Smooth counterexamples to strong unique continuation for a Beltrami system in $\mathbb{C}^2$

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Abstract

We construct an example of a smooth map $\mathbb{C} \to \mathbb{C}^2$ which vanishes to infinite order at the origin, and such that the ratio of the norm of the $\bar{z}$ derivative to the norm of the z derivative also vanishes to infinite order. This gives a counterexample to strong unique continuation for a vector valued analogue of the Beltrami equation.

1 Introduction

We will construct an example of a smooth function $\mathbf{u} : \mathbb{C} \to \mathbb{C}^2$ which has an isolated zero of infinite order at the origin ($\|z^{-k}\mathbf{u}(z)\| \to 0$ as $z \to 0$ for all $k \geq 0$), and where the ratio of norms of derivatives $\|\mathbf{u}_{\bar{z}}\|/\|\mathbf{u}_z\|$ is small, also vanishing to infinite order at $z = 0$. This behavior is obviously different from that of a map $\mathbf{u}$ with $\mathbf{u}_{\bar{z}} \equiv \mathbf{0}$, which would be holomorphic and could not have an isolated zero of infinite order. This vector valued case is also different from the complex scalar case, where solutions $u : \mathbb{C} \to \mathbb{C}$ of the well-known Beltrami equation $u_{\bar{z}} = a(z)u_z$, for small $a(z)$, also cannot vanish to infinite order at an isolated zero ([B], [CH], [AIM], [R]).

More precisely, we will show in Section 4 that in a neighborhood of the origin, $\mathbf{u}(z)$ is a solution of a Beltrami-type system of differential equations, which is linear, elliptic, and has continuous coefficients very close to those of the Cauchy-Riemann system, but does not have the strong unique continuation property. In Theorem 4.1, we verify the coefficients are not Lipschitz continuous. Counterexamples to the weak unique continuation property for elliptic Beltrami systems have been considered by [IVV].

The construction was motivated by an example of Rosay ([R]) and questions posed by [IS], who were considering the unique continuation problem for systems of equations from almost complex geometry.

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In Section 2, we develop a general framework for constructing smooth maps $\mathbb{C} \to \mathbb{C}^2$ vanishing to infinite order. In Section 3, we present both Rosay's example and our new example. In Section 5 we state some open questions.

2 General Setup

2.1 Annular cutoff functions

Start with a real valued function $s(x)$ which is smooth on $\mathbb{R}$, with $s \equiv 0$ on $[0, \frac{1}{4}]$, s increasing on $[\frac{1}{4}, \frac{3}{4}]$, $s(\frac{1}{2}) = \frac{1}{2}$, $s'(\frac{1}{2}) = 2$, $s''(\frac{1}{2}) = 0$, and $s \equiv 1$ on $[\frac{3}{4}, 1]$.

For $r_1 > 0$ and two parameters $0 < r < r_1$ and $0 < \Delta r < r_1 - r$, denote the annulus $A_{r,\Delta r} = \{z = x + iy \in \mathbb{C} : r \leq |z| \leq r + \Delta r\}$ (contained in the disk D_{r_1}), and define a family of functions $\chi_{r,\Delta r} : A_{r,\Delta r} \to \mathbb{R}$ by the formula $\chi_{r,\Delta r}(z) = s\left(\frac{|z|-r}{\Delta r}\right)$. At a particular point $\tilde{z} = (\tilde{x}, \tilde{y}) \in A_{r,\Delta r}$,

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\chi_{r,\Delta r}(x,y)\right]_{(\tilde{x},\tilde{y})} &= \frac{\partial}{\partial x}\left[s\left(\frac{\sqrt{x^2+y^2}-r}{\Delta r}\right)\right]_{(\tilde{x},\tilde{y})} \\
&= s'\left(\frac{\sqrt{\tilde{x}^2+\tilde{y}^2}-r}{\Delta r}\right) \cdot \frac{\tilde{x}}{\sqrt{\tilde{x}^2+\tilde{y}^2}} \cdot \frac{1}{\Delta r} \\
&= s'\left(\frac{|\tilde{z}|-r}{\Delta r}\right) \cdot \frac{\tilde{x}}{|\tilde{z}|\Delta r}.
\end{aligned}$$

The y derivative is similar, and the z, $\bar{z}$ derivatives are complex linear combinations. In particular,

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\chi_{r,\Delta r}(z) &= \overline{\frac{\partial}{\partial z}\chi_{r,\Delta r}(z)} = s'\left(\frac{|z|-r}{\Delta r}\right) \cdot \frac{x+iy}{2|z|\Delta r} &(1)\\
\implies \left|\frac{\partial}{\partial \bar{z}}\chi_{r,\Delta r}(z)\right| &= \left|\frac{\partial}{\partial z}\chi_{r,\Delta r}(z)\right| \leq \frac{m_{01}}{\Delta r} &(2)
\end{aligned}$$

for some constant $m_{01} > 0$ not depending on r_1, r or Δr.

For higher derivatives of $\chi_{r,\Delta r}$, the following Lemma is a simplified version of the Faà di Bruno formula for derivatives of composites.

Lemma 2.1. *For $k \geq 0$, there exist polynomials $p_{abc}(x_1, x_2, x_3)$, $q_{abc}(x_1, x_2, x_3)$ indexed by $a, b, c \geq 0$, $a + b = k$, $c \leq k$, with constant complex coefficients (not depending on r_1, r, Δr, or s), so that*

$$\frac{\partial^a}{\partial z^a}\frac{\partial^b}{\partial \bar{z}^b}\chi_{r,\Delta r}(z) = \sum_{c=0}^{k} s^{(c)}\left(\frac{|z|-r}{\Delta r}\right) \cdot \frac{p_{abc}(z,\bar{z},\Delta r) + |z|q_{abc}(z,\bar{z},\Delta r)}{|z|^{2k}(\Delta r)^k}.$$

Proof. The $k = 0$ case is trivial and the $k = 1$ case is stated above. We record the second derivatives:

$$\begin{aligned}
\frac{\partial^2}{\partial \bar{z}^2}\chi_{r,\Delta r}(z) &= \overline{\frac{\partial^2}{\partial z^2}\chi_{r,\Delta r}(z)} \\
&= s''\left(\frac{|z|-r}{\Delta r}\right)\frac{z^2}{4|z|^2(\Delta r)^2} + s'\left(\frac{|z|-r}{\Delta r}\right)\frac{-z^2}{4|z|^3\Delta r}, \quad (3) \\
\frac{\partial^2}{\partial z \partial \bar{z}}\chi_{r,\Delta r}(z) &= s''\left(\frac{|z|-r}{\Delta r}\right)\frac{1}{4(\Delta r)^2} + s'\left(\frac{|z|-r}{\Delta r}\right)\frac{1}{4|z|\Delta r}.
\end{aligned}$$

The proof for all larger k is by induction on k; the calculation is straightforward and omitted here. ∎

It follows as a consequence of the Lemma that there are positive constants m_{ab} (indexed by $a, b \geq 0$, $a + b = k$, and depending on the choices of s and r_1, but not depending on r, Δr) so that

$$\begin{aligned}
\left|\frac{\partial^a}{\partial z^a}\frac{\partial^b}{\partial \bar{z}^b}\chi_{r,\Delta r}(z)\right| &\leq \frac{m_{ab}}{|z|^{2k}(\Delta r)^k} \\
\implies \max_{z \in A_{r,\Delta r}}\left|\frac{\partial^a}{\partial z^a}\frac{\partial^b}{\partial \bar{z}^b}\chi_{r,\Delta r}(z)\right| &\leq \frac{m_{ab}}{r^{2k}(\Delta r)^k}.
\end{aligned}$$

In various cases, in particular $k = 1$ as in (2), the r^{2k} can be improved (with a smaller exponent), but it is good enough to use later in Lemma 2.3.

2.2 The basic construction of the examples

Let r_n be a real sequence decreasing with limit $= 0$. Denote $\Delta r_n = r_n - r_{n+1}$.

Let A_n denote the closed annulus

$$A_n = A_{r_{n+1},\Delta r_n} = \{z \in \mathbb{C} : r_{n+1} \leq |z| \leq r_n\},$$

so the union is a disk: $D_{r_1} = (\cup A_n) \cup \{0\}$. The annular cutoff functions can be indexed by n: $\chi_n = \chi_{r_{n+1},\Delta r_n} : A_n \to \mathbb{R}$.

For $n \in \mathbb{N}$, let $p(n)$ be an increasing positive integer sequence; set $p(0) = 0$. Let $F(n)$ be a positive real valued sequence; set $F(0) = 1$. Define a function $\mathbf{u} : D_{r_1} \to \mathbb{C}^2$ by $\mathbf{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and on the annulus A_n, for even n:

$$\begin{aligned}
\mathbf{u}(z) &= \begin{bmatrix} u^1(z) \\ u^2(z) \end{bmatrix}, \\
u^1(z) &= F(n)z^{p(n)}, \quad (4) \\
u^2(z) &= \chi_n(z)F(n-1)z^{p(n-1)} + (1 - \chi_n(z))F(n+1)z^{p(n+1)}. \quad (5)
\end{aligned}$$

For odd n, switch the formulas for u^1, u^2.

So far, for any s, r_n, p, F, the function $\mathbf{u}$ is smooth on $D_{r_1} \setminus \{0\}$, and extends holomorphically for $|z| \geq r_1$. We also have that $\mathbf{u}$ and $\mathbf{u}_z$ have non-zero value at every point of $D_{r_1} \setminus \{0\}$; for n even (and switching indices if n is odd):

$$\|\mathbf{u}_z\| \geq \left| \frac{\partial}{\partial z} u^1 \right| = F(n)p(n)|z|^{p(n)-1}. \tag{6}$$

2.3 Smoothness at the origin

An important property of the examples $\mathbf{u} : D_{r_1} \to \mathbb{C}^2$ we want to construct is that they are smooth at (and near) the origin. It is not enough to check only that the components vanish to infinite order. In general, as easily constructed examples would show, for functions $f : \mathbb{R}^N \to \mathbb{R}^M$, f can vanish to infinite order at the origin:

$$\lim_{\vec{x} \to \vec{0}} \frac{f(\vec{x})}{\|\vec{x}\|^k} = \vec{0}$$

for all whole numbers k, but need not be smooth. Our approach to proving smoothness of our examples will be to show u^1, u^2, and all their higher partial derivatives approach 0; this implies vanishing to infinite order, as in the following Lemma.

Lemma 2.2. *Given $f : \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}^1$, suppose f is smooth and for each $j, k = 0, 1, 2, 3, \ldots$,*

$$\lim_{(x,y) \to \vec{0}} \frac{\partial^{k+j} f(x,y)}{\partial x^k \partial y^j} = 0.$$

Then extending f so that $f(0,0) = 0$ defines a smooth function on $\mathbb{R}^2$ that vanishes to infinite order at the origin.

Proof. f is continuous at $\vec{0}$ by hypothesis ($j = k = 0$). To show f is smooth, we only need to show every partial derivative of order ℓ of f exists at $\vec{0}$, and has value 0; then it follows that for $k + j = \ell$, $\frac{\partial^{k+j} f(x,y)}{\partial x^k \partial y^j}$ is continuous at $\vec{0}$.

The proof is by induction on ℓ; suppose for any non-commutative word $x^{k_1} y^{j_1} \cdots x^{k_a} y^{j_b}$ with $k_1 + \cdots k_a + j_1 + \ldots + j_b = k + j = \ell$, $\frac{\partial^\ell}{\partial x^{k_1} y^{j_1} \cdots x^{k_a} y^{j_b}} f(x,y)$ exists at $\vec{0}$ and has value 0. Then, the x-derivative at the origin is (with the y-derivative being similar):

$$\begin{aligned}
& \frac{\partial}{\partial x} \left(\frac{\partial^\ell f(x,y)}{\partial x^{k_1} y^{j_1} \cdots x^{k_a} y^{j_b}} \right) \Bigg]_{(0,0)} \\
= \quad & \lim_{t \to 0} \frac{\frac{\partial^\ell}{\partial x^{k_1} y^{j_1} \cdots x^{k_a} y^{j_b}} f(0+t,0) - \frac{\partial^\ell}{\partial x^{k_1} y^{j_1} \cdots x^{k_a} y^{j_b}} f(0,0)}{t} \\
= \quad & \lim_{t \to 0} \frac{\frac{\partial^\ell}{\partial x^k \partial y^j} f(t,0) - 0}{t}.
\end{aligned}$$

Let $g(t) = \frac{\partial^\ell}{\partial x^k \partial y^j} f(t,0)$ for $t \neq 0$; then $\lim_{t\to 0} g(t) = 0$ by hypothesis, and

$$g'(t) = \frac{\partial^{\ell+1}}{\partial x^{k+1}\partial y^j} f(t,0).$$

L'Hôpital's Rule applies to the above limit:

$$\lim_{t\to 0}\frac{g(t)}{t} = \lim_{t\to 0}\frac{g'(t)}{1} = \lim_{t\to 0}\frac{\partial^{\ell+1}}{\partial x^{k+1}\partial y^j} f(t,0) = 0.$$

The property of vanishing to infinite order follows from Taylor approximation at the origin. ∎

For our examples $\mathbf{u}$, we want to choose r_n, p, and F, so that $\mathbf{u}$ is smooth and vanishes to infinite order at 0. The following criterion for smoothness will be verified for both the Examples in Section 3.

Lemma 2.3. *If* $\frac{(\Delta r_n/r_n)}{(\Delta r_{n+2}/r_{n+2})}$ *is a bounded sequence and, for each integer* $k \geq 0$,

$$\lim_{n\to\infty}\frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} = 0, \tag{7}$$

then $\mathbf{u}$ *is smooth and vanishes to infinite order at the origin.*

Proof. From Lemma 2.2 and the construction of $\mathbf{u}$, it is enough to show, for any non-negative integers a, b, with $a+b=k$, that

$$\max_{z\in A_n}\left|\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial \bar z}\right)^b u^1\right|,\quad \max_{z\in A_n}\left|\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial \bar z}\right)^b u^2\right|$$

both have limit 0 as $n\to\infty$.

The following estimates for the derivatives assume n is even, and sufficiently large compared to k. The derivatives of $u^1 = F(n)z^{p(n)}$ (4) are easy:

$$\begin{aligned}\left|\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial \bar z}\right)^b u^1\right| &\leq \left|F(n)p(n)(p(n)-1)\cdots(p(n)-k+1)z^{p(n)-k}\right|\\ &\leq F(n)(p(n))^k r_n^{p(n)-k} \end{aligned}\tag{8}$$

The derivatives of $u^2(z)$ (5) are considered one term at a time. For the first term:

$$\begin{aligned}&\left(\frac{\partial}{\partial z}\right)^a\left(\frac{\partial}{\partial \bar z}\right)^b\left(\chi_n(z)F(n-1)z^{p(n-1)}\right)\\ =\;& F(n-1)\left(\frac{\partial}{\partial z}\right)^a\left(\left(\left(\frac{\partial}{\partial \bar z}\right)^b\chi_n(z)\right)\cdot z^{p(n-1)}\right)\\ =\;& F(n-1)\sum_{a_1+a_2=a}^{2^a}\left(\left(\frac{\partial}{\partial z}\right)^{a_1}\left(\frac{\partial}{\partial \bar z}\right)^b\chi_n(z)\right)\left(\left(\frac{\partial}{\partial z}\right)^{a_2} z^{p(n-1)}\right)\end{aligned}$$

The sum is over the 2^a terms (with many repeated) that result from applying the product rule a times.

By Lemma 2.1,

$$\left|\left(\frac{\partial}{\partial z}\right)^{a_1}\left(\frac{\partial}{\partial \bar{z}}\right)^{b}\chi_n(z)\right| \le \frac{m_{a_1 b}}{|z|^{2(a_1+b)}(\Delta r_n)^{a_1+b}},$$

and

$$\begin{aligned}\left|\left(\frac{\partial}{\partial z}\right)^{a_2} z^{p(n-1)}\right| &= p(n-1)\cdots(p(n-1)-a_2+1)|z|^{p(n-1)-a_2}\\ &\le (p(n-1))^{a_2}|z|^{p(n-1)-a_2}.\end{aligned}$$

Let $m_k = \max_{a+b\le k} m_{ab}$. Then

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial z}\right)^{a}\left(\frac{\partial}{\partial \bar{z}}\right)^{b}\left(\chi_n(z)F(n-1)z^{p(n-1)}\right)\right|\\ \le\ & F(n-1)2^a \max_{a_1\le a}\left\{\frac{m_{a_1 b}}{|z|^{2(a_1+b)}(\Delta r_n)^{a_1+b}}\right\}\max_{a_2\le a}\left\{(p(n-1))^{a_2}|z|^{p(n-1)-a_2}\right\}\\ \le\ & F(n-1)2^a\frac{m_k}{|z|^{2k}(\Delta r_n)^k}(p(n-1))^a|z|^{p(n-1)-a}\\ \le\ & F(n-1)2^k\frac{m_k}{(\Delta r_n)^k}(p(n-1))^k|z|^{p(n-1)-3k}\\ \le\ & 2^k m_k\frac{F(n-1)(p(n-1))^k r_n^{p(n-1)-3k}}{(\Delta r_n)^k}. \qquad (9)\end{aligned}$$

Similarly for the second term of $u^2(z)$,

$$\begin{aligned}&\left|\left(\frac{\partial}{\partial z}\right)^{a}\left(\frac{\partial}{\partial \bar{z}}\right)^{b}\left((1-\chi_n(z))F(n+1)z^{p(n+1)}\right)\right|\\ \le\ & 2^k m_k\frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-3k}}{(\Delta r_n)^k}. \qquad (10)\end{aligned}$$

So, the criteria for all the derivatives of $\mathbf{u}$ to vanish at the origin are that the expressions (8), (9), and (10) must all have limit 0 as $n\to\infty$. The hypothesis (7) is equivalent to (10)$\to 0$. Comparing (10) to (8) by shifting the index in (8) from n to $n+1$, this scalar multiple of (10) is much larger:

$$\frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-3k}}{(\Delta r_n)^k} > F(n+1)(p(n+1))^k r_{n+1}^{p(n+1)-k},$$

so if (10) has limit 0, then so does (8).

(10)$\to 0$ also implies $F(n+1)r_n^{p(n+1)-k}\to 0$, which is enough to show $\mathbf{u}$ vanishes to infinite order: $\|z^{-k}\mathbf{u}(z)\|\to 0$ as $z\to 0$.

Shifting the index in (9) from n to $n+2$ gives the following quantity (11), which is comparable to (10):

$$\begin{aligned} & 2^k m_k \frac{F(n+1)(p(n+1))^k r_{n+2}^{p(n+1)-3k}}{(\Delta r_{n+2})^k} \\ < \quad & 2^k m_k \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} \cdot \frac{(\Delta r_n/r_n)^k}{(\Delta r_{n+2}/r_{n+2})^k}, \end{aligned} \tag{11}$$

and under the additional hypothesis that $\frac{\Delta r_n/r_n}{\Delta r_{n+2}/r_{n+2}}$ is a bounded sequence, (7) also implies (9)$\to 0$. ∎

2.4 Comparing first derivatives

We want to choose F, p, and r_n so that $\dfrac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|}$ is small, as $z \to 0$. For $z \in A_n$, n even (and switching indices if n is odd), expanding the derivative and using (2) gives:

$$\begin{aligned} \|\mathbf{u}_{\bar{z}}\| &= \left|\frac{\partial}{\partial \bar{z}} u^2\right| \\ &\leq \frac{m_{01}}{\Delta r_n} F(n-1)|z|^{p(n-1)} + \frac{m_{01}}{\Delta r_n} F(n+1)|z|^{p(n+1)} \\ &= \frac{m_{01}}{\Delta r_n}|z|^{p(n)-1}\left[F(n-1)|z|^{p(n-1)-p(n)+1} + F(n+1)|z|^{p(n+1)-p(n)+1}\right]. \end{aligned}$$

Using (6) and introducing a factor $g(n) > 0$, for $z \in A_n$:

$$\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \leq \frac{m_{01} r_n g(n)}{\Delta r_n p(n)} \cdot \frac{\left[F(n-1) r_{n+1}^{p(n-1)-p(n)} + F(n+1) r_n^{p(n+1)-p(n)}\right]}{g(n) F(n)} \tag{12}$$

The first fraction in the product is what we want to make small for large n, depending on p and $\dfrac{\Delta r_n}{r_n}$. The second fraction we would like to make bounded, depending on F and r_n, and an arbitrary fudge factor g. The role of g is to manage the size of F and simplify the calculation proving boundedness of the second factor, possibly at the expense of affecting the rate at which the first factor approaches 0.

3 Examples

Example 3.1. Rosay's example ([R]) has $p(n) = n$, $r_n = 2^{-n+1}$, and $\frac{\Delta r_n}{r_n} = \frac{1}{2}$. Then (12) becomes:

$$\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \leq \frac{m_{01} 2 g(n)}{n} \cdot \frac{\left[F(n-1)2^n + F(n+1)2^{-n+1}\right]}{g(n)F(n)}. \tag{13}$$

The choice, as in ([R]), $F(n) = 2^{n^2/2}$, satisfies the recursive formula

$$F(n-1)2^n = g(n)F(n), \tag{14}$$

with $g(n) = \sqrt{2}$. This simplifies the second factor of the RHS of (13), so it is easily seen to be bounded. The conclusion is $\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \le \frac{C_1}{n}$ for $z \in A_n$, and since $\frac{1}{n} \le \frac{1}{-\log_2 |z|} \le \frac{1}{n-1}$ on A_n,

$$\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \le \frac{C_1}{-\log_2 |z|} \tag{15}$$

for all $z \in D_1 \setminus \{0\}$.

To check that $\mathbf{u}$ is smooth and vanishing to infinite order at the origin, it is enough to verify the condition of Lemma 2.3; for each fixed $k \ge 0$:

$$\lim_{n\to\infty} \frac{2^{(n+1)^2/2}(n+1)^k (2^{-n+1})^{(n+1-4k)}}{(2^{-1})^k} = 0.$$

The goal of the next example is to improve upon the order of vanishing of the ratio (15).

Example 3.2. Consider $r_n = \frac{1}{\ln(n+1)}$, so $r_1 = \frac{1}{\ln(2)} \approx 1.44$, $r_2 = \frac{1}{\ln(3)}$, ... This radius shrinks much more slowly than in Example 3.1. Since

$$\lim_{n\to\infty} \frac{1 - \frac{\ln(n+1)}{\ln(n+2)}}{1/(n\ln(n+2))} = 1,$$

there are constants C_2, $c_2 > 0$ so that for all n:

$$\frac{c_2}{n\ln(n+2)} < \frac{\Delta r_n}{r_n} \quad = \quad 1 - \frac{\ln(n+1)}{\ln(n+2)} < \frac{C_2}{n\ln(n+2)}. \tag{16}$$

Let $p(n) = n^2$; then the inequality (12) becomes:

$$\begin{aligned} \frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \quad \le \quad & \frac{m_{01} \cdot (n\ln(n+2))g(n)}{c_2 n^2} \\ & \cdot \frac{\left[F(n-1)\left[\frac{1}{\ln(n+2)}\right]^{-2n+1} + F(n+1)\left[\frac{1}{\ln(n+1)}\right]^{2n+1}\right]}{g(n)F(n)}. \end{aligned}$$

This motivates, in analogy with the previous Example, this choice of g and a recursive formula for $F(n)$ as in (14):

$$\begin{aligned} g(n) &= \ln(n+2), \\ F(n) &= (\ln(n+2))^{[2n-2]} F(n-1). \end{aligned}$$

So $F(1) = F(0) = 1$, and for $n > 1$,

$$\begin{aligned} F(n) &= (\ln(n+2))^{[2n-2]} \cdot (\ln(n+1))^{[2(n-1)-2]} \cdots (\ln(5))^4 \cdot (\ln(4))^2 \\ &= \left[(\ln(n+2))^{n-1} \cdot (\ln(n+1))^{n-2} \cdots (\ln(5))^2 \cdot (\ln(4))^1\right]^2. \end{aligned}$$

Then the following sequence of ratios is bounded above because it is convergent as $n \to \infty$:

$$\begin{aligned}
& \frac{F(n-1)(\ln(n+2))^{[2n-1]} + F(n+1)(\ln(n+1))^{-[2n+1]}}{g(n)F(n)} \\
= & \frac{F(n-1)(\ln(n+2))^{[2n-1]} + \frac{(\ln(n+3))^{2n}(\ln(n+2))^{[2n-2]}F(n-1)}{(\ln(n+1))^{[2n+1]}}}{\ln(n+2)(\ln(n+2))^{[2n-2]}F(n-1)} \\
= & 1 + \frac{(\ln(n+3))^{2n}}{\ln(n+2)(\ln(n+1))^{[2n+1]}} \leq C_3.
\end{aligned}$$

Here we used the elementary calculus lemma that $\left(\frac{\ln(n+2)}{\ln(n)}\right)^n$ is a bounded sequence.

The estimate for the ratio of derivatives on A_n, for even n, becomes:

$$\begin{aligned}
\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} & \leq \frac{|u^2_{\bar{z}}|}{|u^1_z|} \leq \frac{m_{01}\cdot(n\ln(n+2))\ln(n+2)}{c_2 n^2}\cdot C_3 \\
& = \frac{m_{01}(\ln(n+2))^2 C_3}{c_2 n} = \frac{C_4(\ln(n+2))^2}{n} \qquad (17) \\
& < \frac{C_5(\ln(n+1))^2}{n+2}. \qquad (18)
\end{aligned}$$

For $z \in A_n$,

$$\begin{aligned}
& \frac{1}{\ln(n+2)} \leq |z| \leq \frac{1}{\ln(n+1)} \\
\iff & (\ln(n+1))^2 \leq \frac{1}{|z|^2} \leq (\ln(n+2))^2 \\
\iff & \frac{1}{n+2} \leq \exp(-\frac{1}{|z|}) \leq \frac{1}{n+1},
\end{aligned}$$

so

$$\frac{\|\mathbf{u}_{\bar{z}}\|}{\|\mathbf{u}_z\|} \leq C_5 \frac{1}{|z|^2\exp(\frac{1}{|z|})},$$

for all $z \in D_{r_1}\setminus\{0\}$. It remains to check that $\mathbf{u}$ is smooth, and vanishes to infinite order. The hypothesis on Δr_n of Lemma 2.3 is satisfied (using (16)), so for fixed k, consider the expression:

$$\begin{aligned}
& \frac{F(n+1)(p(n+1))^k r_n^{p(n+1)-4k}}{(\Delta r_n/r_n)^k} \\
= & \frac{\left[(\ln(n+3))^n(\ln(n+2))^{n-1}\cdots\ln(4)\right]^2((n+1)^2)^k\left(\frac{1}{\ln(n+1)}\right)^{(n+1)^2-4k}}{\left(1-\frac{\ln(n+1)}{\ln(n+2)}\right)^k} \\
< & \frac{\left[(\ln(n+3))^n(\ln(n+2))^{n-1}\cdots\ln(4)\right]^2(n+1)^{2k}}{(\ln(n+1))^{(n+1)^2-4k}\left(\frac{c_2}{n\ln(n+2)}\right)^k} \\
< & \frac{\left[(\ln(n+3))^n(\ln(n+2))^{n-1}\cdots\ln(4)\right]^2(n+1)^{3k}(\ln(n+2))^k}{c_2^k(\ln(n+1))^{(n+1)^2-4k}}.
\end{aligned}$$

The last expression has limit zero by the Ratio Test:

$$\begin{aligned}
& \frac{\frac{\left((\ln(n+4))^{n+1}(\ln(n+3))^n\cdots\ln(4)\right)^2}{c_2^k(\ln(n+2))^{(n+2)^2-4k}}\cdot(n+2)^{3k}(\ln(n+3))^k}{\frac{((\ln(n+3))^n\cdots\ln(4))^2}{c_2^k(\ln(n+1))^{(n+1)^2-4k}}\cdot(n+1)^{3k}(\ln(n+2))^k} \\
= & \frac{(\ln(n+4))^{2n+2}(\ln(n+1))^{(n+1)^2-4k}(n+2)^{3k}(\ln(n+3))^k}{(\ln(n+2))^{(n+2)^2-4k}(n+1)^{3k}(\ln(n+2))^k} \\
< & \frac{(\ln(n+4))^{2n+2+k}(n+2)^{3k}}{(\ln(n+2))^{2n+3+k}(n+1)^{3k}} \to 0,
\end{aligned}$$

again using the boundedness of $\left(\frac{\ln(n+2)}{\ln(n)}\right)^n$.

4 A Beltrami-type system

Any smooth map $\mathbf{u}:\mathbb{C}\to\mathbb{C}^2$, $\mathbf{u}=\begin{bmatrix} u^1 \\ u^2 \end{bmatrix}$, satisfies the following Beltrami-type system of first-order differential equations at points where $\mathbf{u}_z \neq \mathbf{0}$:

$$\begin{aligned}
\begin{bmatrix} u^1_{\bar z} \\ u^2_{\bar z} \end{bmatrix} &= \begin{bmatrix} u^1_{\bar z} \\ u^2_{\bar z} \end{bmatrix} \frac{[\overline{u^1_z} \quad \overline{u^2_z}]}{|u^1_z|^2+|u^2_z|^2} \begin{bmatrix} u^1_z \\ u^2_z \end{bmatrix} \\
&= \frac{1}{\|\mathbf{u}_z\|^2} \begin{bmatrix} u^1_{\bar z}\overline{u^1_z} & u^1_{\bar z}\overline{u^2_z} \\ u^2_{\bar z}\overline{u^1_z} & u^2_{\bar z}\overline{u^2_z} \end{bmatrix} \begin{bmatrix} u^1_z \\ u^2_z \end{bmatrix} \qquad (19) \\
&= \mathbf{Q}_{2\times 2}(z) \begin{bmatrix} u^1_z \\ u^2_z \end{bmatrix}.
\end{aligned}$$

For $\mathbf{u}$ constructed as in Section 2, on the annuli A_n with even n, $u^1_{\bar z}\equiv 0$, so the first row of the matrix in (19) is [0 0], and similarly the second row is [0 0] for odd n. Define $\mathbf{Q}(0)$ to be the zero matrix.

For a matrix $\mathbf{Q}(z)$ defined as in (19) by some fixed function $\mathbf{u}$, the operator $\mathcal{L} = \frac{\partial}{\partial \bar{z}} - \mathbf{Q}(z)\frac{\partial}{\partial z}$ is complex linear. If, on some neighborhood of $z = 0$, the $\mathbf{Q}(z)$ entries are defined and small enough, then $\mathcal{L}$ is elliptic (in the sense of [AIM] Section 7.4).

In the following Theorem, we consider $\mathbf{Q}(z)$ for the example $\mathbf{u}(z)$ from Example 3.2. If we restrict $\mathbf{u}$ and $\mathbf{Q}$ to z in some sufficiently small neighborhood of the origin, $\mathbf{u}$ will be a solution of the elliptic equation $\mathcal{L}(\mathbf{u}) = 0$.

Theorem 4.1. *For $\mathbf{u}$ as in Example 3.2, let $q_{ij}(z) = \dfrac{u^i_{\bar{z}}\overline{u^j_z}}{\|\mathbf{u}_z\|^2}$ denote the i, j entry in the matrix $\mathbf{Q}(z)$ from (19).*

- *$q_{ij} \in \mathcal{C}^\infty(D_{r_1} \setminus \{0\}) \cap \mathcal{C}^0(D_{r_1})$;*
- *q_{ij} vanishes to infinite order: $|z^{-k}q_{ij}(z)| \to 0$ as $z \to 0$ for any $k \geq 0$;*
- *The partial derivatives exist at the origin: $\frac{\partial}{\partial x}q_{ij}(0) = \frac{\partial}{\partial y}q_{ij}(0) = 0$;*
- *For any $0 < r < r_1$, q_{22} does not have the Lipschitz property on D_r.*

Proof. The $\mathcal{C}^\infty$ claim follows from the smoothness of $\mathbf{u}$ on D_{r_1} and the nonvanishing of $\mathbf{u}_z$ for $z \neq 0$.

The sum $|q_{11}|^2 + |q_{12}|^2 + |q_{21}|^2 + |q_{22}|^2$ is exactly $\dfrac{\|\mathbf{u}_{\bar{z}}\|^2}{\|\mathbf{u}_z\|^2}$, which vanishes to infinite order as $z \to 0$ for $\mathbf{u}$ as in Example 3.2. It follows that each q_{ij} also vanishes to infinite order, which implies $\mathbf{Q}$ and the entries q_{ij} are continuous at the origin, with the previously assigned values $q_{ij}(0) = 0$. The flatness also implies the existence of all directional derivatives at the origin of $\mathbb{C} = \mathbb{R}^2$; for the x direction,

$$\frac{\partial}{\partial x}q_{ij}\Big]_{(x,y)=(0,0)} = \lim_{x\to 0}\frac{q_{ij}(x,0) - q_{ij}(0,0)}{x} = 0.$$

The last claim takes up the rest of the Proof; the plan is to show there is a sequence of points $x_n \in \mathbb{C}$ approaching 0 so that $\dfrac{\partial}{\partial \bar{z}}\dfrac{u^2_{\bar{z}}\overline{u^2_z}}{\|\mathbf{u}_z\|^2}\Big]_{z=x_n}$ is an unbounded sequence. If q_{22} had a Lipschitz property on D_r, i.e., $|q_{22}(z_1) - q_{22}(z_2)| \leq K|z_1 - z_2|$ for some K and all z_1, z_2, then its derivatives on $D_r \setminus \{0\}$ would be bounded; the unboundedness of the derivative also directly shows $q_{22} \notin \mathcal{C}^1(D_r)$. A real variable analogue of this behavior is the function $\exp(-1/|x|)\cos(\exp(1/|x|))$, which vanishes to infinite order but has unbounded first derivative as $x \to 0$.

It is enough, and simpler, to consider only n which are even and sufficiently large. This will involve some estimates for derivatives that are more precise than (9).

We choose the sequence $x_n = r_{n+1} + \frac{1}{2}\Delta r_n + 0i \in A_n$; then by construction of s and χ_n, $\chi_n(x_n) = \frac{1}{2}$, and (1) gives $\frac{\partial \chi_n}{\partial \bar{z}}(x_n) = \frac{\partial \chi_n}{\partial z}(x_n) = \frac{1}{\Delta r_n}$. From (3),

$$\frac{\partial^2 \chi_n}{\partial z \partial \bar{z}}(x_n) = -\frac{\partial^2 \chi_n}{\partial z^2}(x_n) = -\frac{\partial^2 \chi_n}{\partial \bar{z}^2}(x_n) = \frac{1}{2x_n\Delta r_n}$$

(this is where we use the $s''(\frac{1}{2}) = 0$ assumption, to simplify the calculation).

For r_n as in Example 3.2, $\frac{1}{\ln(n+2)} < x_n < \frac{1}{\ln(n+1)}$, and $\Delta r_n = \frac{1}{\ln(n+1)} - \frac{1}{\ln(n+2)}$ satisfies

$$0 < c_6 n(\ln(n+2))^2 < \frac{1}{\Delta r_n} < C_6 n(\ln(n+2))^2.$$

In the following expression,

$$\begin{aligned}\frac{\partial}{\partial \bar{z}} \frac{u^2_{\bar{z}} \overline{u^2_z}}{\|\mathbf{u}_z\|^2} &= \frac{u^2_{\bar{z}\bar{z}} \overline{u^2_z}}{\|\mathbf{u}_z\|^2} + \frac{u^2_{\bar{z}} \overline{u^2_{zz}}}{\|\mathbf{u}_z\|^2} - \frac{u^2_{\bar{z}} \overline{u^2_z} \frac{\partial}{\partial \bar{z}} (u^1_z \overline{u^1_z} + u^2_z \overline{u^2_z})}{\|\mathbf{u}_z\|^4} \\ = \quad & \frac{u^2_{\bar{z}\bar{z}} \overline{u^2_z}}{\|\mathbf{u}_z\|^2} + \frac{u^2_{\bar{z}} \overline{u^2_{zz}} (u^1_z \overline{u^1_z} + u^2_z \overline{u^2_z}) - u^2_{\bar{z}} \overline{u^2_z} (u^1_z \overline{u^1_{zz}} + u^2_{z\bar{z}} \overline{u^2_z} + u^2_z \overline{u^2_{zz}})}{\|\mathbf{u}_z\|^4},\end{aligned} \quad (20)$$

the terms with the largest magnitude are the ones involving the second z-derivatives, u^1_{zz} and u^2_{zz}. Evaluated at points in the sequence x_n, these terms individually grow at least as fast as some constant multiple of n. However, due to some cancellations, the overall growth rate turns out to be less than n; the Theorem will be proved by showing one of the terms is unbounded and the remaining terms have a slower rate of growth.

The first cancellation is that the second and last terms in the numerator of the second fraction in (20) are exactly opposites. This leaves two terms with zz-derivatives; using the power rule on $u^1 = F(n) z^{n^2}$ in the interior of A_n gives:

$$u^1_{zz} = u^1_z \cdot \frac{n^2 - 1}{z}.$$

(20) becomes:

$$\frac{u^2_{\bar{z}\bar{z}} \overline{u^2_z}}{\|\mathbf{u}_z\|^2} + \frac{u^2_{\bar{z}} u^1_z \overline{u^1_z} \overline{\left(u^2_{zz} - \frac{(n^2-1)u^2_z}{z}\right)}}{\|\mathbf{u}_z\|^4} - \frac{u^2_{\bar{z}} \overline{u^2_z} u^2_{z\bar{z}} \overline{u^2_z}}{\|\mathbf{u}_z\|^4}. \quad (21)$$

We will show that the last of the three terms in (21) is the dominant one.

First, consider the ratio:

$$\begin{aligned} & |u^2_z| / |u^1_z| \\ = \quad & \Bigg| F(n-1)\left(\frac{\partial \chi_n}{\partial z} z^{(n-1)^2} + \chi_n \cdot (n-1)^2 z^{(n-1)^2 - 1}\right) \\ & + F(n+1)\left(-\frac{\partial \chi_n}{\partial z} z^{(n+1)^2} + (1 - \chi_n)(n+1)^2 z^{n^2+2n}\right) \Bigg| \Big/ \left| F(n) n^2 z^{n^2-1} \right|. \end{aligned} \quad (22)$$

The following calculation for $z \in A_n$ is similar to that for the estimate (17).

$$\begin{aligned}\frac{|u^2_z|}{|u^1_z|} \quad \leq \quad & \frac{F(n-1)\left[\left|\frac{\partial \chi_n}{\partial z}\right| |z|^{(n-1)^2} + \chi_n \cdot (n-1)^2 |z|^{n^2-2n}\right]}{F(n) n^2 |z|^{n^2-1}} \\ & + \frac{F(n+1)\left[\left|\frac{\partial \chi_n}{\partial z}\right| |z|^{(n+1)^2} + (1 - \chi_n)(n+1)^2 |z|^{n^2+2n}\right]}{F(n) n^2 |z|^{n^2-1}}\end{aligned}$$

From Example 3.2, recalling $F(n) = (\ln(n+2))^{2n-2}F(n-1)$, $\frac{1}{\ln(n+2)} \leq |z| \leq \frac{1}{\ln(n+1)}$, and $\left|\frac{\partial \chi_n}{\partial z}\right| \leq \frac{m_{01}}{\Delta r_n}$, it follows that there is some constant $C_7 > 1$ so that

$$\frac{|u_z^2|}{|u_z^1|} \leq C_7 \ln(n+2). \tag{23}$$

To estimate the denominators of (21),

$$\begin{aligned} \frac{\|\mathbf{u}_z\|^2}{|u_z^1|^2} &= 1 + \left(\frac{|u_z^2|}{|u_z^1|}\right)^2 \leq 1 + (C_7 \ln(n+2))^2 \\ \Longrightarrow \|\mathbf{u}_z\|^2 &\leq C_8(\ln(n+2))^2 |u_z^1|^2. \end{aligned}$$

So we get a lower bound for the third term in (21),

$$\frac{\left|u_{\bar{z}}^2 \overline{u_z^2} u_{z\bar{z}}^2 \overline{u_z^2}\right|}{\|\mathbf{u}_z\|^4} \geq \frac{|u_{\bar{z}}^2| \left|u_z^2\right|^2 \left|u_{z\bar{z}}^2\right|}{C_8^2(\ln(n+2))^4 |u_z^1|^4}, \tag{24}$$

and consider (24) one factor at a time.

$$\frac{|u_{\bar{z}}^2|}{|u_z^1|} = \frac{\left|\frac{\partial \chi_n}{\partial \bar{z}} \cdot F(n-1) z^{(n-1)^2} - \frac{\partial \chi_n}{\partial \bar{z}} \cdot F(n+1) z^{(n+1)^2}\right|}{\left|F(n) n^2 z^{n^2-1}\right|} \tag{25}$$

$$\begin{aligned} \left.\frac{|u_{\bar{z}}^2|}{|u_z^1|}\right]_{z=x_n} &= \frac{1}{\Delta r_n}\left|\frac{1}{(\ln(n+2))^{2n-2} n^2 x_n^{2n-2}} - \frac{(\ln(n+3))^{2n} x_n^{2n+2}}{n^2}\right| \\ &\geq c_6 n (\ln(n+2))^2 \left(\frac{(\ln(n+1))^{2n-2}}{(\ln(n+2))^{2n-2} n^2} - \frac{(\ln(n+3))^{2n}}{n^2 (\ln(n+1))^{2n+2}}\right) \\ &\geq \frac{c_4 (\ln(n+2))^2}{n}. \end{aligned} \tag{26}$$

This lower bound (26) is comparable to the upper bound (17), which is used in the next step to find a lower bound for $\frac{|u_z^2|}{|u_z^1|}$ (22). Note that the expression (22) has two terms which, using the equality of $\frac{\partial \chi_n}{\partial \bar{z}}$ and $\frac{\partial \chi_n}{\partial z}$ when evaluated at x_n, match two of the terms from $u_{\bar{z}}^2$ in (25):

$$\begin{aligned} \left.\frac{|u_z^2|}{|u_z^1|}\right]_{z=x_n} &= \left|\left.\frac{u_{\bar{z}}^2}{u_z^1}\right]_{x_n} + \frac{(n-1)^2}{2(\ln(n+2))^{2n-2} n^2 x_n^{2n-1}}\right. \\ &\qquad \left. + \frac{(\ln(n+3))^{2n}(n+1)^2 x_n^{2n+1}}{2n^2}\right| \\ &\geq \frac{(n-1)^2 (\ln(n+1))^{2n-1}}{2(\ln(n+2))^{2n-2} n^2} - \left|\left.\frac{u_{\bar{z}}^2}{u_z^1}\right]_{x_n}\right| \\ &\qquad - \frac{(\ln(n+3))^{2n}(n+1)^2}{2n^2 (\ln(n+1))^{2n+1}} \\ &\geq c_9 \ln(n+2) - C_4 \frac{(\ln(n+2))^2}{n} - C_{10} \frac{1}{\ln(n+1)} \\ &\geq c_7 \ln(n+2). \end{aligned} \tag{27}$$

This lower bound is comparable to the upper bound (23). The remaining factor from (24) involves second derivatives:

$$\begin{aligned}
& |u^2_{z\bar{z}}|/|u^1_z| \\
= & \left|F(n-1)\left(\frac{\partial^2\chi_n}{\partial z\partial\bar{z}}\cdot z^{(n-1)^2}+2\frac{\partial\chi_n}{\partial\bar{z}}\cdot(n-1)^2 z^{n^2-2n}\right)\right. \\
& \left.-F(n+1)\left(\frac{\partial^2\chi_n}{\partial z\partial\bar{z}}\cdot z^{(n+1)^2}+\frac{\partial\chi_n}{\partial\bar{z}}\cdot(n+1)^2 z^{n^2+2n}\right)\right| / |F(n)n^2 z^{n^2-1}| \\
\geq & \frac{F(n-1)}{F(n)n^2}\left(\left|\frac{\partial\chi_n}{\partial\bar{z}}\right|(n-1)^2|z|^{-2n+1}-\left|\frac{\partial^2\chi_n}{\partial z\partial\bar{z}}\right||z|^{-2n+2}\right) \\
& -\frac{F(n+1)}{F(n)n^2}\left(\left|\frac{\partial^2\chi_n}{\partial z\partial\bar{z}}\right||z|^{2n+2}+\left|\frac{\partial\chi_n}{\partial\bar{z}}\right|(n+1)^2|z|^{2n+1}\right).
\end{aligned}$$

Evaluating at x_n (again, for sufficiently large n),

$$\begin{aligned}
\left.\frac{|u^2_{z\bar{z}}|}{|u^1_z|}\right]_{x_n} \geq & \frac{1}{(\ln(n+2))^{2n-2}n^2}\left(\frac{(n-1)^2}{\Delta r_n x_n^{2n-1}}-\frac{1}{2x_n\Delta r_n}\frac{1}{x_n^{2n-2}}\right) \\
& -\frac{(\ln(n+3))^{2n}}{n^2}\left(\frac{x_n^{2n+2}}{2x_n\Delta r_n}+\frac{1}{\Delta r_n}(n+1)^2 x_n^{2n+1}\right) \\
\geq & \frac{1}{(\ln(n+2))^{2n-2}n^2}\left(c_6 n(\ln(n+2))^2(n-1)^2(\ln(n+1))^{2n-1}\right. \\
& \left.-\frac{1}{2}C_6 n(\ln(n+2))^2(\ln(n+2))^{2n-1}\right) \\
& -\frac{(\ln(n+3))^{2n}C_6(\ln(n+2))^2}{n^2}\left(\frac{1}{2(\ln(n+1))^{2n+1}}\right. \\
& \left.+\frac{(n+1)^2}{(\ln(n+1))^{2n+1}}\right) \\
\geq & c_{11}n(\ln(n+2))^3. \qquad (28)
\end{aligned}$$

So, the term from (24) is bounded below by a product including factors from (26), (27), and (28):

$$\begin{aligned}
\left.\frac{\left|u^2_{\bar{z}}(\overline{u^2_z})^2 u^2_{z\bar{z}}\right|}{\|\mathbf{u}_z\|^4}\right]_{x_n} \geq & \frac{c_4(\ln(n+2))^2(c_7\ln(n+2))^2 c_{11}n(\ln(n+2))^3}{nC_8^2(\ln(n+2))^4} \\
\geq & c_{12}(\ln(n+2))^3. \qquad (29)
\end{aligned}$$

From the second term in (21), we consider the following quantity:

$$\begin{aligned}
& u_{zz}^2 - \frac{(n^2-1)u_z^2}{z} \\
= \; & F(n-1)\left(\frac{\partial^2 \chi_n}{\partial z^2} z^{(n-1)^2} + 2\frac{\partial \chi_n}{\partial z}(n-1)^2 z^{n^2-2n}\right. \\
& \left. + \chi_n \cdot (n-1)^2 (n^2-2n) z^{n^2-2n-1}\right) \\
& + F(n+1)\left(-\frac{\partial^2 \chi_n}{\partial z^2} z^{(n+1)^2} - 2\frac{\partial \chi_n}{\partial z}(n+1)^2 z^{n^2+2n}\right. \\
& \left. + (1-\chi_n) \cdot (n+1)^2 (n^2+2n) z^{n^2+2n-1}\right) \\
& - F(n-1)(n^2-1)\left(\frac{\partial \chi_n}{\partial z} z^{n^2-2n} + \chi_n \cdot (n-1)^2 z^{n^2-2n-1}\right) \\
& - F(n+1)(n^2-1)\left(-\frac{\partial \chi_n}{\partial z} z^{n^2+2n} + (1-\chi_n) \cdot (n+1)^2 z^{n^2+2n-1}\right).
\end{aligned}$$

The cancellation of the n^4 quantities is the key step. The ratio

$$\begin{aligned}
& \left| u_{zz}^2 - \frac{(n^2-1)u_z^2}{z} \right| / |u_z^1| \\
= \; & \left| F(n-1)\left(\frac{\partial^2 \chi_n}{\partial z^2} z^{(n-1)^2} + \frac{\partial \chi_n}{\partial z}(n^2-4n+3) z^{n^2-2n}\right.\right. \\
& \left. - \chi_n \cdot (n-1)^2 (2n-1) z^{n^2-2n-1}\right) \\
& + F(n+1)\left(\frac{\partial^2 \chi_n}{\partial z^2} z^{(n+1)^2} - \frac{\partial \chi_n}{\partial z}(n^2+4n+3) z^{n^2+2n}\right. \\
& \left.\left. + (1-\chi_n)(n+1)^2 (2n+1) z^{n^2+2n-1}\right) \right| / \left| F(n) n^2 z^{n^2-1} \right|
\end{aligned}$$

has an upper bound on the x_n sequence:

$$\begin{aligned}
&\left.\frac{\left|u^2_{zz}-\frac{(n^2-1)u^2_z}{z}\right|}{|u^1_z|}\right]_{x_n}\\
\leq\ & \frac{1}{(\ln(n+2))^{2n-2}n^2}\left(\frac{1}{2x_n\Delta r_n x_n^{2n-2}}+\frac{n^2-4n+3}{\Delta r_n x_n^{2n-1}}+\frac{(n-1)^2(2n-1)}{2x_n^{2n}}\right)\\
&+\frac{(\ln(n+3))^{2n}}{n^2}\left(\frac{x_n^{2n+2}}{2x_n\Delta r_n}+\frac{(n^2+4n+3)x_n^{2n+1}}{\Delta r_n}+\frac{(n+1)^2(2n+1)x_n^{2n}}{2}\right)\\
\leq\ & \frac{1}{(\ln(n+2))^{2n-2}n^2}\left(C_6 n(\ln(n+2))^2(\frac{1}{2}+n^2-4n+3)(\ln(n+2))^{2n-1}\right.\\
&\left.+\frac{1}{2}(n-1)^2(2n-1)(\ln(n+2))^{2n}\right)\\
&+\frac{(\ln(n+3))^{2n}}{n^2}\left(C_6 n(\ln(n+2))^2(\frac{1}{2}+n^2-4n+3)\frac{1}{(\ln(n+1))^{2n+1}}\right.\\
&\left.+\frac{(n+1)^2(2n+1)}{(\ln(n+1))^{2n}}\right)\\
\leq\ & C_{13}n(\ln(n+2))^3. \qquad (30)
\end{aligned}$$

The middle term from (21), evaluated at points x_n, has factors bounded by (17), (27), and (30):

$$\begin{aligned}
&\left.\left|\frac{u^2_{\bar z}u^1_z\overline{u^1_z}\left(\overline{u^2_{zz}-(n^2-1)u^2_z/z}\right)}{\|\mathbf{u}\|^4}\right|\right]_{x_n}\\
\leq\ & \left.\frac{|u^2_{\bar z}||u^1_z|^2\left|u^2_{zz}-(n^2-1)u^2_z/z\right|}{|u^2_z|^4}\right]_{x_n}\\
=\ & \left.\frac{|u^2_{\bar z}|}{|u^1_z|}\right]_{x_n}\left.\frac{|u^1_z|^4}{|u^2_z|^4}\right]_{x_n}\left.\frac{\left|u^2_{zz}-(n^2-1)u^2_z/z\right|}{|u^1_z|}\right]_{x_n}\\
\leq\ & C_4\frac{(\ln(n+2))^2}{n}\frac{1}{(c_7\ln(n+2))^4}C_{13}n(\ln(n+2))^3\\
\leq\ & C_{14}\ln(n+2). \qquad (31)
\end{aligned}$$

The first term from (21) involves the second $\bar{z}$-derivative:

$$\begin{aligned}
\frac{|u^2_{\bar{z}\bar{z}}|}{|u^1_z|} &= \frac{\left|\frac{\partial^2\chi_n}{\partial\bar{z}^2}F(n-1)z^{(n-1)^2} - \frac{\partial^2\chi_n}{\partial\bar{z}^2}F(n+1)z^{(n+1)^2}\right|}{\left|F(n)n^2 z^{n^2-1}\right|} \\
&\leq \frac{F(n-1)}{F(n)n^2}\left|\frac{\partial^2\chi_n}{\partial\bar{z}^2}\right||z|^{-2n+2} + \frac{F(n+1)}{F(n)n^2}\left|\frac{\partial^2\chi_n}{\partial\bar{z}^2}\right||z|^{2n+2}. \\
\left.\frac{|u^2_{\bar{z}\bar{z}}|}{|u^1_z|}\right]_{x_n} &\leq \frac{1}{2x_n\Delta r_n n^2}\left(\frac{1}{(\ln(n+2))^{2n-2}}\cdot\frac{1}{x_n^{2n-2}} + (\ln(n+3))^{2n}x_n^{2n+2}\right) \\
&\leq \frac{C_6 n(\ln(n+2))^2}{2n^2}\left(\frac{(\ln(n+2))^{2n-1}}{(\ln(n+2))^{2n-2}} + \frac{(\ln(n+3))^{2n}}{(\ln(n+1))^{2n+1}}\right) \\
&\leq C_{15}\frac{(\ln(n+2))^3}{n}.
\end{aligned}$$

The first term from (21) also approaches 0 for large n:

$$\left.\frac{|u^2_{\bar{z}\bar{z}}\overline{u^2_z}|}{\|\mathbf{u}_z\|^2}\right]_{x_n} \leq \left.\frac{|u^2_{\bar{z}\bar{z}}|}{|u^1_z|}\right]_{x_n}\left.\frac{|u^2_z|}{\|\mathbf{u}_z\|}\right]_{x_n} \leq C_{15}\frac{(\ln(n+2))^3}{n}.$$

The conclusion from (20) and (21) is:

$$\begin{aligned}
\left.\left|\frac{\partial}{\partial\bar{z}}\frac{u^2_{\bar{z}}\overline{u^2_z}}{\|\mathbf{u}_z\|^2}\right|\right]_{x_n} &\geq c_{12}(\ln(n+2))^3 - C_{14}\ln(n+2) - C_{15}\frac{(\ln(n+2))^3}{n} \\
&> \frac{c_{16}}{x_n^3}.
\end{aligned}$$

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5 Remarks and Questions

Remark 5.1. The regularity of the coefficients is an important consideration in the analysis of unique continuation properties for some PDEs (for example, [LNW] for strong UCP, and [IVV] for weak UCP), which is why we presented the detailed Proof of Theorem 4.1. However, we do not yet understand the sharpness of Example 3.2 and Theorem 4.1 for this particular unique continuation problem. The question that naturally arises is: would improved regularity of $\mathbf{Q}(z)$ (in addition to, or instead of, the flatness property) imply a strong unique continuation property, or, oppositely, is there some counterexample where $\mathbf{Q}(z)$ is smooth?

Remark 5.2. [R] shows how Example 3.1 can be modified so that the origin is a non-isolated zero of $\mathbf{u}$; it is a matter of replacing quantities z^N in (4), (5) by $z^{N-1}(z-a_n)$ for a sequence a_n and re-working the cutoff functions χ_n. Our Example 3.2 can be modified in an analogous way but we have not worked out all the details.

Remark 5.3. By a construction analogous to (19), the function $\mathbf{u}$ from Example 3.2 also satisfies a real linear, elliptic equation of the form $\mathbf{u}_{\bar{z}} = \widetilde{\mathbf{Q}}_{2\times 2}\overline{\mathbf{u}_z}$. $\widetilde{\mathbf{Q}}(z)$ is not the same as $\mathbf{Q}(z)$ but also has entries vanishing to infinite order.

Remark 5.4. Another differential inequality, considered by [R], is

$$\|\mathbf{u}_{\bar{z}}\| \leq K\|\mathbf{u}\|^{\alpha}\|\mathbf{u}_z\|,$$

for $0 < \alpha < 1$. Our attempts to use the construction of Section 2 to find smooth functions $\mathbf{u}$ satisfying the inequality and vanishing to infinite order at an isolated zero have not yet met any success. [R] proves a weak unique continuation property for $\alpha = \frac{1}{2}$, but the strong property remains an open question.

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