# The Attraction of Surfaces of Revolution

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In my lectures for the first-year calculus sequence, I state and solve physics problems. After the section on surface area, the following problem generated some interest:

*Assuming an inverse square law of attraction, what is the force exerted by a massive surface of revolution on a point mass $m$ located on the axis of symmetry?*

An important special case is the attractive force of gravity exerted by a spherical shell on a point mass $m$. Since any line through the center is an axis of symmetry, $m$ can be anywhere in space.

For the general case, here are some preliminary assumptions:

1. The surface of revolution is defined by a nonnegative function $f(x)$ on a closed interval $[a, b]$, such that $f'$ exists on $(a, b)$. The graph of $f$ is revolved around the $x$-axis as in the Figure.

2. The surface's mass is distributed evenly, in the sense that it has a constant "planar density," $d \geq 0$. The units on $d$ might be kilograms per square meter, for example, to distinguish it from linear or spatial density.

3. The "inverse square law" refers to a force exerted on a point mass $m$ by another point mass $M$ separated by distance $r > 0$. Then the magnitude of the force is $GmMr^{-2}$, for a positive constant $G$. $M$, $m$ will be assumed nonnegative, and the direction of the force on $m$ is toward $M$.

4. To simplify calculation, the point mass $m$ can be assumed to be at the origin, by translating $f$ left or right if necessary.

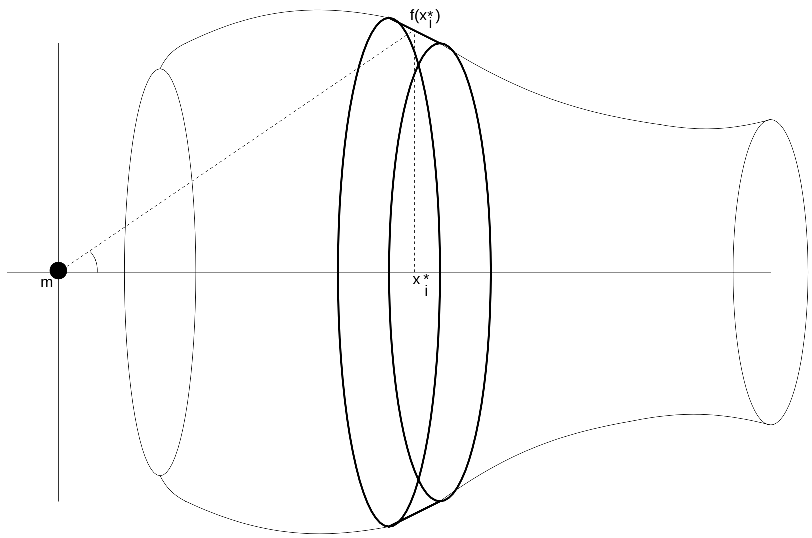


Figure 1

To start with the solution to the physics problem, we slice the surface with planes parallel to the $yz$-plane, and review a (sketchy) derivation of the integral formula for surface area.

The Riemann sum procedure is to partition $[a, b]$ into $n$ subintervals $[x_{i-1}, x_i]$, with length $\Delta x_i$, for $i = 1$ to $n$, and then select a sample point $x_i^*$ — the midpoint will be convenient. The graph of $f$ can be approximated by $n$ line segments $L_i$ connecting $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$. Revolving each segment $L_i$ gives a truncated cone $C_i$, which approximates a slice $S_i$ of the surface of revolution. Each $C_i$ has surface area

$$\pi(f(x_{i-1}) + f(x_i))\sqrt{(\Delta x_i)^2 + (\Delta f_i)^2},$$

where $\Delta f_i$ abbreviates $f(x_i) - f(x_{i-1})$. (This well-known formula for the area of a truncated cone can be derived without calculus.) The average $\frac{1}{2}(f(x_{i-1}) + f(x_i))$ is the distance from $(x_i^*, 0)$ to the midpoint of $L_i$, which can be approximated by $f(x_i^*)$. Then, the approximate area of $C_i$, and the slice $S_i$, is $2\pi f(x_i^*)\sqrt{1 + \left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i$, and the mass of $S_i$, denoted $M_i$, is approximately the density times this area:

$$M_i \approx d \cdot 2\pi f(x_i^*)\sqrt{1+\left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i.$$

The total area of the surface is the $n \to \infty$ (and $\max \Delta x_i \to 0$) limit of the sum of the approximate areas, and its total mass, denoted $M(f) = \sum M_i$, is equal to $d$ times this area:

$$M(f) = d \cdot \int_a^b 2\pi f(x)\sqrt{1+\left(\frac{df}{dx}\right)^2}dx.$$

The force $F_i$, exerted by each slice $S_i$ on the mass $m$ at the origin, will be directed along the $x$-axis. This is obvious by the rotational symmetry, and also follows from the following approximation of $F_i$ as a vector sum. The slice $S_i$ can itself be subdivided "radially" into $2N$ pieces by $N$ planes through the $x$-axis. When $n$ and $N$ are large, each of these pieces can be treated as a point with mass $\frac{M_i}{2N}$, for the purposes of approximating $F_i$ using the inverse square law. Every piece of $S_i$ will be represented by one of its points, with horizontal coordinate $x_i^*$ and at distance $f(x_i^*)$ from the $x$-axis, so that the line from $m$ to this point is at an angle $\theta_i$ with the $x$-axis. The force exerted on $m$ by the piece has approximate magnitude $Gm\frac{M_i}{2N}(\sqrt{(x_i^*)^2+(f(x_i^*))^2})^{-2}$. Its horizontal component (along the $x$-axis) is $\cos(\theta_i)$ times the magnitude, and its radial component is $\sin(\theta_i)$ times the magnitude. The force exerted by the opposite piece (rotating the piece and its representative point by $180°$) has the same horizontal component, but an oppositely directed radial component. In the sum over $2N$ pieces, the radial components all cancel, and the approximate horizontal components total to

$$\begin{aligned}
F_i &\approx Gm\frac{M_i}{(x_i^*)^2+(f(x_i^*))^2}\cos(\theta_i)\\
&\approx Gm\frac{d2\pi f(x_i^*)\sqrt{1+\left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i}{(x_i^*)^2+(f(x_i^*))^2}\frac{x_i^*}{\sqrt{(x_i^*)^2+(f(x_i^*))^2}}\\
&= 2\pi Gmd\frac{x_i^* f(x_i^*)\sqrt{1+\left(\frac{\Delta f_i}{\Delta x_i}\right)^2}}{((x_i^*)^2+(f(x_i^*))^2)^{(3/2)}}\Delta x_i.
\end{aligned}$$

The second step uses the earlier approximation for $M_i$, and the ratio for the cosine: $\cos(\theta_i) = \frac{x_i^*}{\sqrt{(x_i^*)^2+(f(x_i^*))^2}}$. This formula for $F_i$ is actually a signed quantity, with the formula for the cosine taking into account the direction of the force acting on $m$: to the right for $x_i^* > 0$, and to the left for $x_i^* < 0$.

So, in the $n \to \infty$ limit, the answer to the physics question is

$$\int_a^b 2\pi Gmd \frac{x f(x)\sqrt{1+(f'(x))^2}}{(x^2+(f(x))^2)^{(3/2)}} dx,$$

assuming that this definite integral exists, which (mathematically) is a non-trivial condition required of $f$.

As an application of this formula, consider a sphere with center $(c, 0)$ (on the positive $x$-axis, $c > 0$) and radius $R > 0$. Using the above formula, with $f(x) = \sqrt{R^2 - (x-c)^2}$, and $[a, b] = [c-R, c+R]$ gives $f'(x) = \frac{c-x}{\sqrt{R^2-(x-c)^2}}$, and total force

$$\begin{aligned}
F &= \int_{c-R}^{c+R} 2\pi Gmd \frac{x\sqrt{R^2-(x-c)^2}\sqrt{1+\left(\frac{c-x}{\sqrt{R^2-(x-c)^2}}\right)^2}}{(x^2+R^2-(x-c)^2)^{(3/2)}} dx \\
&= 2\pi GmdR \int_{c-R}^{c+R} \frac{x}{(R^2+2xc-c^2)^{(3/2)}} dx \\
&= \frac{2\pi GmdR}{c^2} \frac{R^2+xc-c^2}{\sqrt{R^2+2xc-c^2}} \Bigg|_{c-R}^{c+R} \\
&= \begin{cases} \frac{2\pi GmdR^2}{c^2}\left(\frac{c+R}{\sqrt{(c+R)^2}} + \frac{c-R}{\sqrt{(c-R)^2}}\right) & \text{if } c \neq R \\ 2\pi Gmd & \text{if } c = R. \end{cases} \\
&= \begin{cases} 4\pi GmdR^2c^{-2} & \text{if } c > R \\ 0 & \text{if } c < R \\ 2\pi Gmd & \text{if } c = R. \end{cases}
\end{aligned}$$

The total mass of the sphere is $M(f) = 4\pi R^2 d$, and if this mass were concentrated at the center $(c, 0)$ with $c > R$, the force on the mass $m$ at $(0, 0)$ would be $GmM(f)c^{-2}$. This is the same as the above integral, so we have a single-variable derivation of a result of Newton, that the external gravitational attraction of a sphere is equal to the attractive force of a point with the same mass at the sphere's center. This was part of Newton's argument that a solid ball has the same property.

The same integral also demonstrates the fact that if the particle of mass $m$ is inside the sphere, so $c < R$, then it feels no force acting in any direction. (This fact was interesting and surprising to many students.) At $c = R$, the particle is on the sphere, and the force is $\frac{1}{2}GmM(f)c^{-2}$; plotting $F$ as a function of $c$, there is a discontinuity at $c = R$. The $c = 0$ and $c < 0$ cases follow from similar calculations.

Other surfaces of revolution for which the above integral formula might be tractable are cylinders, $f(x) = K$, truncated cones, $f(x) = kx + K$, or funnel shapes, $f(x) = k/x$, over intervals where $f(x) \geq 0$. The construction also could be applied to a repelling force.