

**COMMENTS ON:  
THE ALGEBRA AND GEOMETRY OF STEINER AND OTHER  
QUADRATICALLY PARAMETRIZABLE SURFACES**

ADAM COFFMAN

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1. ERRATA

The following typos appear in the published paper, [CSS].

- (p. 270) The derivative, equation (34), should read

$$\frac{\partial f_3}{\partial x} = [x_2(x_3^2 + x_4^2) - 2x_1x_4^2, x_1(x_3^2 + x_4^2), \\ 2x_1x_2x_3 - 4x_3(x_3^2 + x_4^2), 2x_1x_2x_4 - 2x_1^2x_4 - 4(x_3^2 + x_4^2)x_4].$$

This corrects the lower right  $2 \times 2$  block of the Hessian, equation (35), to

$$\begin{array}{cc} 2x_1x_2 - 12x_3^2 - 4x_4^2 & -8x_3x_4 \\ -8x_3x_4 & 2x_1x_2 - 2x_1^2 - 12x_4^2 - 4x_3^2 \end{array} \Bigg|.$$

- (p. 273) Below equation (58) should read “As in the case of  $\Sigma_7$  and  $\Sigma_8$ ”
- (p. 279) **Theorem 7.** should read “*The order of  $\Sigma$  is  $4 - \nu_\Sigma$ .*”
- (p. 281) Above equation (105) should read “Since  $\mathcal{P}(\Sigma) \not\subseteq \mathbb{M}_2$ ,  $e \neq 0$ .”
- (p. 284) Step (2) in Section 6. should read “ $\det\{\lambda\mathbf{M} + \mu\mathbf{N}\}$ .”

2. UPDATES

The references mention unpublished notes of A. Schwartz and C. Stanton — a version from 1988 is available from me, on request. Another pre-1996 source on this topic, with some details on the matrix calculations leading to the classification theorem, is [C<sub>1</sub>], which I made available online in 2018.

It should have been mentioned that Coffman’s research was supported in part by a National Science Foundation Research Experience for Undergraduates program in the summer of 1990.

Since publication, the following related article has come to our attention: [D]. The topic of projections of the real Veronese variety has more recently been considered in [C<sub>3</sub>]. Also see my web page on Steiner surfaces, currently at this address: [C<sub>2</sub>].

3. CITATIONS

Our article is cited in these academic papers: [A<sub>1</sub>], [A<sub>2</sub>], [A<sub>3</sub>], [ABB], [AMT], [AS<sub>1</sub>], [AS<sub>2</sub>], [BJKL], [BCF], [BEG], [CFRV], [EGL<sub>1</sub>], [EGL<sub>2</sub>], [GS], [G], [HJS], [H], [HW<sub>1</sub>], [HK], [HW<sub>2</sub>], [HW<sub>3</sub>], [LG], [PA], [PO], [POS], [PR], [PT], [P], [RJ], [S], [SPS], [WG], [WC], [WCD], [Zanella], [Z<sub>1</sub>], [Z<sub>2</sub>], [Z<sub>3</sub>], as well as these books: [F], [KI], and this computer technical manual: [T].

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DEPARTMENT OF MATHEMATICAL SCIENCES, PURDUE UNIVERSITY FORT WAYNE, FORT WAYNE,  
IN 46805-1499

*E-mail address:* CoffmanA@pfw.edu