The Attraction of Surfaces of Revolution

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July 24, 2000

In my lectures for the first-year calculus sequence, I state and solve physics problems. After the section on surface area, the following problem generated some interest:

Assuming an inverse square law of attraction, what is the force exerted by a massive surface of revolution on a point mass m located on the axis of symmetry?

An important special case is the attractive force of gravity exerted by a spherical shell on a point mass m. Since any line through the center is an axis of symmetry, m can be anywhere in space.

For the general case, here are some preliminary assumptions:

1. The surface of revolution is defined by a nonnegative function $f(x)$ on a closed interval $[a, b]$, such that f' exists on (a, b). The graph of f is revolved around the x-axis as in the Figure.

2. The surface's mass is distributed evenly, in the sense that it has a constant "planar density," $d \geq 0$. The units on d might be kilograms per square meter, for example, to distinguish it from linear or spatial density.

3. The "inverse square law" refers to a force exerted on a point mass m by another point mass M separated by distance $r > 0$. Then the magnitude of the force is $GmMr^{-2}$, for a positive constant G. M, m will be assumed nonnegative, and the direction of the force on m is toward M.

4. To simplify calculation, the point mass m can be assumed to be at the origin, by translating f left or right if necessary.

Figure 1

To start with the solution to the physics problem, we slice the surface with planes parallel to the yz-plane, and review a (sketchy) derivation of the integral formula for surface area.

The Riemann sum procedure is to partition $[a, b]$ into n subintervals $[x_{i-1}, x_i]$, with length Δx_i, for $i = 1$ to n, and then select a sample point x_i^* — the midpoint will be convenient. The graph of f can be approximated by n line segments L_i connecting $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$. Revolving each segment L_i gives a truncated cone C_i, which approximates a slice S_i of the surface of revolution. Each C_i has surface area

$$\pi(f(x_{i-1}) + f(x_i))\sqrt{(\Delta x_i)^2 + (\Delta f_i)^2},$$

where Δf_i abbreviates $f(x_i) - f(x_{i-1})$. (This well-known formula for the area of a truncated cone can be derived without calculus.) The average $\frac{1}{2}(f(x_{i-1}) + f(x_i))$ is the distance from $(x_i^*, 0)$ to the midpoint of L_i, which can be approximated by $f(x_i^*)$. Then, the approximate area of C_i, and the slice S_i, is $2\pi f(x_i^*)\sqrt{1 + \left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i$, and the mass of S_i, denoted M_i, is approximately the density times this area:

$$M_i \approx d \cdot 2\pi f(x_i^*)\sqrt{1 + \left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i.$$

The total area of the surface is the $n \to \infty$ (and $\max \Delta x_i \to 0$) limit of the sum of the approximate areas, and its total mass, denoted $M(f) = \sum M_i$, is equal to d times this area:

$$M(f) = d \cdot \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{df}{dx}\right)^2}dx.$$

The force F_i, exerted by each slice S_i on the mass m at the origin, will be directed along the x-axis. This is obvious by the rotational symmetry, and also follows from the following approximation of F_i as a vector sum. The slice S_i can itself be subdivided "radially" into $2N$ pieces by N planes through the x-axis. When n and N are large, each of these pieces can be treated as a point with mass $\frac{M_i}{2N}$, for the purposes of approximating F_i using the inverse square law. Every piece of S_i will be represented by one of its points, with horizontal coordinate x_i^* and at distance $f(x_i^*)$ from the x-axis, so that the line from m to this point is at an angle θ_i with the x-axis. The force exerted on m by the piece has approximate magnitude $Gm\frac{M_i}{2N}(\sqrt{(x_i^*)^2 + (f(x_i^*))^2})^{-2}$. Its horizontal component (along the x-axis) is $\cos(\theta_i)$ times the magnitude, and its radial component is $\sin(\theta_i)$ times the magnitude. The force exerted by the opposite piece (rotating the piece and its representative point by 180°) has the same horizontal component, but an oppositely directed radial component. In the sum over $2N$ pieces, the radial components all cancel, and the approximate horizontal components total to

$$\begin{aligned} F_i &\approx Gm\frac{M_i}{(x_i^*)^2 + (f(x_i^*))^2}\cos(\theta_i) \\ &\approx Gm\frac{d2\pi f(x_i^*)\sqrt{1+\left(\frac{\Delta f_i}{\Delta x_i}\right)^2}\Delta x_i}{(x_i^*)^2 + (f(x_i^*))^2}\frac{x_i^*}{\sqrt{(x_i^*)^2 + (f(x_i^*))^2}} \\ &= 2\pi Gmd\frac{x_i^* f(x_i^*)\sqrt{1+\left(\frac{\Delta f_i}{\Delta x_i}\right)^2}}{((x_i^*)^2 + (f(x_i^*))^2)^{(3/2)}}\Delta x_i. \end{aligned}$$

The second step uses the earlier approximation for M_i, and the ratio for the cosine: $\cos(\theta_i) = \frac{x_i^*}{\sqrt{(x_i^*)^2+(f(x_i^*))^2}}$. This formula for F_i is actually a signed quantity, with the formula for the cosine taking into account the direction of the force acting on m: to the right for $x_i^* > 0$, and to the left for $x_i^* < 0$.

So, in the $n \to \infty$ limit, the answer to the physics question is

$$\int_a^b 2\pi Gmd\frac{xf(x)\sqrt{1+(f'(x))^2}}{(x^2 + (f(x))^2)^{(3/2)}}dx,$$

assuming that this definite integral exists, which (mathematically) is a non-trivial condition required of f.

As an application of this formula, consider a sphere with center $(c, 0)$ (on the positive x-axis, $c > 0$) and radius $R > 0$. Using the above formula, with $f(x) = \sqrt{R^2 - (x-c)^2}$, and $[a, b] = [c-R, c+R]$ gives $f'(x) = \frac{c-x}{\sqrt{R^2-(x-c)^2}}$, and total force

$$
\begin{aligned}
F &= \int_{c-R}^{c+R} 2\pi G m d \frac{x\sqrt{R^2-(x-c)^2}\sqrt{1+\left(\frac{c-x}{\sqrt{R^2-(x-c)^2}}\right)^2}}{(x^2+R^2-(x-c)^2)^{(3/2)}} dx \\
&= 2\pi G m d R \int_{c-R}^{c+R} \frac{x}{(R^2+2xc-c^2)^{(3/2)}} dx \\
&= \frac{2\pi G m d R}{c^2} \frac{R^2+xc-c^2}{\sqrt{R^2+2xc-c^2}} \Bigg|_{c-R}^{c+R} \\
&= \begin{cases} \frac{2\pi G m d R^2}{c^2}\left(\frac{c+R}{\sqrt{(c+R)^2}} + \frac{c-R}{\sqrt{(c-R)^2}}\right) & \text{if } c \neq R \\ 2\pi G m d & \text{if } c = R. \end{cases} \\
&= \begin{cases} 4\pi G m d R^2 c^{-2} & \text{if } c > R \\ 0 & \text{if } c < R \\ 2\pi G m d & \text{if } c = R. \end{cases}
\end{aligned}
$$

The total mass of the sphere is $M(f) = 4\pi R^2 d$, and if this mass were concentrated at the center $(c, 0)$ with $c > R$, the force on the mass m at $(0, 0)$ would be $GmM(f)c^{-2}$. This is the same as the above integral, so we have a single-variable derivation of a result of Newton, that the external gravitational attraction of a sphere is equal to the attractive force of a point with the same mass at the sphere's center. This was part of Newton's argument that a solid ball has the same property.

The same integral also demonstrates the fact that if the particle of mass m is inside the sphere, so $c < R$, then it feels no force acting in any direction. (This fact was interesting and surprising to many students.) At $c = R$, the particle is on the sphere, and the force is $\frac{1}{2}GmM(f)c^{-2}$; plotting F as a function of c, there is a discontinuity at $c = R$. The $c = 0$ and $c < 0$ cases follow from similar calculations.

Other surfaces of revolution for which the above integral formula might be tractable are cylinders, $f(x) = K$, truncated cones, $f(x) = kx + K$, or funnel shapes, $f(x) = k/x$, over intervals where $f(x) \geq 0$. The construction also could be applied to a repelling force.