad - \sum_{a \text{ even},\ a>0} f_3^a z_1^{a-2}(z_1 + x_2)\bar{z}_1 x_2 + \sum p_{3B}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}(z_1 + x_2)\bar{z}_1, \end{aligned}$$

which determines p_{3B}. The only terms left are those with even powers of $\bar{z}_1$:

$$\begin{aligned} 0 &= e_{3A} + p_{3A} - 2\bar{z}_1 \overline{p_{1D}} \\ &= \sum_{b \text{ even}} e_3^{abc} z_1^a \bar{z}_1^b x_2^c + \sum p_{3A}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma} - 2\bar{z}_1 \overline{\sum_{\alpha \text{ odd}} p_{1D}^{\alpha\beta\gamma} z_1^\alpha x_2^\beta \bar{z}_1^{2\gamma}}, \end{aligned}$$

which determines p_{3A}.

The conclusion is that there exist formal series p_1, p_2, p_3, p_4 that solve the system (3.6), for any given E_2, e_3, e_4, and that the coefficients of the solutions are complex linear combinations of the coefficients of E_2, e_3, e_4 and their complex conjugates. As remarked at the beginning of the Proof, since the linear system has a formal solution, the non-linear system (3.5) has a formal solution also. ∎

Conjecture 3.2. *Given M with the non-degenerate quadratic normal form (type* (I)*, as in* (3.1)*), there exist convergent series p_1, p_2, p_3, p_4 which are solutions of the system of equations* (3.5)*, so that M is analytically equivalent to the polynomial model.* ∎

Remark. As evidence in favor of the conjecture, note that among all the algebraic manipulations of the $\vec{e}$ series in the above Proof, including the decompositions into subseries, none drastically shrinks the radius of convergence. The form of the solution $\vec{p}$ of the linear problem suggests that its domain contains some polydisc in $\mathbb{C}^4$ with radius lengths that depend on the radius lengths of the polydisc of convergence of the given quantity $\vec{e}$, in such a way that the new defining equations in the transformed coordinates will have a domain not much smaller than the domain of the original equations (after applying a suitable version of Lemma 2.3). When similar situations have arisen in other normal form problems ([M], [C$_2$]), the technique of rapid convergence has been used to prove that the sequence of compositions of approximate solutions converges to a holomorphic normalizing transformation on an open neighborhood of $\vec{0}$.

4. Some degenerate cases

In this Section we consider M satisfying the first non-degeneracy condition but not the second, so after a coordinate transformation putting the quadratic terms into normal form (II), the defining equations become

$$\begin{aligned} y_2 &= E_2(z_1,\bar z_1,x_2) \\ z_3 &= \bar z_1^2 + e_3(z_1,\bar z_1,x_2) \\ z_4 &= z_1\bar z_1 + e_4(z_1,\bar z_1,x_2), \end{aligned} \tag{4.1}$$

where E_2, e_3, e_4 have degree 3. This differs from the non-degenerate case only in that it is missing a $\bar z_1 x_2$ term.

Theorem 4.1. *Given a manifold M with the type* (II) *normal form, exactly one of the following two cases must hold:*

(1) *There exists a unique integer $N \geq 3$ and a holomorphic coordinate change such that the defining equations in the new coordinates are*

$$\begin{aligned} y_2 &= O(N+1) \\ z_3 &= \bar z_1^2 + O(N+1) \\ z_4 &= z_1\bar z_1 + x_2^{N-1}\bar z_1 + O(N+1). \end{aligned}$$

(2) *For any integer $n \geq 2$, there exists a holomorphic coordinate change such that the defining equations in the new coordinates are*

$$\begin{aligned} y_2 &= O(n+1) \\ z_3 &= \bar z_1^2 + O(n+1) \\ z_4 &= z_1\bar z_1 + O(n+1). \end{aligned}$$

In case (2), M is formally equivalent to the real algebraic variety $\{y_2 = 0, z_3 = \bar z_1^2, z_4 = z_1\bar z_1\}$. It could be denoted the "$N = \infty$" case.

Proof. By hypothesis, there is some $N \geq 3$ and some coordinate system where the defining equations for M are:

$$\begin{aligned} y_2 &= E_2, \\ z_3 &= \bar z_1^2 + e_3 \\ z_4 &= z_1\bar z_1 + e_4, \end{aligned}$$

where (E_2, e_3, e_4) has degree N. A coordinate transformation of the form (2.3), where p_1 has weight $N-1$, and p_2, p_3, p_4 have weight N, gives:

$$\begin{aligned} \operatorname{Im}(\tilde z_2) &= \operatorname{Im}(z_2+p_2) = E_2 + \operatorname{Im}(p_2(z_1, x_2+iE_2, \bar z_1^2+e_3, z_1\bar z_1+e_4)) \\ &= E_2 + \operatorname{Im}(p_2(z_1,x_2,\bar z_1^2, z_1\bar z_1)) + O(2N-1) \\ \tilde z_3 - \bar{\tilde z}_1^2 &= e_3 + p_3 - 2\bar z_1\overline{p_1} - \overline{p_1}^2 \\ &= e_3 + p_3(z_1,x_2,\bar z_1^2,z_1\bar z_1) - 2\bar z_1\overline{p_1(z_1,x_2,\bar z_1^2,z_1\bar z_1)} + O(2N-2) \\ \tilde z_4 - \tilde z_1\bar{\tilde z}_1 &= e_4 + p_4 - z_1\overline{p_1} - \bar z_1 p_1 - p_1\overline{p_1} \\ &= e_4 + p_4(z_1,x_2,\bar z_1^2,z_1\bar z_1) - \bar z_1 p_1(z_1,x_2,\bar z_1^2,z_1\bar z_1) \\ &\quad - z_1\overline{p_1(z_1,x_2,\bar z_1^2,z_1\bar z_1)} + O(2N-2). \end{aligned}$$

Any term in E_2 of the form $z_1^a\bar z_1^b x_2^c$ with $a > b$ can be eliminated by some term of p_2 of the form $z_1^a z_2^c z_3^{b/2}$ if b is even, or $z_1^{a-1}z_2^c z_3^{(b-1)/2}z_4$ if b is odd. Since E_2

is real-valued, the $a > b$ terms are matched by their complex conjugates, which are canceled by $\overline{p_2}$. Similarly, the $z_1^a \bar{z}_1^a x_2^c$ terms, with real coefficients, can also be eliminated by p_2 terms with a either even or odd.

Consider p_1 of the form $p_{1A} + p_{1B}$ with $p_{1A} = \sum p_{1A}^{jk} z_1^{2j} z_2^k$, $p_{1B} = \sum p_{1B}^{jk} z_2^k z_3^{j>1}$. The degree N terms of e_3 that have the form $\bar{z}_1^{odd} x_2^k$ can be eliminated by p_{1A} terms of the form $z_1^{odd-1} z_2^k$. Then the degree N terms of $e_4 - \bar{z}_1 p_{1A}$ that have the form $\bar{z}_1^{odd} x_2^k$ can be eliminated by p_{1B} terms of the form $z_2^k z_4^{(odd-1)/2}$, except for the $\bar{z}_1 x_2^{N-1}$ term, since that component of p_1 was already used, and the remaining terms of $e_4 - z_1\overline{p_1} - \bar{z}_1 p_{1A}$ can be eliminated by p_4. The remaining terms of $e_3 - 2\bar{z}_1\overline{p_{1B}}$, none of which are of the form $\bar{z}_1^{odd} x_2^k$, can be eliminated by p_3. A linear transformation of the form $\tilde{z}_1 = c_1 z_1$, $\tilde{z}_2 = z_2$, $\tilde{z}_3 = \overline{c_1}^2 z_3$, $\tilde{z}_4 = c_1\overline{c_1} z_4$, will not alter the coefficients on the quadratic terms, but can normalize the coefficient of $\bar{z}_1 x_2^{N-1}$ in e_4 to either 1, in which case the existence of N in Case (1) is established, or else 0, in which case we return to the beginning of the Proof with N increased by one, to establish either Case (1) for some larger N, or Case (2) by induction.

For the uniqueness from Case (1), suppose the defining equations are of the form

$$\begin{aligned} y_2 &= E_2, \\ z_3 &= \bar{z}_1^2 + e_3 \\ z_4 &= z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + e_4, \end{aligned}$$

where (E_2, e_3, e_4) has degree $N+1$. A coordinate transformation of the form (2.3), with p_1 weight 2, p_2, p_3, p_4 weight 3, gives:

$$\begin{aligned} \mathrm{Im}(\tilde{z}_2) &= \mathrm{Im}(z_2 + p_2) \\ &= E_2 + \mathrm{Im}(p_2(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + e_4)) \\ &= E_2 + \mathrm{Im}(p_2(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1)) + O(N+1) \\ \tilde{z}_3 - \bar{\tilde{z}}_1^2 &= e_3 + p_3 - 2\bar{z}_1\overline{p_1} - \overline{p_1}^2 \\ &= e_3 + p_3(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1) - 2\bar{z}_1\overline{p_1(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1)} - \overline{p_1}^2 + O(N+1) \\ \tilde{z}_4 - \tilde{z}_1\bar{\tilde{z}}_1 &= x_2^{N-1}\bar{z}_1 + e_4 + p_4 - z_1\overline{p_1} - \bar{z}_1 p_1 - p_1\overline{p_1} \\ &= x_2^{N-1}\bar{z}_1 + e_4 + p_4(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1) - \bar{z}_1 p_1(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1) \\ &\quad - z_1\overline{p_1(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1)} - p_1\overline{p_1} + O(N+1). \end{aligned} \tag{4.2}$$

If there were some coordinate change taking the degree N normal form to the normal form of some other degree N', we can assume (by switching) that $N' > N$, so the coordinate change would eliminate all the terms of degree $\le N$, including the $\bar{z}_1 x_2^{N-1}$ term, from (4.2). Note that to get to (4.2) from the line before it, we are replacing $p_k(z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + e_4)$ by $p_k(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1)$, but even when p_k has weight less than N, the difference between these quantities has degree at least $N+1$. The previous part of the Proof showed that the z_2^k term of p_1 is the only term out of p_1, p_2, p_3, p_4 that can eliminate a $\bar{z}_1 x_2^{N-1}$ term from e_3 or e_4 from the simplified equations, and it cannot always do both. However, to establish uniqueness, we must consider the non-linear components also — there is a possibility that some lower weight terms of $\vec{p}$ could contribute to the $\bar{z}_1 x_2^{N-1}$ term. Considering the products $\overline{p_1}^2$, and $p_1\overline{p_1}$, the only way a term of the form $\bar{z}_1 x_2^{N-1}$ could appear is if there were terms $p_1^a z_2^a$ and $p_1^b z_1 z_2^b$, with $a + b = N - 1$. However, a non-zero coefficient on the $p_1^a z_2^a$ term would contribute a term of the form $\bar{z}_1 x_2^a$

for some $a < N-1$ to both the e_3 and e_4 expressions, which could not be canceled by any of the other terms from any p_k. ∎

It follows from the above Proof that for manifolds with a type (II) quadratic normal form, the value of N, $3 \le N \le \infty$, is an invariant of M under formal coordinate changes, and each equivalence class indexed by $3 \le N \le \infty$ is obviously non-empty, with a polynomial model as a representative manifold.

Theorem 4.2. *Given $N \ge 3$, suppose a manifold M is in the normal form from Case (1) of Theorem 4.1:*

$$\begin{aligned} y_2 &= O(N+1) \\ z_3 &= \bar{z}_1^2 + O(N+1) \\ z_4 &= z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + O(N+1). \end{aligned}$$

Then, for any $n \ge N$, there exists a holomorphic coordinate change such that the defining equations in the new coordinates are

$$\begin{aligned} y_2 &= O(n+1) \\ z_3 &= \bar{z}_1^2 + O(n+1) \\ z_4 &= z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + O(n+1). \end{aligned}$$

That is, M is formally equivalent to the real algebraic variety $\{y_2 = 0, z_3 = \bar{z}_1^2, z_4 = z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1\}$.

Proof. The $n = N$ case is trivial and also the start of an induction: we will show that if the defining equations are of the form

$$\begin{aligned} y_2 &= E_2(z_1, \bar{z}_1, x_2) \\ z_3 &= \bar{z}_1^2 + e_3(z_1, \bar{z}_1, x_2) \\ z_4 &= z_1\bar{z}_1 + x_2^{N-1}\bar{z}_1 + e_4(z_1, \bar{z}_1, x_2). \end{aligned}$$

with (E_2, e_3, e_4) degree $n > N$, then there exists a transformation of the form (2.3) which eliminates the degree n terms, so that the (E_2, e_3, e_4) expression in the new coordinates has degree at least $n+1$.

For points $\vec{z} = (z_1, x_2 + iE_2, \bar{z}_1^2 + e_3, (z_1 + x_2^{N-1})\bar{z}_1 + e_4)$ on M,

$$\begin{aligned} \mathrm{Im}(\tilde{z}_2) &= \mathrm{Im}(z_2 + p_2(\vec{z})) \\ &= E_2(z_1, \bar{z}_1, x_2) + \mathrm{Im}(p_2(\vec{z})), \end{aligned}$$

$$\begin{aligned} \tilde{z}_3 - \bar{\tilde{z}}_1^2 &= z_3 + p_3(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2 \\ &= e_3 + p_3 - 2\bar{z}_1\overline{p_1} - \overline{p_1}^2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\tilde{z}_4 - (\tilde{z}_1 + \tilde{x}_2^{N-1})\bar{\tilde{z}}_1 \\ = {}& z_4 + p_4(\vec{z}) - (z_1 + p_1(\vec{z}) + (x_2 + \mathrm{Re}(p_2(\vec{z})))^{N-1})\overline{(z_1 + p_1(\vec{z}))} \\ = {}& e_4 + p_4 - (z_1 + x_2^{N-1})\overline{p_1} - \bar{z}_1 p_1 - \bar{z}_1((x_2 + \mathrm{Re}(p_2))^{N-1} - x_2^{N-1}) \\ & -p_1\overline{p_1} - ((x_2 + \mathrm{Re}(p_2))^{N-1} - x_2^{N-1})\overline{p_1}. \end{aligned} \tag{4.4}$$

Since we only want to eliminate the degree n terms, we will be able to ignore any expressions of degree $\ge n+1$. However, the inhomogeneity of the "normal form" part of the equations makes this more complicated than the situation from Section 3. In particular, we will need some of the terms of $\vec{p}$ to have weight less than n.

To start, we make the following choices for the form of the coordinate transformation:

$$\begin{aligned}
p_1(z_1,z_2,z_3) &= p_{1A}z_2^{n-1}+p_{1B}(z_1,z_2)+p_{1C}z_1z_2^{n-N}+p_{1D}(z_2,z_3)\\
p_{1B} &= \sum_{k+2j=n-1,j>0} p_{1B}^{jk}z_1^{2j}z_2^k\\
p_{1D} &= \sum_{k+2j=n-1,j>0} p_{1D}^{kj}z_2^kz_3^j\\
p_2(z_1,z_2,z_3,z_4) &= p_{2A}(z_1,z_2,z_3)+z_4p_{2B}(z_1,z_2,z_3)\\
p_3(z_1,z_2,z_3,z_4) &= p_{3A}z_2^{n-N}z_3+p_{3B}z_2^{2(n-N)}z_3\\
&\quad +p_{3C}(z_1,z_2,z_3)+z_4p_{3D}(z_1,z_2,z_3)\\
p_4(z_1,z_2,z_3,z_4) &= p_{4A}z_2^{n-N}z_4+p_{4B}z_2^{2(n-N)}z_4\\
&\quad +p_{4C}(z_1,z_2,z_3)+z_4p_{4D}(z_1,z_2,z_3).
\end{aligned}$$

The weight of p_{1C} is $n-N+1$, which is in the interval $[2,n-2]$, and the weights of p_{3A} and p_{4A} are one higher, $n-N+2$. The weights of p_{3B} and p_{4B} are each $2(n-N+1)$, which is in the interval $[4,2n-4]$. The p_{1A}, p_{1B}, p_{1D} parts of p_1 have weight $n-1$, and the p_2, $p_{3C}+z_4p_{3D}$ and $p_{4C}+z_4p_{4D}$ quantities are chosen to have weight n. If the weight, $2(n-N+1)$, of p_{3B} and p_{4B} is greater than n, these quantities can be ignored for the purposes of this Proof. If $2(n-N+1)=n$, then a $z_2^{2(n-N)}z_3$ term could be included in the p_{3C} expression and a $z_2^{2(n-N)}z_4$ in the z_4p_{3D} expression, so there is again no need to consider separately the p_{3B} and p_{4B} terms.

Considering the lowest weight quantity, the difference between $p_{1C}z_1(x_2+iE_2)^{n-N}$ and $p_{1C}z_1x_2^{n-N}$ has degree $(n-N+1)-1+n=2n-N\geq n+1$. Similarly, the differences $p_{3A}(x_2+iE_2)^{n-N}(\bar{z}_1^2+e_3)-p_{3A}x_2^{n-N}\bar{z}_1^2$ and $p_{4A}(x_2+iE_2)^{n-N}(z_1\bar{z}_1+x_2^{N-1}\bar{z}_1+e_4)-p_{4A}x_2^{n-N}(z_1\bar{z}_1+x_2^{N-1}\bar{z}_1)$ have degree $(n-N+2)-2+n=2n-N\geq n+1$. The E_2, e_3, e_4 quantities similarly contribute only high degree terms to the expressions with higher weights, so in (4.3) and (4.4), we can replace $\vec{z}$ with $(z_1,x_2,\bar{z}_1^2,(z_1+x_2^{N-1})\bar{z}_1)$, and include the differences in a "$+O(n+1)$" quantity.

The quantity $((x_2+\mathrm{Re}(p_2(\vec{z})))^{N-1}-x^{N-1})$ from (4.4) has degree $N-2+n\geq n+1$, so the terms with that as a factor can also be included in the $O(n+1)$ quantity. Expanding $\overline{p_1(z_1,x_2+iE_2,z_1\bar{z}_1+e_3)}^2$, the only term that might have degree n or less is $\overline{(p_{1C}z_1x_2^{n-N})}^2$ and similarly, the only term of

$$p_1(z_1,x_2+iE_2,z_1\bar{z}_1+e_3)\overline{p_1(z_1,x_2+iE_2,z_1\bar{z}_1+e_3)}$$

that might have degree n or less is $(p_{1C}z_1x_2^{n-N})\overline{(p_{1C}z_1x_2^{n-N})}$. All the other parts of the products have higher degree, such as $\overline{p_{1A}p_{1C}}$, which has degree $(n-1)+(n-N+1)=2n-N\geq n+1$. There is also a term $x_2^{N-1}\overline{p_1(\vec{z})}$ in (4.4), and the part that has the form

$$x_2^{N-1}\overline{(p_{1A}(x_2+iE_2)^{n-1}+p_{1B}(z_1,x_2+iE_2)+p_{1D}(x_2+iE_2,z_1\bar{z}_1+e_3))}$$

has degree $N-1+n-1\geq n+1$. So, (4.3) and (4.4) simplify to:

$$\begin{aligned}
\tilde{z}_3-\bar{\tilde{z}}_1^2 &= e_3(z_1,\bar{z}_1,x_2)+p_3(z_1,x_2,\bar{z}_1^2,(z_1+x_2^{N-1})\bar{z}_1)\\
&\quad -2\bar{z}_1\overline{p_1(z_1,x_2,\bar{z}_1^2)}-\overline{(p_{1C}z_1x_2^{n-N})}^2+O(n+1),
\end{aligned}$$

$$\begin{aligned}\tilde z_4-(\tilde z_1+\tilde x_2^{N-1})\bar{\tilde z}_1 &= e_4(z_1,\bar z_1,x_2)+p_4(z_1,x_2,\bar z_1^2,(z_1+x_2^{N-1})\bar z_1)\\ &\quad -z_1\overline{p_1(z_1,x_2,\bar z_1^2)}-\bar z_1 p_1(z_1,x_2,\bar z_1^2)\\ &\quad -x_2^{N-1}\overline{(p_{1C}z_1x_2^{n-N})}\\ &\quad -(p_{1C}z_1x_2^{n-N})\overline{(p_{1C}z_1x_2^{n-N})}+O(n+1).\end{aligned}$$

When substituted into the weight n expressions $z_4p_{2B}(z_1,z_2,z_3)$, $z_4p_{3D}(z_1,z_2,z_3)$, and $z_4p_{4D}(z_1,z_2,z_3)$, the quantity $z_4=z_1\bar z_1+x_2^{N-1}\bar z_1+e_4$ can be replaced with $z_1\bar z_1$, since the difference has degree $n-2+N\ge n+1$. Similarly, $p_{4B}(x_2+iE_2)^{2(n-N)}(z_1\bar z_1+x_2^{N-1}\bar z_1+e_4)$ differs from $p_{4B}x_2^{2(n-N)}z_1\bar z_1$ only by a quantity with degree $n+1$. However, in the weight $n-N+2$ expression $p_{4A}(x_2+iE_2)^{n-N}(z_1\bar z_1+x_2^{N-1}\bar z_1+e_4)$, the $x_2^{N-1}\bar z_1$ part does contribute a degree n term, so we can't replace z_4 with just $z_1\bar z_1$, but the difference $p_{4A}(x_2+iE_2)^{n-N}(z_1\bar z_1+x_2^{N-1}\bar z_1+e_4)-p_{4A}x_2^{n-N}(z_1\bar z_1+x_2^{N-1}\bar z_1)$ has degree $2n-N\ge n+1$.

Finally, breaking e_3 and e_4 into convenient subseries, we get the simplified (but still non-linear if $2(n-N)+2\le n$) expressions:

$$\mathrm{Im}(\tilde z_2) = E_2(z_1,\bar z_1,x_2)+\mathrm{Im}(p_2(z_1,x_2,\bar z_1^2,z_1\bar z_1))+O(n+1),$$

$$\tilde z_3-\bar{\tilde z}_1^2$$

$$= \sum_{b\text{ even}} e_3^{abc}z_1^a\bar z_1^bx_2^c+p_{3C}(z_1,x_2,\bar z_1^2) \tag{4.5}$$

$$+\sum_{b\text{ odd},\ a>0} e_3^{abc}z_1^a\bar z_1^bx_2^c-2\bar z_1\overline{\sum_{j>0}p_{1D}^{kj}x_2^k\bar z_1^{2j}}+z_1\bar z_1p_{3D}(z_1,x_2,\bar z_1^2) \tag{4.6}$$

$$+\sum_{b\text{ odd},\ b>1} e_3^{0bc}\bar z_1^bx_2^c-2\bar z_1\overline{\sum_{j>0}p_{1B}^{jk}z_1^{2j}x_2^k} \tag{4.7}$$

$$+e_3^{0,1,n-1}\bar z_1x_2^{n-1}-2\bar z_1\overline{(p_{1A}x_2^{n-1})} \tag{4.8}$$

$$+p_{3A}x_2^{n-N}\bar z_1^2-2\bar z_1\overline{(p_{1C}z_1x_2^{n-N})} \tag{4.9}$$

$$+p_{3B}x_2^{2(n-N)}\bar z_1^2-\overline{(p_{1C}z_1x_2^{n-N})}^2+O(n+1), \tag{4.10}$$

$$\tilde z_4-(\tilde z_1+\tilde x_2^{N-1})\bar{\tilde z}_1$$

$$= \sum_{b\text{ even}} e_4^{abc}z_1^a\bar z_1^bx_2^c-z_1\overline{(p_{1A}x_2^{n-1})}-z_1\overline{\sum_{j>0}p_{1B}^{jk}z_1^{2j}x_2^k}$$

$$-z_1\overline{\sum_{j>0}p_{1D}^{kj}x_2^k\bar z_1^{2j}}+p_{4C}(z_1,x_2,\bar z_1^2) \tag{4.11}$$

$$+\sum_{b\text{ odd},\ a>0} e_4^{abc}z_1^a\bar z_1^bx_2^c-\bar z_1\sum_{j>0}p_{1B}^{jk}z_1^{2j}x_2^k+z_1\bar z_1p_{4D}(z_1,x_2,\bar z_1^2) \tag{4.12}$$

$$+\sum_{b\text{ odd},\ b>1} e_4^{0bc}\bar z_1^bx_2^c-\bar z_1\sum_{j>0}p_{1D}^{kj}x_2^k\bar z_1^{2j} \tag{4.13}$$

$$+e_4^{0,1,n-1}\bar z_1x_2^{n-1}-\bar z_1p_{1A}x_2^{n-1}-x_2^{N-1}\overline{(p_{1C}z_1x_2^{n-N})}+p_{4A}x_2^{n-N}x_2^{N-1}\bar z_1 \tag{4.14}$$

$$+p_{4A}x_2^{n-N}z_1\bar z_1-\bar z_1p_{1C}z_1x_2^{n-N}-z_1\overline{(p_{1C}z_1x_2^{n-N})} \tag{4.15}$$

$$+p_{4B}x_2^{2(n-N)}z_1\bar z_1-(p_{1C}z_1x_2^{n-N})\overline{(p_{1C}z_1x_2^{n-N})}+O(n+1), \tag{4.16}$$

We note first that it is easy to find $p_2(z_1,z_2,z_3,z_4)$ such that in a coordinate system where $\tilde{z}_2 = z_2 + p_2$, $\text{Im}(\tilde{z}_2) = O(n+1)$. As in the Proof of Theorem 4.1, any (formally) real-valued series $E_2(z_1,\bar{z}_1,x_2)$ can be canceled by an expression of the form $\text{Im}(p_2(z_1,x_2,\bar{z}_1^2,z_1\bar{z}_1))$.

The other two expressions above have been grouped together to see how to choose the coefficients of $\vec{p}$. So that there are no terms of degree $< n$, we have to choose $p_{3A} = 2\overline{p_{1C}}$ to eliminate line (4.9) and $p_{4A} = p_{1C} + \overline{p_{1C}}$ for line (4.15). If $2n - 2N + 2 < n$, we have to choose $p_{3B} = \overline{p_{1C}}^2$ to eliminate line (4.10) and $p_{4B} = p_{1C}\overline{p_{1C}}$ for line (4.16). If $2n - 2N + 2 > n$, the quadratic quantities can be put with the $O(n+1)$ (as previously mentioned, there is no need for p_{3B}, p_{4B}). If $2n - 2N + 2 = n$, then $\overline{(p_{1C}z_1x_2^{n-N})}^2$ has the same form as the terms in (4.5) and $(p_{1C}z_1x_2^{n-N})\overline{(p_{1C}z_1x_2^{n-N})}$ has the same form as the terms in (4.12).

To cancel the degree n terms in (4.8), $p_{1A} = \frac{1}{2}\overline{e_3^{0,1,n-1}}$, and p_{1B} is similarly determined by (4.7). The choices for p_{1A} and p_{4A} turn (4.14) into

$$(e_4^{0,1,n-1} - \frac{1}{2}\overline{e_3^{0,1,n-1}} - \overline{p_{1C}} + (p_{1C} + \overline{p_{1C}}))\bar{z}_1 x_2^{n-1},$$

so $p_{1C} = \frac{1}{2}\overline{e_3^{0,1,n-1}} - e_4^{0,1,n-1}$. Line (4.13) determines p_{1D}, and then the remaining lines, (4.5), (4.6), (4.11), (4.12) (including the quadratic terms if $2n - 2N + 2 = n$) can clearly be eliminated by some p_{3C}, p_{3D}, p_{4C}, p_{4D}. ∎

Remark. In the $N \geq 3$ cases, no conjecture is offered on the analytic equivalence of M and the real algebraic model. Unlike the approximate solution from $N = 2$ case of the previous Section, which nearly doubled the degree of the defining equations $\vec{e}$, the above construction only increases the degree of the defining equation by one. The construction does not seem to be as well-suited as that of the $N = 2$ case for a straightforward application of the rapid convergence technique.

5. Conclusion

For $N \geq 2$, let M_N denote the following real algebraic affine subvariety of $\mathbb{C}^n$:

$$M_N = \{y_2 = 0,\ z_3 = \bar{z}_1^2, z_4 = z_1\bar{z}_1 + \bar{z}_1 x_2^{N-1}\}.$$

Note that M_N is totally real at every point except the origin, where there is a CR singularity with type (I) if $N = 2$, or type (II) if $N > 2$. Similarly, denote

$$M_\infty = \{y_2 = 0,\ z_3 = \bar{z}_1^2, z_4 = z_1\bar{z}_1\}.$$

This variety has the structure of a product $S \times \mathbb{R} \subseteq \mathbb{C}^3 \times \mathbb{C}$, where S is the CR singular real surface $\{(z_1,z_3,z_4) : z_3 = \bar{z}_1^2, z_4 = z_1\bar{z}_1\} \subseteq \mathbb{C}^3$, considered in [C$_2$]. M_∞ has a CR singularity of type (II) at every point on the real line $\{(0,x_2,0,0)\}$, and is totally real at points not on the line.

The results of Sections 2, 3, 4 can be summarized as follows.

Proposition 5.1. *Given a real analytic threefold M in $\mathbb{C}^4$ with a CR singularity at $\vec{0}$, exactly one of the following cases holds:*

(1) *There exists a unique integer $N \geq 2$ such that for any $n \geq N$, there exists a biholomorphic coordinate transformation of a neighborhood of $\vec{0}$, so that*

in the new coordinate system, the defining equations of M are

$$\begin{aligned} y_2 &= O(n+1) \\ z_3 &= \bar{z}_1^2 + O(n+1) \\ z_4 &= z_1\bar{z}_1 + \bar{z}_1 x_2^{N-1} + O(n+1), \end{aligned}$$

so M is formally equivalent to the real algebraic model M_N, or,

(2) *For any $n \geq 3$, there exists a biholomorphic coordinate transformation of a neighborhood of $\vec{0}$, so that in the new coordinate system, the defining equations of M are*

$$\begin{aligned} y_2 &= O(n+1) \\ z_3 &= \bar{z}_1^2 + O(n+1) \\ z_4 &= z_1\bar{z}_1 + O(n+1), \end{aligned}$$

so M is formally equivalent to the real algebraic model M_∞, or,

(3) *There exists a biholomorphic coordinate transformation of a neighborhood of $\vec{0}$, so that in the new coordinate system, the defining equations of M are*

$$\begin{aligned} y_2 &= Q_2 + O(3) \\ z_3 &= q_3 + O(3) \\ z_4 &= q_4 + O(3), \end{aligned}$$

where (Q_2, q_3, q_4) is one of the types (III)–(XII) *from Section* 2.

$\blacksquare$

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