# REAL CONGRUENCE OF COMPLEX MATRIX PENCILS AND COMPLEX PROJECTIONS OF REAL VERONESE VARIETIES

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Abstract. Quadratically parametrized maps from a real projective space to a complex projective space are constructed as projections of the Veronese embedding. A classification theorem relates equivalence classes of projections to real congruence classes of complex symmetric matrix pencils. The images of some low-dimensional cases include certain quartic curves in the Riemann sphere, models of the real projective plane in complex projective 4-space, and some normal form varieties for real submanifolds of complex space with CR singularities.

## 1. Introduction

One way to construct a smooth map from one projective space to another is by a "rational parametrization." This article will consider maps of the form

$$[u_0 : u_1 : \ldots : u_m] \mapsto [P_0 : P_1 : \ldots : P_n],$$

where the $u_j$ are real homogeneous coordinates, and each $P_k$ is a homogeneous quadratic polynomial in the $u_j$ variables with complex coefficients. Outside the common zero locus of the $P_k$, such a parametrization defines a smooth map $\mathbb{R}P^m \to \mathbb{C}P^n$, which is a restriction of a holomorphic map $\mathbb{C}P^m \to \mathbb{C}P^n$. A natural classification of such maps is to say that two are equivalent if they are related by a real linear coordinate change in the domain and a complex linear transformation of the target. However, working with $\mathbb{R}$ and $\mathbb{C}$ simultaneously will require some attention to detail, so a rigorous but elementary construction is carried out in the next Section, and some differences between real and complex geometry will be pointed out. After working out the theory, the practical approach to the equivalence of these quadratic parametrizations will be its relationship to the congruence of matrix pencils (Theorems 2.6, 2.15), and to a classification program of [W] (Propositions 4.3, 6.4). Some low dimensional cases, where the real projective line and plane are mapped to complex projective spaces, will be considered in detail. Section 4 makes some observations on parametric curves in the Riemann sphere and establishes a complete list of equivalence classes. Section 5 states and proves a new classification of two-dimensional spaces of $2 \times 2$ complex symmetric matrices, up to real congruence, and gives a geometric interpretation. Section 6 classifies quadratically parametrized maps from the real projective plane to $\mathbb{C}P^4$, most of which are totally real embeddings, but some will have singularities or a complex tangent. Section 7 briefly discusses the special case where the coefficients of $P_k$ are real, so they define

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maps $\mathbb{R}P^m \to \mathbb{R}P^n$. Section 8 shows a connection between quadratic parametrizations and the Hopf bundle over a complex projective space, and also surveys some real varieties that have appeared in the literature on real submanifolds of complex manifolds, which admit quadratic rational parametrizations.

## 2. The projective geometric construction

2.1. **General background.** The first steps in our description of maps from $\mathbb{R}P^m$ to $\mathbb{C}P^n$ will review (and fix some notation for) some well-known constructions in projective geometry over arbitrary fields.

Let $\mathbb{K}$ and $\mathbb{F}$ be fields, and let $m \geq 0$ be an integer. The projective $m$-space over the field $\mathbb{K}$, $\mathbb{K}P^m$, is the set of one-dimensional subspaces in $\mathbb{K}^{m+1}$. Denote the usual projection $\pi^m_{\mathbb{K}} : \mathbb{K}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{K}P^m$, so that a non-zero column vector $\mathbf{z}$ spans the line $\pi^m_{\mathbb{K}}(\mathbf{z})$. A line $z \in \mathbb{K}P^m$ with representative non-zero vector $\mathbf{z} = (z_0, z_1, \ldots, z_m)^T$ will have homogeneous coordinates $[z_0 : z_1 : \ldots : z_m]$.

Let $\mathbf{f} : \mathbb{K}^{m+1} \to \mathbb{F}^{N+1}$ be any function. Given $\mathbf{z} \in \mathbb{K}^{m+1} \setminus \{\mathbf{0}\}$, suppose $\mathbf{f}$ has the following two properties: first,

$$\mathbf{f}(\mathbf{z}) \neq \mathbf{0}, \tag{2.1}$$

and second, for any $\lambda \in \mathbb{K} \setminus \{0\}$, there exists $\mu \in \mathbb{F} \setminus \{0\}$ so that

$$\mathbf{f}(\lambda \cdot \mathbf{z}) = \mu \cdot \mathbf{f}(\mathbf{z}). \tag{2.2}$$

Then $\mathbf{f}$ will also have these two properties at every non-zero scalar multiple of $\mathbf{z}$. If $U \subseteq \mathbb{K}^{m+1} \setminus \{\mathbf{0}\}$ is the set of points where $\mathbf{f}$ has the two properties, then we will say "$\mathbf{f}$ induces a map from $\mathbb{K}P^m$ to $\mathbb{F}P^N$ which is well-defined on the set $\pi^m_{\mathbb{K}}(U)$," and we will denote the induced map, which takes $\pi^m_{\mathbb{K}}(\mathbf{z})$ to $\pi^N_{\mathbb{F}}(\mathbf{f}(\mathbf{z}))$, by $f : z \mapsto f(z)$. It should also be mentioned that the map of projective spaces induced by a composition of maps is equal to the composition of the induced maps.

As an example with $\mathbb{K} = \mathbb{F}$, if $\mathbf{f} : \mathbb{K}^{m+1} \to \mathbb{K}^{N+1}$ is $\mathbb{K}$-linear, then $f$ is well-defined on the lines not contained in the kernel of $\mathbf{f}$. If $\mathbf{f} : \mathbb{K}^{m+1} \to \mathbb{K}^{m+1}$ is $\mathbb{K}$-linear and invertible, then $f$ is well-defined on all of $\mathbb{K}P^m$, and also invertible. Let $GL(m+1, \mathbb{K}) \subseteq M(m+1, \mathbb{K})$ denote the subset of nonsingular matrices in the set of $(m+1) \times (m+1)$ matrices with entries in $\mathbb{K}$. Let $PGL(m+1, \mathbb{K})$ denote the set of one-dimensional subspaces of $M(m+1, \mathbb{K})$ which are subsets of $GL(m+1, \mathbb{K}) \cup \{\mathbf{0}\}$. The following construction defines a group action of $PGL(m+1, \mathbb{K})$ on $\mathbb{K}P^m$. For any nonsingular matrix $\mathbf{A}$, there is a corresponding invertible $\mathbb{K}$-linear transformation, which in turn induces an automorphism of $\mathbb{K}P^m$, denoted $A$. Any non-zero scalar multiple of $\mathbf{A}$ induces the same map $A : \mathbb{K}P^m \to \mathbb{K}P^m$, so this notation is consistent with the above conventions: a nonsingular matrix $\mathbf{A}$ spans a line $A \in PGL(m+1, \mathbb{K})$, and the automorphism of $\mathbb{K}P^m$ induced by $\mathbf{A}$ will be denoted $A : z \mapsto A \cdot z$.

Define a map from $\mathbb{K}^{m+1}$ to $\mathbb{K}^{(m+1)(m+2)/2}$, so that for $\mathbf{z} = (z_0, \ldots, z_m)^T$,

$$\mathbf{v}_{\mathbb{K}} : \mathbf{z} \mapsto (z_0^2, z_0z_1, z_1^2, z_0z_2, z_1z_2, z_2^2, \ldots, z_0z_m, z_1z_m, \ldots, z_m^2)^T.$$

The components of the map are all the $(m+1)(m+2)/2$ quadratic monomials $z_iz_j$. It satisfies (2.1) and (2.2) at every non-zero vector, so it induces a well-defined map

$$\begin{aligned} v_{\mathbb{K}} : \mathbb{K}P^m &\to \mathbb{K}P^{m(m+3)/2} : \\ z &\mapsto [z_0^2 : z_0z_1 : z_1^2 : z_0z_2 : z_1z_2 : z_2^2 : \ldots : z_0z_m : z_1z_m : \ldots : z_m^2], \end{aligned}$$

called the Veronese map.

Define a $\mathbb{K}$-linear invertible map from the space of $d \times d$ symmetric matrices, $S(d, \mathbb{K}) \subseteq M(d, \mathbb{K})$, to the space of column $d(d+1)/2$-vectors by stacking the columns of the upper triangular part of the matrix:

$$\begin{array}{rcl} \mathbf{vech} : S(d, \mathbb{K}) & \to & \mathbb{K}^{d(d+1)/2} : \\ \mathbf{M}_{d\times d} = (m_{ij})_{i,j=1\ldots d} & \mapsto & \begin{pmatrix} m_{11} \\ m_{12} \\ m_{22} \\ \vdots \\ m_{i\leq j} \\ \vdots \\ m_{1d} \\ \vdots \\ m_{dd} \end{pmatrix}_{(d(d+1)/2)\times 1} . \end{array}$$

This is a "vectorization" map for symmetric matrices (following the terminology of [Searle]). Denote its inverse by $\mathbf{k} : \mathbb{K}^{d(d+1)/2} \to S(d, \mathbb{K})$.

The composition of the maps $\mathbf{v}_{\mathbb{K}}$ and $\mathbf{k}$ (in the case $d = m+1$) has the following interpretation in terms of matrix multiplication:

$$\mathbf{k} \circ \mathbf{v}_{\mathbb{K}} : \mathbb{K}^{m+1} \to S(m+1, \mathbb{K}) : \mathbf{z} \mapsto \mathbf{z} \cdot \mathbf{z}^T. \tag{2.3}$$

$\mathbf{z}^T$ is a row vector, the transpose of $\mathbf{z}$, so the product $\mathbf{z} \cdot \mathbf{z}^T$ is a rank $\leq 1$ symmetric $(m+1) \times (m+1)$ matrix.

2.2. **Complex projective geometry.** We continue here with some elementary constructions, as in the previous Subsection, but with $\mathbb{K} = \mathbb{C}$, so we are in the familiar territory of complex projective geometry. We also will consider projective spaces with their usual topological and analytic structure — for example, the Veronese map $v_{\mathbb{C}} : \mathbb{C}P^m \to \mathbb{C}P^{m(m+3)/2}$ is a holomorphic embedding of complex manifolds. It will be convenient to abbreviate $\mathbf{v}_{\mathbb{C}} = \mathbf{v}$ and $v_{\mathbb{C}} = v$.

The next ingredients in the construction are an integer $n$ such that $0 \leq n \leq (m+1)(m+2)/2 - 1 = m(m+3)/2$, and a $(n+1) \times (m+1)(m+2)/2$ matrix $\mathbf{P}$ with complex entries and full rank $n+1 \leq (m+1)(m+2)/2$, called the coefficient matrix. The linear transformation $\mathbb{C}^{(m+1)(m+2)/2} \to \mathbb{C}^{n+1}$ (also denoted $\mathbf{P}$) induces a "projection" map $P : \mathbb{C}P^{m(m+3)/2} \to \mathbb{C}P^n$, which is well-defined for all elements $z$ except those lines in the kernel of $\mathbf{P}$. Let $\mathbb{C}P^n$ have homogeneous coordinates $[Z_0 : \ldots : Z_n]$.

So, the composition $P \circ v$ is a well-defined map $\mathbb{C}P^m \to \mathbb{C}P^n$ if the image of $\mathbf{v}$ contains no lines in the kernel of $\mathbf{P}$. When the $(n+1) \times (m+1)(m+2)/2$ entries of the matrix $\mathbf{P}$ are used as complex coefficients $p_k^{i,j}$ of quadratic polynomials

$$P_k = \sum_{0 \leq i \leq j \leq m} p_k^{i,j} z_i z_j,$$

the map $P \circ v$ is of the form

$$[z_0 : \ldots : z_m] \mapsto [P_0 : \ldots : P_n].$$

**Example 2.1.** The $m = 1$, $n = 1$ case is in the assumed dimension range. A $2 \times 3$ matrix $\mathbf{P}$ with rank 2 has a kernel equal to a line in $\mathbb{C}^3$, or a single point $x \in \mathbb{C}P^2$. $P \circ v : \mathbb{C}P^1 \to \mathbb{C}P^1$ is well-defined if the image of $v([z_0 : z_1]) = [z_0^2 : z_0 z_1 : z_1^2]$, a complex curve in $\mathbb{C}P^2$, misses the point $x$. Otherwise, $P \circ v$ is defined on all but one point of the domain $\mathbb{C}P^1$.

Even if it is well-defined, the composition $P \circ v$ may not be one-to-one, and may also have singular points, where its (complex) Jacobian has rank less than $m$.

**Theorem 2.2.** *Suppose* $\mathbf{P}$ *and* $\mathbf{Q}$ *are coefficient matrices so that the induced maps* $P \circ v$ *and* $Q \circ v$ *are equal, and well-defined at every point of* $\mathbb{C}P^m$. *Then, there exists a non-zero constant* $\nu \in \mathbb{C}$ *so that* $\mathbf{P} = \nu \cdot \mathbf{Q}$.

*Proof.* The equality of the maps induced by $\mathbf{P} \circ \mathbf{v}$ and $\mathbf{Q} \circ \mathbf{v}$ means that there exists some function $\mathbf{f} : \mathbb{C}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{C} \setminus \{0\}$ so that

$$(\mathbf{P} \circ \mathbf{v})(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \cdot (\mathbf{Q} \circ \mathbf{v})(\mathbf{z}).$$

Applying this equality, and the fact that $\mathbf{v}(\mu \cdot \mathbf{z}) = \mu^2 \cdot \mathbf{v}(\mathbf{z})$, to $\lambda \cdot \mathbf{z}$, $\lambda \neq 0$,

$$\begin{aligned} (\mathbf{P} \circ \mathbf{v})(\lambda \cdot \mathbf{z}) &= \mathbf{f}(\lambda \cdot \mathbf{z}) \cdot (\mathbf{Q} \circ \mathbf{v})(\lambda \cdot \mathbf{z}) \\ \implies \lambda^2 \cdot (\mathbf{P} \circ \mathbf{v})(\mathbf{z}) &= \lambda^2 \cdot \mathbf{f}(\lambda \cdot \mathbf{z}) \cdot (\mathbf{Q} \circ \mathbf{v})(\mathbf{z}) \\ \implies \lambda^2 \cdot \mathbf{f}(\mathbf{z}) \cdot (\mathbf{Q} \circ \mathbf{v})(\mathbf{z}) &= \lambda^2 \cdot \mathbf{f}(\lambda \cdot \mathbf{z}) \cdot (\mathbf{Q} \circ \mathbf{v})(\mathbf{z}). \end{aligned}$$

Now, the hypothesis that $Q \circ v$ is well-defined implies $(\mathbf{Q} \circ \mathbf{v})(\mathbf{z}) \neq \mathbf{0}$, so $\mathbf{f}(\mathbf{z}) = \mathbf{f}(\lambda \cdot \mathbf{z})$. Since $\mathbf{f}$ satisfies properties (2.1) and (2.2), it defines a function $\underline{\mathbf{f}} : \mathbb{C}P^m \to \mathbb{C}$ (it also induces a map $f : \mathbb{C}P^m \to \mathbb{C}P^0$, but this is different and will not be needed). $\underline{\mathbf{f}}$ can be given an explicit expression when restricted to an affine neighborhood, say $U_0 = \{z_0 = 1\}$:

$$\underline{\mathbf{f}}([1 : z_1 : \ldots : z_m]) = \mathbf{f}((1, z_1, \ldots, z_m)).$$

For each $u \in U_0$, with a representative vector $\mathbf{u}$, $(\mathbf{Q} \circ \mathbf{v})(\mathbf{u}) \neq \mathbf{0}$ implies there is some component $Q_k$, $k = 0, \ldots, n$, so that

$$Q_k(\mathbf{z}) = q_k^{0,0} + \sum_{j=1}^{m} q_k^{0,j} z_j + \sum_{1 \le i \le j \le m} q_k^{i,j} z_i z_j$$

is non-zero for $\mathbf{z}$ in a small neighborhood of $\mathbf{u}$ in $U_0$. In that neighborhood,

$$\underline{\mathbf{f}}(z) = \mathbf{f}(\mathbf{z}) = \frac{P_k(\mathbf{z})}{Q_k(\mathbf{z})}$$

is holomorphic in $z_1, \ldots, z_m$. Similarly, $\underline{\mathbf{f}}$ is holomorphic in every affine neighborhood, but a holomorphic map $\underline{\mathbf{f}} : \mathbb{C}P^m \to \mathbb{C}$ must be constant. The result follows since the image of $\mathbf{v}$ clearly spans $\mathbb{C}^{(m+1)(m+2)/2}$. ∎

**Example 2.3.** In general, to establish that $\mathbf{P} = \nu \cdot \mathbf{Q}$, it is not enough to check that $P \circ v = Q \circ v$ only on some open set. For example, with $m = n = 1$, the coefficient matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

defines a composite map $P \circ v : [z_0 : z_1] \mapsto [z_0^2 : z_0 z_1]$, which is not defined at the point $[0 : 1]$. For

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the composite map is $Q \circ v : [z_0 : z_1] \mapsto [z_0 z_1 : z_1^2]$. It is not defined at the point $[1 : 0]$, but $(Q \circ v)(z) = (P \circ v)(z)$ for every $z$ in $\mathbb{C}P^1$ except two.

The following Proposition is recalled from [CLO] §8.5, and it gives another interesting property of maps which are defined at every point.

**Proposition 2.4.** *Let $F : \mathbb{C}P^m \to \mathbb{C}P^n$ be defined by homogeneous polynomials $[f_0 : \ldots : f_n]$, of the same degree in $z_0, \ldots z_m$, with no common zeros. Then, the image $F(\mathbb{C}P^m)$ is an irreducible projective algebraic variety in $\mathbb{C}P^n$.* ∎

The Proposition applies to maps of the form $P \circ v$ when the degree is two, and in some later examples the implicit polynomial equations defining the image $(P \circ v)(\mathbb{C}P^m) \subseteq \mathbb{C}P^n$ will be given. The corresponding claim for maps between real projective spaces is false, as shown by the Whitney umbrella surface and other examples ([A], [CLO], [CSS]) where the real parametric image does not fill up a real variety.

**Definition 2.5.** For fixed integers $m$, $n$, two coefficient matrices $\mathbf{P}$ and $\mathbf{Q}$ are "c-equivalent" if there exist matrices $\mathbf{A} \in GL(m+1, \mathbb{C})$, $\mathbf{B} \in GL(n+1, \mathbb{C})$ such that for all $\mathbf{z} \in \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$,

$$\mathbf{Q} \cdot (\mathbf{v}(\mathbf{z})) = \mathbf{B} \cdot \mathbf{P} \cdot (\mathbf{v}(\mathbf{A} \cdot \mathbf{z})).$$

The following Theorem relates c-equivalence to congruence of matrix pencils. Similar classification theorems, with similar proofs, appear in [CSS] and [$C_1$].

**Theorem 2.6.** $\mathbf{P}$ *and* $\mathbf{Q}$ *are c-equivalent if and only if there exists* $\mathbf{A} \in GL(m+1, \mathbb{C})$ *such that the following* $(m(m+3)/2-n)$*-dimensional subspaces of* $S(m+1, \mathbb{C})$ *are equal:*

$$\mathbf{k}(\ker(\mathbf{P})) = \mathbf{A} \cdot (\mathbf{k}(\ker(\mathbf{Q}))) \cdot \mathbf{A}^T.$$

*Proof.* The map

$$\mathbf{z} \mapsto \mathbf{vech}(\mathbf{A} \cdot (\mathbf{k}(\mathbf{z})) \cdot \mathbf{A}^T)$$

is a $\mathbb{C}$-linear invertible map $\mathbb{C}^{(m+1)(m+2)/2} \to \mathbb{C}^{(m+1)(m+2)/2}$. It, and its representation as a square matrix, will be denoted $[\mathbf{A} \otimes \mathbf{A}]$.

Using Equation (2.3),

$$\begin{aligned}
(\mathbf{k} \circ \mathbf{v})(\mathbf{A} \cdot \mathbf{z}) &= (\mathbf{A} \cdot \mathbf{z}) \cdot (\mathbf{A} \cdot \mathbf{z})^T \\
&= \mathbf{A} \cdot \mathbf{z} \cdot \mathbf{z}^T \cdot \mathbf{A}^T \\
&= \mathbf{A} \cdot ((\mathbf{k} \circ \mathbf{v})(\mathbf{z})) \cdot \mathbf{A}^T \\
&= \mathbf{k}([\mathbf{A} \otimes \mathbf{A}] \cdot (\mathbf{v}(\mathbf{z}))).
\end{aligned}$$

Since $\mathbf{k}$ is an isomorphism,

$$\mathbf{v}(\mathbf{A} \cdot \mathbf{z}) = [\mathbf{A} \otimes \mathbf{A}] \cdot (\mathbf{v}(\mathbf{z})). \tag{2.4}$$

(For present purposes, $[\mathbf{A} \otimes \mathbf{A}]$ is merely a convenient label; see [Searle] or [$C_3$] for the connections between vectorization of matrices and tensor products.)

So, from the definition of c-equivalence,

$$\mathbf{Q} \cdot (\mathbf{v}(\mathbf{z})) = \mathbf{B} \cdot \mathbf{P} \cdot (\mathbf{v}(\mathbf{A} \cdot \mathbf{z})) = (\mathbf{B} \cdot \mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}]) \cdot (\mathbf{v}(\mathbf{z})),$$

and since the image of $\mathbf{v} : \mathbb{C}^{m+1} \to \mathbb{C}^{(m+1)(m+2)/2}$ spans the target space, $\mathbf{Q}$ and $\mathbf{P}$ are c-equivalent if and only if there exist $\mathbf{A}$, $\mathbf{B}$ so that

$$\mathbf{Q} = \mathbf{B} \cdot \mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}].$$

This equation says $\mathbf{Q}$ and $\mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}]$ are "row-equivalent" matrices, and therefore there exists such an invertible $\mathbf{B}$ if and only if $\ker(\mathbf{Q}) = \ker(\mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}])$. This equality of subspaces of $\mathbb{C}^{(m+1)(m+2)/2}$ is equivalent to the equality of subspaces of $S(m+1,\mathbb{C})$:

$$\mathbf{k}(\ker(\mathbf{Q})) = \mathbf{k}(\ker(\mathbf{P}\cdot[\mathbf{A}\otimes\mathbf{A}])).$$

Suppose $\mathbf{K}\in\mathbf{k}(\ker(\mathbf{P}\cdot[\mathbf{A}\otimes\mathbf{A}]))$. This is equivalent to

$$\mathbf{0} = (\mathbf{P}\cdot[\mathbf{A}\otimes\mathbf{A}])(\mathbf{vech}(\mathbf{K})) = \mathbf{P}\cdot\mathbf{vech}(\mathbf{A}\cdot\mathbf{K}\cdot\mathbf{A}^T),$$

by definition of $[\mathbf{A}\otimes\mathbf{A}]$, or, equivalently,

$$\mathbf{vech}(\mathbf{A}\cdot\mathbf{K}\cdot\mathbf{A}^T)\in\ker(\mathbf{P}) \iff \mathbf{A}\cdot\mathbf{K}\cdot\mathbf{A}^T\in\mathbf{k}(\ker(\mathbf{P})).$$

This proves the claim of the Theorem. ∎

**Corollary 2.7.** *Given matrices* $\mathbf{P}$ *and* $\mathbf{Q}$*, let* $P$ *and* $Q$ *be the induced projections. If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are c-equivalent, then there exist automorphisms* $A\in PGL(m+1,\mathbb{C})$*,* $B\in PGL(n+1,\mathbb{C})$ *such that*

$$(Q\circ v)(z) = B\cdot((P\circ v)(A\cdot z))$$

*for all* $z\in\mathbb{C}P^m$ *where both sides are defined. Conversely, if there exist* $A$ *and* $B$ *such that* $Q$ *and* $P$ *satisfy the above equation at every point* $z\in\mathbb{C}P^m$*, then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are c-equivalent.*

*Proof.* The first implication is easy: if there exist $\mathbf{A}$ and $\mathbf{B}$ so that $\mathbf{Q}\circ\mathbf{v} = \mathbf{B}\circ\mathbf{P}\circ\mathbf{v}\circ\mathbf{A}$, then they induce $A$ and $B$ so that the composite maps of projective spaces are equal where they are defined. For the converse, suppose there exist such maps $A$, $B$, and let $\mathbf{A}$, $\mathbf{B}$ be matrix representatives. By the Proof of the previous Theorem, $\mathbf{B}\circ\mathbf{P}\circ\mathbf{v}\circ\mathbf{A} = \mathbf{B}\circ\mathbf{P}\circ[\mathbf{A}\otimes\mathbf{A}]\circ\mathbf{v}$, so $\mathbf{Q}$ and $\mathbf{B}\circ\mathbf{P}\circ[\mathbf{A}\otimes\mathbf{A}]$ are linear maps whose composites with $\mathbf{v}$ induce equal maps on all of $\mathbb{C}P^m$. By Theorem 2.2, there exists a constant $\nu\neq 0$ so that

$$\mathbf{Q} = \nu\cdot\mathbf{B}\circ\mathbf{P}\circ[\mathbf{A}\otimes\mathbf{A}],$$

and again, from the previous Proof, this proves $\mathbf{P}$ and $\mathbf{Q}$ are c-equivalent. ∎

So, if $\mathbf{P}$ and $\mathbf{Q}$ are c-equivalent, it is not too abusive to also call the compositions $P\circ v$ and $Q\circ v$ "c-equivalent maps" $\mathbb{C}P^m\to\mathbb{C}P^n$. The geometric idea is that the compositions $Q\circ v$ and $P\circ v$ are related by a reparametrization $A$ of the domain, $\mathbb{C}P^m$, and a coordinate change $B$ of the target, $\mathbb{C}P^n$.

**Example 2.8.** In the $m=2$, $n=3$ case, the image of $P\circ v:\mathbb{C}P^2\to\mathbb{C}P^3$ is called a complex Steiner surface. The c-equivalence classes of such maps were known classically ([A], [Salmon], [Sommerville]).

2.3. **Real projective geometry.** The maps to be introduced in this Subsection are the inclusion:

$$\begin{aligned}\boldsymbol{\delta}:\mathbb{R}^{m+1} &\to \mathbb{C}^{m+1}:\\ (u_0,\ldots,u_m)^T &\mapsto (u_0+0i,\ldots,u_m+0i)^T,\end{aligned}$$

and the real linear involution of $\mathbb{C}^{m+1}$ defined by entrywise complex conjugation:

$$\begin{aligned}\mathbf{C}:\mathbb{C}^{m+1} &\to \mathbb{C}^{m+1}:\\ (z_0,\ldots,z_m)^T &\mapsto (\bar{z}_0,\ldots,\bar{z}_m)^T.\end{aligned}$$

The image of $\boldsymbol{\delta}$ is exactly the fixed point set of $\mathbf{C}$. For non-zero vectors, $\boldsymbol{\delta}$ satisfies (2.1) and (2.2), with $\mathbb{K} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$, and $\mathbf{C}$ is not complex linear but still satisfies (2.1) and (2.2), so both maps induce well-defined maps of projective spaces:

$$\delta : \mathbb{R}P^m \to \mathbb{C}P^m, \ C : \mathbb{C}P^m \to \mathbb{C}P^m.$$

**Theorem 2.9.** *The induced map $\delta$ is a smooth embedding, and its image is a regular smooth submanifold of $\mathbb{C}P^m$ which is equal to the fixed point set of the induced involution $C$.*

*Proof.* For this Theorem, $\mathbb{C}P^m$ is considered only as a differentiable manifold of real dimension $2m$. The map $\delta$ is a smooth immersion because it is a smooth immersion on affine coordinate charts, for example, the induced map restricted to the $\{u_0 \neq 0\}$ neighborhood ($\cong \mathbb{R}^m$) maps to the $\{z_0 \neq 0\}$ neighborhood ($\mathbb{C}^m \cong \mathbb{R}^{2m}$) in the target:

$$\begin{array}{rcl} \delta : \mathbb{R}^m & \to & \mathbb{C}^m : \\ [1 : u_1 : \ldots : u_m] & \mapsto & [1 + 0i : u_1 + 0i : \ldots : u_m + 0i]. \end{array}$$

The induced map is one-to-one: suppose $u, w \in \mathbb{R}P^m$, and $\delta(u) = \delta(w)$. Then, $u$ and $w$ are spanned by non-zero vectors $\mathbf{u}, \mathbf{w} \in \mathbb{R}^{m+1}$, and by definition of the induced map, there exists a non-zero complex scalar $\mu$ so that

$$\boldsymbol{\delta}(\mathbf{u}) = \mu \cdot \boldsymbol{\delta}(\mathbf{w}) \in \mathbb{C}^{m+1}.$$

The vector $\mathbf{u} = (u_0, \ldots, u_m)^T$ has some non-zero entry $u_j \in \mathbb{R}$, which satisfies $u_j + 0i = \mu \cdot (w_j + 0i)$, so $w_j$ is also non-zero, and in particular, $\mu = \frac{u_j}{w_j} \in \mathbb{R} \subseteq \mathbb{C}$. This implies $\mathbf{u}$ is a real scalar multiple of $\mathbf{w}$:

$$(u_0 + i0, \ldots, u_m + i0)^T = \mu \cdot (w_0 + i0, \ldots, w_m + i0)^T,$$

so the real lines $u$ and $w$ are equal. This is enough to show that the image of the compact manifold $\mathbb{R}P^m$ is a regular submanifold in the sense of [B] §III.5.

It is obvious that for any element $[u_0 : \ldots : u_m] \in \mathbb{R}P^m$, its image under $\delta$ is fixed by the involution $C$. Suppose, conversely, that $z = [z_0 : \ldots : z_m]$ is a fixed point, so there exists some non-zero complex scalar $\mu$ such that

$$(\bar{z}_0, \ldots, \bar{z}_m)^T = \mu \cdot (z_0, \ldots, z_m)^T.$$

Then, for some non-zero entry $z_j \in \mathbb{C}$, $\mu = \frac{\bar{z}_j}{z_j}$ is an element of the unit circle $S^1 \subseteq \mathbb{C}$. Twice the "real part" of $\mathbf{z}$ is:

$$\mathbf{z} + \mathbf{C}(\mathbf{z}) = (z_0 + \bar{z}_0, \ldots, z_m + \bar{z}_m)^T = (1 + \mu)\mathbf{z}.$$

If $\frac{\mu_1}{\mu_2} \neq -1$, this shows that $\mathbf{z}$ is a complex scalar multiple of a non-zero vector with real entries, so $z$ is in the image of $\delta$. If $\mu = -1$, the following $\mathbf{C}$-invariant, non-zero complex scalar multiple of $\mathbf{z}$ will work instead:

$$i \cdot (\mathbf{z} - \mathbf{C}(\mathbf{z})) = i \cdot (\mathbf{z} + \mathbf{z}) \neq \mathbf{0}.$$

∎

The composition $v \circ \delta : \mathbb{R}P^m \to \mathbb{C}P^{m(m+3)/2}$ is also a smooth embedding. It has the following form, for $u = [u_0 : \ldots : u_m]$:

$$u \mapsto [u_0^2 : u_0u_1 : u_1^2 : u_0u_2 : u_1u_2 : u_2^2 : \ldots : u_0u_m : u_1u_m : \ldots : u_m^2].$$

The image happens to be contained in the image of another inclusion

$$\delta' : \mathbb{R}P^{m(m+3)/2} \to \mathbb{C}P^{m(m+3)/2},$$

and it is the "real Veronese variety" named in the Title. For a coefficient matrix $\mathbf{P}$, the composition $P\circ v\circ\delta:\mathbb{R}P^m\to\mathbb{C}P^n$ is smooth at points where it is well-defined, but it is not necessarily one-to-one or nonsingular. It is possible that $P\circ v\circ\delta$ is well-defined, or an embedding, even if $P\circ v$ is neither. As mentioned in the Introduction, the composition $P\circ v\circ\delta$ is of the form:

$$u\mapsto[P_0:P_1:\ldots:P_n],$$

with complex coefficients $p_k^{i,j}$ on quadratic terms in real variables:

$$P_k=\sum_{0\le i\le j\le m}p_k^{i,j}u_iu_j.$$

Maps of the form $P\circ v\circ\delta$ will be the main objects of interest in subsequent Sections. These real analytic parametrizations do not behave exactly like the complex analytic maps $P\circ v$. For instance, a minor modification of Example 2.3 shows that an analogue of Theorem 2.2 fails.

**Example 2.10.** The coefficient matrix

$$\mathbf{P}=\begin{pmatrix}1&i&0\\0&1&i\end{pmatrix}$$

defines a composite map $P\circ v\circ\delta:[u_0:u_1]\mapsto[(u_0+iu_1)u_0:(u_0+iu_1)u_1]$, and

$$\mathbf{Q}=\begin{pmatrix}1&-i&0\\0&1&-i\end{pmatrix}$$

defines a map $Q\circ v\circ\delta:[u_0:u_1]\mapsto[(u_0-iu_1)u_0:(u_0-iu_1)u_1]$. These maps $\mathbb{R}P^1\to\mathbb{C}P^1$ agree at every point in the domain, but the matrices are not related by scalar multiplication, and the maps $P\circ v$, $Q\circ v$ are not defined on all of $\mathbb{C}P^1$.

Some examples (Examples 4.8, 4.10) will show that an analogue of Proposition 2.4 fails in general. The image of a map $P\circ v\circ\delta$ will be contained in some real algebraic variety in $\mathbb{C}P^n$, but may not be equal to it, even if $P\circ v$ is defined everywhere on $\mathbb{C}P^m$.

The rest of this Section will develop a notion of equivalence for coefficient matrices $\mathbf{P}$ which will be useful in studying the geometry of maps $P\circ v\circ\delta$.

It is easy to see that a matrix $\mathbf{A}\in M(m+1,\mathbb{C})$ has all real entries if and only if $\mathbf{A}=\mathbf{C}\circ\mathbf{A}\circ\mathbf{C}$. In fact, such matrices are the only ones that fix the image of $\boldsymbol{\delta}$.

**Lemma 2.11.** *Suppose* $\mathbf{z}=\mathbf{C}(\mathbf{z})$ *implies* $\mathbf{A}\cdot\mathbf{z}=\mathbf{C}(\mathbf{A}\cdot\mathbf{z})$. *Then* $\mathbf{A}=\mathbf{C}\circ\mathbf{A}\circ\mathbf{C}$.

*Proof.* For all $\mathbf{z}$ such that $\mathbf{z}=\mathbf{C}(\mathbf{z})$, $\mathbf{A}\cdot\mathbf{z}=\mathbf{C}(\mathbf{A}\cdot\mathbf{z})=\mathbf{C}(\mathbf{A}\cdot\mathbf{C}(\mathbf{z}))$. $\mathbf{C}\circ\mathbf{A}\circ\mathbf{C}$ is complex linear, and it is equal to the complex linear transformation $\mathbf{A}$, because they agree on the set $\boldsymbol{\delta}(\mathbb{R}^{m+1})$, which spans $\mathbb{C}^{m+1}$. ∎

The Lemma shows that the inclusion $\boldsymbol{\delta}'':M(m+1,\mathbb{R})\to M(m+1,\mathbb{C})$ defines a bijection between matrices with real entries and complex linear transformations that leave invariant $\boldsymbol{\delta}(\mathbb{R}^{m+1})$. If $\underline{\mathbf{A}}$ denotes a real matrix, with corresponding complex matrix $\boldsymbol{\delta}''(\underline{\mathbf{A}})=\underline{\mathbf{A}}+i\cdot\underline{\mathbf{0}}=\mathbf{A}$, then

$$\boldsymbol{\delta}\circ\underline{\mathbf{A}}=\mathbf{A}\circ\boldsymbol{\delta}:\mathbb{R}^{m+1}\to\mathbb{C}^{m+1}. \tag{2.5}$$

Given $\underline{\mathbf{A}}$, $\mathbf{A}$ is the only complex linear transformation satisfying (2.5); if also $\boldsymbol{\delta}\circ\underline{\mathbf{A}}=\mathbf{A_1}\circ\boldsymbol{\delta}$, then $\mathbf{A_1}$ must equal $\mathbf{A}$, since they agree on $\boldsymbol{\delta}(\mathbb{R}^{m+1})$.

A fact similar to Lemma 2.11 applies to automorphisms of complex projective space.

**Theorem 2.12.** *Given an automorphism $A$ of $\mathbb{C}P^m$, suppose $A$ fixes $\delta(\mathbb{R}P^m)$ as a set:*

$$z = C(z) \implies A \cdot z = C(A \cdot z).$$

*Then, there exists $\mathbf{A} \in GL(m+1,\mathbb{C})$ so that $\mathbf{A}$ induces $A$, and $\mathbf{A} = \mathbf{C} \circ \mathbf{A} \circ \mathbf{C}$.*

*Proof.* There is some invertible matrix $\mathbf{A_0}$ that induces $A$. For any nonzero real vector $\mathbf{u} \in \boldsymbol{\delta}(\mathbb{R}^{m+1}) \setminus \{\mathbf{0}\}$, the equation $u = C(u)$ holds, so $A \cdot u = C(A \cdot u)$, and there is some complex scalar $\mu \neq 0$, so that

$$\mathbf{C}(\mathbf{A_0} \cdot \mathbf{u}) = \mathbf{C}(\mathbf{A_0} \cdot \mathbf{C}(\mathbf{u})) = \mu \cdot \mathbf{A_0} \cdot \mathbf{u}.$$

This implies that there is some function $\mathbf{f} : \boldsymbol{\delta}(\mathbb{R}^{m+1}) \setminus \{\mathbf{0}\} \to \mathbb{C} \setminus \{0\}$ so that

$$\mathbf{f}(\mathbf{u}) \cdot \mathbf{u} = \mathbf{A_0}^{-1} \cdot \mathbf{C}(\mathbf{A_0} \cdot \mathbf{C}(\mathbf{u})).$$

Notice the composite real linear transformation $\mathbf{A_0}^{-1} \circ \mathbf{C} \circ \mathbf{A_0} \circ \mathbf{C}$ is in fact complex linear, with some matrix representative $\mathbf{S} \in GL(m+1,\mathbb{C})$, and the above equation just says that every nonzero real vector is an eigenvector of $\mathbf{S}$.

There exist basis vectors $\mathbf{u}^0, \ldots, \mathbf{u}^m \in \boldsymbol{\delta}(\mathbb{R}^{m+1})$, so that any $\mathbf{z} \in \mathbb{C}^{m+1}$ is a unique complex linear combination of the basis elements. Using the linearity of $\mathbf{S}$, and the eigenvalue equation,

$$\begin{aligned} \mathbf{S}(\mathbf{u}^0 + \cdots + \mathbf{u}^m) &= \mathbf{S}(\mathbf{u}^0) + \cdots + \mathbf{S}(\mathbf{u}^m) \\ \mathbf{f}(\mathbf{u}^0 + \cdots + \mathbf{u}^m) \cdot (\mathbf{u}^0 + \cdots + \mathbf{u}^m) &= \mathbf{f}(\mathbf{u}^0) \cdot \mathbf{u}^0 + \cdots + \mathbf{f}(\mathbf{u}^m) \cdot \mathbf{u}^m, \end{aligned}$$

and if $\lambda = \mathbf{f}(\mathbf{u}^0 + \cdots + \mathbf{u}^m)$, then $\mathbf{f}(\mathbf{u}^0) = \cdots = \mathbf{f}(\mathbf{u}^m) = \lambda$, by the uniqueness of the coefficients. Since the complex linear transformations $\mathbf{S}$ and $\lambda \cdot \mathbf{I}$ agree on a basis, they are equal.

It follows that $\lambda \cdot \mathbf{A_0} = \mathbf{C} \circ \mathbf{A_0} \circ \mathbf{C}$, and multiplying both sides by $\bar{\lambda}$ gives

$$\bar{\lambda} \cdot \lambda \cdot \mathbf{A_0} = \bar{\lambda} \cdot \mathbf{C} \circ \mathbf{A_0} \circ \mathbf{C} = \mathbf{C} \circ (\lambda \cdot \mathbf{A_0}) \circ \mathbf{C} = \mathbf{C} \circ (\mathbf{C} \circ \mathbf{A_0} \circ \mathbf{C}) \circ \mathbf{C} = \mathbf{A_0},$$

so $\bar{\lambda} \cdot \lambda = 1$, and $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Let $\mathbf{A} = e^{i\theta/2} \cdot \mathbf{A_0}$, so that

$$\mathbf{C} \circ \mathbf{A} \circ \mathbf{C} = \mathbf{C} \circ (e^{i\theta/2} \cdot \mathbf{A_0}) \circ \mathbf{C} = e^{-i\theta/2} \cdot \mathbf{C} \circ \mathbf{A_0} \circ \mathbf{C} = e^{-i\theta/2} \cdot (e^{i\theta} \mathbf{A_0}) = \mathbf{A}.$$

∎

The uniqueness statement from Equation (2.5) also has a projective version.

**Theorem 2.13.** *If $\underline{\mathbf{A}} \in GL(m+1,\mathbb{R})$ induces an automorphism $\underline{A}$ of $\mathbb{R}P^m$, and $\mathbf{A_1} \in GL(m+1,\mathbb{C})$ induces an automorphism $A_1$ of $\mathbb{C}P^m$ such that $A_1 \circ \delta = \delta \circ \underline{A}$, then there is a non-zero complex constant $\lambda$ so that $\mathbf{A_1} = \lambda \cdot \boldsymbol{\delta}''(\underline{\mathbf{A}})$.*

*Proof.* Let $\mathbf{A} = \boldsymbol{\delta}''(\underline{\mathbf{A}})$, so that $\boldsymbol{\delta} \circ \underline{\mathbf{A}} = \mathbf{A} \circ \boldsymbol{\delta}$. Then, $A_1 \circ \delta = \delta \circ \underline{A}$ implies there is some function $\mathbf{f} : \mathbb{R}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{C} \setminus \{\mathbf{0}\}$ so that $(\mathbf{A_1} \circ \boldsymbol{\delta})(\mathbf{u}) = \mathbf{f}(\mathbf{u}) \cdot (\boldsymbol{\delta} \circ \underline{\mathbf{A}})(\mathbf{u}) = \mathbf{f}(\mathbf{u}) \cdot (\mathbf{A} \circ \boldsymbol{\delta})(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$. As in the Proof of the previous Theorem, this means every nonzero real vector is an eigenvector of $\mathbf{A_1}^{-1} \circ \mathbf{A}$, so $\mathbf{A_1} = \lambda \cdot \mathbf{A}$. ∎

It follows from Theorems 2.12 and 2.13 that $A$ fixes $\delta(\mathbb{R}P^m)$ if and only if $A = C \circ A \circ C$, and that every such automorphism is uniquely determined by its restriction to $\delta(\mathbb{R}P^m)$.

**Definition 2.14.** For fixed integers $m$, $n$, two (complex) coefficient matrices $\mathbf{P}$ and $\mathbf{Q}$ are "r-equivalent" if there exist matrices $\mathbf{A} \in GL(m+1,\mathbb{C})$, $\mathbf{B} \in GL(n+1,\mathbb{C})$ such that $\mathbf{A} = \mathbf{C} \circ \mathbf{A} \circ \mathbf{C}$, and for all $\mathbf{z} \in \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$,

$$\mathbf{Q} \cdot (\mathbf{v}(\mathbf{z})) = \mathbf{B} \cdot \mathbf{P} \cdot (\mathbf{v}(\mathbf{A} \cdot \mathbf{z})).$$

**Theorem 2.15.** *Given* $\mathbf{P}$ *and* $\mathbf{Q}$*, the following are equivalent.*

1. $\mathbf{P}$ *and* $\mathbf{Q}$ *are r-equivalent.*
2. *There exist* $\mathbf{A} \in GL(m+1,\mathbb{C})$*,* $\mathbf{B} \in GL(n+1,\mathbb{C})$ *such that* $\mathbf{A} = \mathbf{C} \circ \mathbf{A} \circ \mathbf{C}$*, and for all* $\mathbf{u} \in \mathbb{R}^{m+1}$*,*
$$\mathbf{Q} \cdot ((\mathbf{v} \circ \boldsymbol{\delta})(\mathbf{u})) = \mathbf{B} \cdot \mathbf{P} \cdot (\mathbf{v}(\mathbf{A} \cdot \boldsymbol{\delta}(\mathbf{u}))).$$
3. *There exist* $\underline{\mathbf{A}} \in GL(m+1,\mathbb{R})$*,* $\mathbf{B} \in GL(n+1,\mathbb{C})$ *such that for all* $\mathbf{u} \in \mathbb{R}^{m+1}$*,*
$$\mathbf{Q} \cdot ((\mathbf{v} \circ \boldsymbol{\delta})(\mathbf{u})) = \mathbf{B} \cdot \mathbf{P} \cdot ((\mathbf{v} \circ \boldsymbol{\delta})(\underline{\mathbf{A}} \cdot \mathbf{u})).$$
4. *There exists* $\mathbf{A} \in GL(m+1,\mathbb{C})$ *such that* $\mathbf{A} = \mathbf{C} \circ \mathbf{A} \circ \mathbf{C}$ *, and*
$$\mathbf{k}(\ker(\mathbf{P})) = \mathbf{A} \cdot (\mathbf{k}(\ker(\mathbf{Q}))) \cdot \mathbf{A}^T.$$

*Proof.* That (1) implies (2) follows from Definition 2.14, the equivalence of (2) and (3) follows from Lemma 2.11 and Equation (2.5), and (1) and (4) are equivalent by Theorem 2.6. Assuming (2), and using the identity (2.4) from Theorem 2.6, gives

$$\mathbf{Q} \cdot ((\mathbf{v} \circ \boldsymbol{\delta})(\mathbf{u})) = \mathbf{B} \cdot \mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}] \cdot ((\mathbf{v} \circ \boldsymbol{\delta})(\mathbf{u}))$$

for all $\mathbf{u} \in \mathbb{R}^{m+1}$. Since $(\mathbf{v} \circ \boldsymbol{\delta})(\mathbb{R}^{m+1})$ spans $\mathbb{C}^{(m+1)(m+2)/2} (\cong S(m+1,\mathbb{C}))$, $\mathbf{Q} = \mathbf{B} \cdot \mathbf{P} \cdot [\mathbf{A} \otimes \mathbf{A}]$, which, by the calculations from the Proof of Theorem 2.6, implies (4). ∎

**Corollary 2.16.** *Given matrices* $\mathbf{P}$ *and* $\mathbf{Q}$*, let* $P$ *and* $Q$ *be the induced projections. If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are r-equivalent, then there exist automorphisms* $A \in PGL(m+1,\mathbb{C})$*,* $B \in PGL(n+1,\mathbb{C})$ *such that* $A = C \circ A \circ C$*, and*

$$(Q \circ v)(z) = B \cdot ((P \circ v)(A \cdot z))$$

*for all* $z \in \mathbb{C}P^m$ *where both sides are defined. Conversely, if there exist* $A$ *and* $B$ *such that* $A = C \circ A \circ C$*, and* $Q$ *and* $P$ *satisfy the above equation at every point* $z \in \mathbb{C}P^m$*, then* $\mathbf{P}$ *and* $\mathbf{Q}$ *are r-equivalent.*

*Proof.* The argument is identical to the Proof of Corollary 2.7, except that Theorem 2.12 must be used for the converse, when picking a matrix representing $A$. ∎

However, in contrast to Corollary 2.7, some examples (Examples 5.7, 5.8) will show that the existence of $\underline{A}$ and $B$ such that

$$Q \circ v \circ \delta = B \circ P \circ v \circ \delta \circ \underline{A} : \mathbb{R}P^m \to \mathbb{C}P^n$$

at every point of $\mathbb{R}P^m$ is not enough to establish the r-equivalence of $\mathbf{P}$ and $\mathbf{Q}$.

## 3. Equivalence of parametrizations

The remaining Sections will consider maps of the form $P \circ v \circ \delta : \mathbb{R}P^m \to \mathbb{C}P^n$, for specific choices of $m$ and $n$. To get an idea of which $m$, $n$ will pose interesting, yet tractable, r-equivalence classification problems, consider the following naïve dimension count.

Recall that coefficient matrices have size $(n+1) \times (m+1)(m+2)/2$, and are full rank, with complex scalar multiples of a matrix $\mathbf{P}$ defining exactly the same projection $P$. So, the "parameter space" of projection matrices is a dense open subset of $\mathbb{C}P^{(n+1)(m+1)(m+2)/2-1}$, which has real dimension $(n+1)(m+1)(m+2) - 2$. The group acting on the matrix space, whose orbits are the r-equivalence classes, is $PGL(m+1,\mathbb{R}) \times PGL(n+1,\mathbb{C})$, which has real dimension $\big((m+1)^2 - 1\big) +$

$2\left((n+1)^2-1\right)$. The difference between these two dimensions is the expected number of real moduli:

$$\begin{aligned}\mathcal{M}(m,n) &= (n+1)(m+1)(m+2)-2-(m(m+2)+2n(n+2))\\ &= m^2n+3mn+m-2n-2n^2.\end{aligned}$$

The following table lists some values of $\mathcal{M}(m,n)$, with $0\le n<(m+1)(m+2)/2-1$. It also shows the dimension of the kernel of an associated coefficient matrix $\mathbf{P}$.

| $m$ | $n$ | $\mathcal{M}(m,n)$ | $\dim_{\mathbb{C}}(\mathbf{k}(\ker(\mathbf{P})))$ in $S(m+1,\mathbb{C})$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 2 |
| 2 | 4 | 2 | 1 |
| 2 | 3 | 8 | 2 |
| 2 | 2 | 10 | 3 |
| 2 | 1 | 8 | 4 |
| 2 | 0 | 2 | 5 |
| 3 | 8 | 3 | 1 |
| $\vdots$ | | | |

The last row is the case $m=3$, $n=8$, where the map $v\circ\delta:\mathbb{R}P^3\to\mathbb{C}P^9$ can be composed with a $9\times10$ projection matrix $\mathbf{P}$ to get a map $P\circ v\circ\delta:\mathbb{R}P^3\to\mathbb{C}P^8$. Classifying these matrices up to r-equivalence is equivalent to the classification of non-zero $4\times4$ complex symmetric matrices, up to complex scalar multiplication and real congruence. In fact, this congruence problem is solved by [W], and the generic congruence classes are described by a three-dimensional set of real parameters, as expected by $\mathcal{M}(3,8)=3$.

The simplest cases, to be examined in the next Sections, are the first three rows in the above table. We will use the real congruence problem to find representatives of each r-equivalence class, and to see how the algebraic invariants of $\mathbf{P}$ correspond to geometric properties of the maps $P\circ v\circ\delta$ and $P\circ v$.

Some of the interesting geometric features that the image $(P\circ v\circ\delta)(\mathbb{R}P^m)$ may have are differential-topological singularities (Examples 4.6, 4.8, 6.7), or a locus of self-intersection (Examples 4.9, 6.8). As remarked after Theorem 2.9, such points do not occur in the image of $v\circ\delta$, but they could occur after the projection by $P$.

In addition to the differential topology of maps $P\circ v\circ\delta$, it will also be important to consider their interaction with the complex structure on the target space $\mathbb{C}P^n$. A real submanifold $M$ ($\dim_{\mathbb{R}}=m$) of a complex manifold ($\dim_{\mathbb{C}}=n$ with complex structure operator $J$ on the tangent bundle), if it is in general position, will satisfy the following property at most points $x$: $\dim_{\mathbb{C}}(T_xM\cap JT_xM)=\max\{0,m-n\}$. The points $x\in M$ where the tangent space contains a complex subspace of greater dimension than this minimum are called "CR singular" points.

The image of $v\circ\delta:\mathbb{R}P^m\to\mathbb{C}P^{m(m+3)/2}$ is a real submanifold, and at each point, the tangent space contains no complex lines, so it is called "totally real." There could be CR singular points after the projection by $P$, and loci of such points will be another interesting feature to look for when classifying maps $P\circ v\circ\delta$. Sections 6 and 8 will consider several examples of real submanifolds of complex manifolds with $2\le m\le n$, so any point where the tangent space contains a complex line will be a CR singular point. If $u$ is an element of $\mathbb{R}P^m$, and $P\circ v$ is nonsingular at $\delta(u)$, then $P\circ v$ will be a complex analytic diffeomorphism of a neighborhood of

$\delta(u)$ onto a neighborhood in $(P \circ v)(\mathbb{C}P^m)$, and since $\delta(\mathbb{R}P^m)$ is totally real near $\delta(u)$, the image $(P \circ v \circ \delta)(\mathbb{R}P^m)$ will also be totally real near $(P \circ v \circ \delta)(u)$. So, the only candidates for CR singularities in the image of $P \circ v \circ \delta$ will be images of singular points of $P \circ v$, and this phenomenon will be observed in Sections 6 and 8.

## 4. Parametric curves in the Riemann sphere

In the case $m = n = 1$, a $2 \times 3$ matrix $\mathbf{P}$ with rank 2 determines a map $P \circ v \circ \delta : \mathbb{R}P^1 \to \mathbb{C}P^1$, as in Examples 2.1, 2.3, 2.10. Although $P \circ v \circ \delta$ may not be defined on all of $\mathbb{R}P^1$, the image will be a real curve in the Riemann sphere $\mathbb{C}P^1$, with a homogeneous parametric equation of the form

$$[u_0 : u_1] \mapsto [p_0^{00}u_0^2 + p_0^{01}u_0u_1 + p_0^{11}u_1^2 : p_1^{00}u_0^2 + p_1^{01}u_0u_1 + p_1^{11}u_1^2].$$

Restricted to the $\{[1 : u]\}$ real affine line in $\mathbb{R}P^1$, and the $\{[1 : Z]\}$ complex affine line in $\mathbb{C}P^1$, the equation is

$$Z = \frac{p_1^{00} + p_1^{01}u + p_1^{11}u^2}{p_0^{00} + p_0^{01}u + p_0^{11}u^2}.$$

If $Z = X + iY$, the parametric equations are rational functions of $u \in \mathbb{R}$ of degree at most four:

$$\begin{aligned} X &= \frac{\mathrm{Re}((p_1^{00} + p_1^{01}u + p_1^{11}u^2)(\overline{p_0^{00}} + \overline{p_0^{01}}u + \overline{p_0^{11}}u^2))}{|p_0^{00} + p_0^{01}u + p_0^{11}u^2|^2}, \\ Y &= \frac{\mathrm{Im}((p_1^{00} + p_1^{01}u + p_1^{11}u^2)(\overline{p_0^{00}} + \overline{p_0^{01}}u + \overline{p_0^{11}}u^2))}{|p_0^{00} + p_0^{01}u + p_0^{11}u^2|^2}. \end{aligned}$$

The images of such quartic parametrizations are various interesting "special plane curves," but their equivalence classes, under projective transformations of the domain and range, will turn out to have simple representatives. Real curves in $\mathbb{C}P^1$ parametrized by rational functions with complex coefficients are discussed in [MM], which describes a notion essentially the same as r-equivalence, and arrives at the same parameter count, $\mathcal{M}(1, 1) = 1$.

To get started with the classification, consider the c-equivalence problem for $2 \times 3$ coefficient matrices. By Theorem 2.6, it will be enough to recall that the only invariant of one-dimensional subspaces of $S(2, \mathbb{C})$ under complex congruence is the rank: 1 or 2. These correspond to the two cases of Example 2.1, whether the kernel of the coefficient matrix is contained in the image of $\mathbf{v}$ or not.

**Example 4.1.** Given $\mathbf{P}$, if $\mathbf{k}(\ker(\mathbf{P}))$ is a line spanned by a matrix of rank 1, then this line is in the image of $\mathbf{v}$, and it is congruent to

$$\{\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \lambda \in \mathbb{C}\}.$$

$\mathbf{P}$ is c-equivalent to

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as in Example 2.3. The composite map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$, $Q \circ v : [z_0 : z_1] \mapsto [z_0z_1 : z_1^2]$, is well-defined at every point except $[1 : 0]$.

**Example 4.2.** Given $\mathbf{P}$, if $\mathbf{k}(\ker(\mathbf{P}))$ is a line spanned by a matrix of rank 2, then this line is congruent to

$$\{\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C}\},$$

and $\mathbf{P}$ is c-equivalent to

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The composite map from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ is $Q \circ v : [z_0 : z_1] \mapsto [z_0^2 - z_1^2 : z_0 z_1]$. Maps in this c-equivalence class are defined at every point, and are two-to-one except at two singular points.

Under the smaller group, where only "real" changes of variables are allowed, there will be more equivalence classes. The rank 1 case, where $P \circ v$ is undefined at one point, will split into two cases, depending on whether this point is in the image of $\delta$ (Example 4.4) or not (Example 4.5). The rank 2 case will split into some one-parameter families, as expected from $\mathcal{M}(1,1) = 1$.

The following classification of one-dimensional matrix pencils is recalled from [W].

**Proposition 4.3.** *If $\mathbf{K}$ is a non-zero matrix in $S(2, \mathbb{C})$, then there is exactly one matrix in the list below equal to $\lambda \cdot \mathbf{A} \cdot \mathbf{K} \cdot \mathbf{A}^T$ for some nonsingular real matrix $\mathbf{A}$ and non-zero complex scalar $\lambda$.*

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;

(2) $\begin{pmatrix} 0 & 1/2 \\ 1/2 & i \end{pmatrix}$;

(3) $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha = \cos(\theta) + i \sin(\theta)$, $0 \leq \theta \leq \pi$;

(4) $\begin{pmatrix} -it^2 & 1 \\ 1 & i \end{pmatrix}$, $0 < t \leq 1$.

$\blacksquare$

**Example 4.4.** The first normal form in the above list was mentioned in Example 4.1. The representative coefficient matrix $\mathbf{Q}$ induces a map from the real projective line to $\mathbb{C}P^1$, $Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0 u_1 : u_1^2]$, which is not defined at $[1 : 0]$, and on the affine neighborhood $\{[u : 1]\}$, it is the map $u \mapsto [u : 1]$.

**Example 4.5.** The other rank 1 matrix from Proposition 4.3 is in case (4), with $t = 1$. A representative coefficient matrix, i.e., a matrix whose kernel is spanned by $\mathbf{vech}(\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix})$, is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & i \end{pmatrix}.$$

The induced map is

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 + u_1^2 : u_0 u_1 + i u_1^2],$$

which takes $[1:0]$ to $[1:0]$, and restricts to a parametric map $\mathbb{R} \to \mathbb{R}^2$, in the $\{[u:1]\}$, $\{[1:X+iY]\}$ neighborhoods:

$$\begin{aligned} X &= \frac{u}{u^2+1}, \\ Y &= \frac{1}{u^2+1}. \end{aligned}$$

The image of $Q \circ v \circ \delta$ is the circle $X^2 + (Y - \frac{1}{2})^2 = \frac{1}{4}$ in this neighborhood.

We can conclude so far that if $\mathbf{P}$ is a coefficient matrix whose kernel is spanned by a rank 1 complex matrix, then it is r-equivalent to one of the above representatives. Since the action of $PGL(2,\mathbb{C})$ on $\mathbb{C}P^1$ takes lines and circles to lines and circles, the image of $P \circ v \circ \delta$ will be a circle (or line) in the Riemann sphere, possibly with one point deleted.

**Example 4.6.** The exceptional rank 2 matrix, $\begin{pmatrix} 0 & 1/2 \\ 1/2 & i \end{pmatrix}$ from case (2) of Proposition 4.3, has a representative coefficient matrix,

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & i \end{pmatrix}.$$

The induced map is

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 : 2u_0u_1 + iu_1^2],$$

which takes $[0:1]$ to $[0:1]$, and restricts to a parametric map $\mathbb{R} \to \mathbb{R}^2$, in the $\{[1:u]\}$, $\{[1:X+iY]\}$ neighborhoods: $X = 2u$, $Y = u^2$. The image of $Q \circ v \circ \delta$ is the parabola $Y = X^2/4$ in this neighborhood. The point at infinity is a cusp singularity, which is visible in other affine neighborhoods; inverting the parabola in its focus, for example, gives a "cardioid," and more generally the image of a parabola is a "cuspidal biquadratic" ([MM], [CF]).

**Example 4.7.** One of the rank 2 matrices from the list is in case (3), with $\alpha = 1$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and a representative coefficient matrix is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The induced map is

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 - u_1^2 : u_0u_1].$$

This mapping is two-to-one over the whole domain: $(Q \circ v \circ \delta)([u_0 : u_1]) = (Q \circ v \circ \delta)([-u_1 : u_0])$, and the image is $\delta(\mathbb{R}P^1)$.

Example 4.7 shows that r-equivalence classes cannot necessarily be distinguished by inspecting the image of $Q \circ v \circ \delta$, since the image of the map from Example 4.5 was also projectively equivalent to a line.

**Example 4.8.** The other end of case (3) is at $\alpha = -1$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with representative coefficient matrix

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The induced map is

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 + u_1^2 : u_0 u_1].$$

This mapping is two-to-one at most points: $(Q\circ v\circ\delta)([u_0 : u_1]) = (Q\circ v\circ\delta)([u_1 : u_0])$, except at two singular points: $(Q \circ v \circ \delta)([1 : 1]) = [2 : 1]$, $(Q \circ v \circ \delta)([1 : -1]) = [2 : -1]$. The image is contained in $\delta(\mathbb{R}P^1)$; the map $[u_0 : u_1] \mapsto [1 : \frac{u_0 u_1}{u_0^2+u_1^2}]$ is a projection of the circle onto the interval $-\frac{1}{2} \leq X \leq \frac{1}{2}$.

**Example 4.9.** The remaining matrices from case (3), with $\alpha = e^{i\theta}$, $0 < \theta < \pi$, correspond to representative coefficient matrices of the form:

$$\mathbf{Q} = \begin{pmatrix} \alpha & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For each $\alpha$, the induced map is

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [\alpha u_0^2 - u_1^2 : u_0 u_1].$$

This mapping is one-to-one except at a double point, where $[0 : 1]$ and $[1 : 0]$ both go to a "node," $[1 : 0] \in \mathbb{C}P^1$. In one affine neighborhood, the image is a hyperbola, $[u : 1] \mapsto [\alpha u - \frac{1}{u} : 1]$, with parametric and implicit equations:

$$X = \cos(\theta)u - \frac{1}{u}, \quad Y = \sin(\theta)u \implies \cos(\theta)Y^2 = \sin(\theta)XY + \sin^2(\theta).$$

In another affine neighborhood, the image is a lemniscate ([CF]), $[1 : u] \mapsto [1 : \frac{u}{\alpha-u^2}]$, with parametric and implicit equations:

$$\begin{aligned} & X = \frac{u(\cos(\theta) - u^2)}{u^4 - 2\cos(\theta)u^2 + 1}, \quad Y = \frac{-u\sin(\theta)}{u^4 - 2\cos(\theta)u^2 + 1} \\ \implies \quad & \cos(\theta)Y^2 + \sin(\theta)XY - \sin(\theta)^2(X^2 + Y^2)^2 = 0. \end{aligned}$$

The tangent cone at the origin is the union of the $X$-axis and the line with slope $-\tan(\theta)$, so that one loop of the figure is in the interior of the angle formed by the positive $X$-axis and the ray measured $\pi - \theta$ counterclockwise.

Since the angle between these two tangent lines is a conformal invariant, the r-equivalence classes of self-intersecting immersions $P \circ v \circ \delta$ can be distinguished by looking at a neighborhood of the node. As $\theta$ approaches $0^+$ or $\pi^-$, the curve folds in on itself to give the two-to-one maps from the previous Examples.

**Example 4.10.** The last family of equivalence classes from Proposition 4.3 is in case (4), with $0 < t < 1$. Representative coefficient matrices are of the form:

$$\mathbf{Q}_t = \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & i \end{pmatrix}.$$

For each $t$, the induced map is

$$Q_t \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 + t^2 u_1^2 : u_0 u_1 + i u_1^2].$$

This mapping from $\mathbb{R}P^1$ to $\mathbb{C}P^1$ is one-to-one, and in one affine neighborhood, the image is an ellipse, $[u : 1] \mapsto [1 : \frac{u+i}{1+t^2u^2}]$, with parametric and implicit equations:

$$X = \frac{u}{1 + t^2u^2}, \quad Y = \frac{1}{1 + t^2u^2} \implies \frac{X^2}{(\frac{1}{2t})^2} + \frac{(Y - \frac{1}{2})^2}{(\frac{1}{2})^2} = 1.$$

For any $t$, the minor axis of the ellipse is the segment from $(0,0)$ to $(0,1)$, and the major axis is parallel to the $X$-axis. Its eccentricity is $\sqrt{1-t^2}$, and as $t \to 1^-$, the ellipse approaches the circle from Example 4.5.

Similarity transformations of this affine neighborhood will preserve the eccentricity of the ellipse, but the image in other affine neighborhoods (or under the action of $PGL(2,\mathbb{C})$) will not necessarily be an ellipse. Unlike the previous Examples with the hyperbolas or the parabola, there are no distinguished points in the image where the r-equivalence class can be detected by a local conformal invariant.

One thing that can be said about the curves in this class is that the eccentricity is an invariant in the following sense: if there is a linear fractional transformation of $\mathbb{C}$ that takes one (non-circular) ellipse into another ellipse, then that transformation is a similarity. This claim is proved in [CF]. So, unlike the circles from Examples 4.5 and 4.7, the r-equivalence classes of parametrized ellipses can be distinguished by looking at their images.

The real implicit equations for images of the ellipse will have an isolated node, for example, the inversion of the ellipse $X^2/A^2 + Y^2/B^2 = 1$ in the unit circle will be the quartic

$$(X^2+Y^2)^2 - (\frac{X^2}{A^2} + \frac{Y^2}{B^2}) = 0,$$

which contains both the parametric image and the point at the origin.

To summarize, each r-equivalence class of maps $P \circ v \circ \delta : \mathbb{R}P^1 \to \mathbb{C}P^1$ has a representative whose image in at least one affine neighborhood is equal to a real conic curve, or contained in some straight line, and all irreducible real affine conics appear at least once in this way. The set of all possible images in the Riemann sphere includes Möbius transformations of conics, and circles and lines (possibly with one point or an arc deleted). Classically (see [K], [MP], [P]), the term “nodal biquadratic” has been used to refer to inversive images of ellipses and hyperbolas, and (as previously mentioned) “cuspidal biquadratic” to refer to inversive images of parabolas. Various cases of conics transformed by inversions have interesting names as special plane curves; these are surveyed in [CF], which displays some pictures and gives further references on the images of conics in the inversive plane.

Another interesting observation is that an analogue of Proposition 4.3 was used in [C$_1$] to classify immersions of the complex projective line in $\mathbb{C}P^2$, and there are (at least superficially) some geometric similarities between the corresponding equivalence classes.

## 5. Maps from the projective line to a point

As the heading of this Section suggests, maps from $\mathbb{R}P^1$ to $\mathbb{C}P^0$, of the form

$$u = [u_0 : u_1] \mapsto [P_0] = [p_0^{00}u_0^2 + p_0^{01}u_0u_1 + p_0^{11}u_1^2],$$

will not have a very interesting image. This is the $m = 1$, $n = 0$ case of the construction from Section 2, and the $1 \times 3$ complex matrix $\mathbf{P} = (p_0^{00}, p_0^{01}, p_0^{11})$ is non-zero, so the map $P \circ v \circ \delta$ will be defined on at least one point of $\mathbb{R}P^1$, with image $\mathbb{C}P^0 = \{[1]\}$. However, $\mathcal{M}(1,0) = 1$ suggests there will be infinitely many r-equivalence classes, and in fact the geometric phenomenon detected by r-equivalence is the configuration of the points in $\mathbb{C}P^1$ where $P \circ v$ is undefined.

This case also illustrates the problem of finding a normal form for a complex symmetric pencil of matrices, under real congruence. The previous Section used Proposition 4.3 to classify one-dimensional complex subspaces of the three-dimensional space $S(2,\mathbb{C})$, but the two-dimensional subspaces are not covered by [W].

To begin the classification of two-dimensional subspaces, first consider the complex congruence classes, which by Theorem 2.6 correspond to c-equivalence classes of maps $P\circ v : \mathbb{C}P^1 \to \mathbb{C}P^0$. The following normal form for two basis elements of a subspace is recalled from [CW].

**Proposition 5.1.** *If $L$ is a two-dimensional subspace of $S(2,\mathbb{C})$, then there is exactly one subspace in the list below equal to the subspace $\{\mathbf{A}\cdot\mathbf{M}\cdot\mathbf{A}^T : \mathbf{M}\in L\}$ for some nonsingular complex matrix $\mathbf{A}$.*

(1) $\{\lambda\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+\mu\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} : \lambda,\mu\in\mathbb{C}\}$;

(2) $\{\lambda\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\mu\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} : \lambda,\mu\in\mathbb{C}\}$.

$\blacksquare$

The congruence class of a pencil can be distinguished by its intersection with the affine Veronese variety, the image of $\mathbf{k}\circ\mathbf{v}$, which by Equation (2.3) is the locus of singular matrices in $S(2,\mathbb{C})$. Given a plane $L$, its congruence class can be determined by picking any matrices $\mathbf{K}_1$ and $\mathbf{K}_2$ which span $L$, and counting the number of distinct roots $[\lambda:\mu]$ of the characteristic polynomial $\det(\lambda\cdot\mathbf{K}_1+\mu\cdot\mathbf{K}_2)$ (cf [CSS] §4).

**Example 5.2.** It is easy to check that the singular matrices in the pencil $L = \{\lambda\cdot\begin{pmatrix}1&0\\0&0\end{pmatrix}+\mu\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\}$ are exactly the scalar multiples of $\begin{pmatrix}1&0\\0&0\end{pmatrix} = (\mathbf{k}\circ\mathbf{v})((1,0)^T)$. So if $\mathbf{vech}(L)=\ker(\mathbf{Q})$, then, projectively, there is one point, $[1:0]$, where $Q\circ v$ is undefined. One such coefficient matrix is $\mathbf{Q}=(0,0,1)_{1\times 3}$, which defines a parametric map

$$Q\circ v : [z_0 : z_1]\mapsto [z_1^2].$$

**Example 5.3.** The singular matrices in the pencil $L = \{\lambda\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}+\mu\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\}$ form exactly two lines: the scalar multiples of $\begin{pmatrix}1&1\\1&1\end{pmatrix} = (\mathbf{k}\circ\mathbf{v})((1,1)^T)$, and $\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = (\mathbf{k}\circ\mathbf{v})((1,-1)^T)$. So if $\mathbf{vech}(L)=\ker(\mathbf{Q})$, then, projectively, there are two points, $[1:1]$ and $[1:-1]$, where $Q\circ v$ is undefined. One such coefficient matrix is $\mathbf{Q}=(1,0,-1)_{1\times 3}$, which defines a parametric map

$$Q\circ v : [z_0 : z_1]\mapsto [z_0^2 - z_1^2].$$

So, maps $(P\circ v)(z) = [P_0]$ fall into two c-equivalence classes, where the quadratic homogeneous polynomial $P_0$ has a double root, or two distinct roots in $\mathbb{C}P^1$ (cf [H], Example 10.8). At this point, before actually finding normal forms for the r-equivalence classes, it is possible to predict how the c-equivalence classes will be partitioned. By the discussion from Subsection 2.3, the action of $PGL(2,\mathbb{R})$ on the domain $\mathbb{C}P^1$ fixes the real line $\delta(\mathbb{R}P^1)$. It is easy to see that it acts transitively on both $\delta(\mathbb{R}P^1)$ and its complement in $\mathbb{C}P^1$. Let $z$ and $z'$ be the roots of $P_0$ in $\mathbb{C}P^1$, where $P \circ v$ is undefined. In the first c-equivalence class, $z = z'$. If the double root is on the real line, it can be moved to some certain point on the line, say, $[1 : 0]$, and if it is not on the real line, then it can be moved to some other certain point, say $[1 : i]$. In the second c-equivalence class, where $z \neq z'$, there are two cases. If the two points are on the real line, they can be moved to a certain pair, say $[1 : 0]$ and $[0 : 1]$. If one of the points is not on the real line, then it can be moved to $[1 : i]$, and the other point can be anywhere else on the Riemann sphere. There is a one-parameter subgroup of $PGL(2, \mathbb{C})$ which fixes both $[1 : i]$ and the real line (a pole and equator of the sphere), so the other point can be moved (rotated) to some semicircular meridian of longitude. This intuitive description of r-equivalence classes is justified by the following theorem of linear algebra.

**Theorem 5.4.** *If $L$ is a two-dimensional subspace of $S(2, \mathbb{C})$, then there is exactly one subspace in the list below equal to the subspace $\{\mathbf{A} \cdot \mathbf{M} \cdot \mathbf{A}^T : \mathbf{M} \in L\}$ for some nonsingular real matrix $\mathbf{A}$.*

(1) $\{\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mu \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$;

(2) $\{\lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mu \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$;

(3) $\{\lambda \cdot \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} + \mu \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$;

(4) $\{\lambda \cdot \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} + \mu \cdot \begin{pmatrix} 1 & it \\ it & -t^2 \end{pmatrix}\}$, $-1 \leq t < 1$.

*Proof.* Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be a basis for the given subspace $L$. There is always at least one element $\mathbf{K} \in L$ that has rank 1: it could be $\mathbf{K} = \mathbf{K}_1$, or if not, then

$$\det(\lambda \cdot \mathbf{K}_1 + \mathbf{K}_2) = \lambda^2 \det(\mathbf{K}_1) + \ldots + \det(\mathbf{K}_2) = 0$$

has one or two complex solutions, so there is some $\lambda = \lambda_0$, so that the linear combination $\mathbf{K} = \lambda_0 \cdot \mathbf{K}_1 + \mathbf{K}_2$ is singular but non-zero. By Proposition 4.3, there is some real matrix $\mathbf{A}_1$ so that $\mathbf{A}_1 \cdot \mathbf{K} \cdot \mathbf{A}_1^T$ is equal to a complex scalar multiple of either $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$. So, the proof continues in two parts.

For the first part, $L$ is congruent to a subspace which is spanned by $\mathbf{K}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and some non-zero matrix of the form $\mathbf{K}_4 = \begin{pmatrix} 0 & \beta \\ \beta & \gamma \end{pmatrix}$. If $\gamma = 0$, then a complex rescaling of $\mathbf{K}_4$ gives case (1) of the Theorem. If $\gamma \neq 0$, then, by a complex rescaling, assume $\gamma = 1$. If (after the rescaling) $\beta \in \mathbb{R}$, then let $\mathbf{A}_2 = \begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix}$, so $\mathbf{A}_2 \cdot \mathbf{K}_3 \cdot \mathbf{A}_2^T = \mathbf{K}_3$, and

$$\mathbf{A}_2 \cdot \mathbf{K}_4 \cdot \mathbf{A}_2^T = \begin{pmatrix} -\beta^2 & 0 \\ 0 & 1 \end{pmatrix},$$

and some linear combination of $\mathbf{K}_3$ and $\mathbf{A}_2 \cdot \mathbf{K}_4 \cdot \mathbf{A}_2^T$ gives case (2) of the Theorem. If $\gamma = 1$ and $\mathrm{Im}(\beta) \neq 0$, let

$$\mathbf{A}_2 = \begin{pmatrix} \frac{1}{\mathrm{Im}(\beta)} & -\frac{\mathrm{Re}(\beta)}{\mathrm{Im}(\beta)} \\ 0 & -1 \end{pmatrix}.$$

Then, $\mathbf{A}_2 \cdot \mathbf{K}_3 \cdot \mathbf{A}_2^T = (\frac{1}{\mathrm{Im}(\beta)})^2 \cdot \mathbf{K}_3$, and

$$\mathbf{A}_2 \cdot \mathbf{K}_4 \cdot \mathbf{A}_2^T = \begin{pmatrix} -\frac{\mathrm{Re}(\beta)(\mathrm{Re}(\beta)+2i\cdot\mathrm{Im}(\beta))}{(\mathrm{Im}(\beta))^2} & -i \\ -i & 1 \end{pmatrix}.$$

Some complex linear combination of $\mathbf{K}_3$ and $i \cdot \mathbf{A}_2 \cdot \mathbf{K}_4 \cdot \mathbf{A}_2^T$ will be equal to the matrix $\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$, giving the $t = 0$ pencil from case (4) of the Theorem. By finding the set of rank 1 matrices in each case, Proposition 5.1 says that the pencils from (1) and (2) fall into different complex congruence classes, so they are also not congruent by any real matrix. Similarly, (1) and any of the pencils in case (4) are not congruent. To show that (2) and any of the pencils in case (4) are not congruent, it is enough, and left to the reader, to check that $\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$ cannot be diagonalized by any real matrix.

The next part to consider is where $L$ is congruent to a subspace spanned by $\mathbf{K}_5 = \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}$, and some other matrix $\mathbf{K}_6 = \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}$. Since

$$\det(\lambda \cdot \mathbf{K}_5 + \mu \cdot \mathbf{K}_6) = \mu \cdot ((i\alpha - i\delta - 2\beta) \cdot \lambda + (\alpha\delta - \beta^2) \cdot \mu),$$

the singular matrices in $L$ will form exactly one line if and only if $\beta = \frac{i}{2}(\alpha - \delta)$. In this case,

$$-\frac{i}{2}(\alpha - \delta) \cdot \mathbf{K}_5 + \begin{pmatrix} \alpha & \frac{i}{2}(\alpha - \delta) \\ \frac{i}{2}(\alpha - \delta) & \delta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\alpha + \delta) & 0 \\ 0 & \frac{1}{2}(\alpha + \delta) \end{pmatrix},$$

and rescaling gives case (3) of the Theorem. The matrix pencils from cases (3) and (1) are equivalent under complex congruence, but not under real congruence, since such a transformation preserves rank, and would have to take the line spanned by $\mathbf{K}_5$ to the line spanned by $\mathbf{K}_3$, contradicting Proposition 4.3.

The final possibility is that $L$ is spanned by $\mathbf{K}_5$ and some other rank 1 matrix, which could be a multiple of $\mathbf{K}_3 = (\mathbf{k} \circ \mathbf{v})((1,0)^T)$, or a multiple of some other singular matrix, $(\mathbf{k} \circ \mathbf{v})((\beta, 1)^T)$. Since the $\mathbf{K}_3$ case was already considered, let $\mathbf{K}_7 = \begin{pmatrix} \beta^2 & \beta \\ \beta & 1 \end{pmatrix}$, for some $\beta = a + ib \neq -i$ (if $\beta = -i$, $\mathbf{K}_5$ and $\mathbf{K}_7$ are not linearly independent). Let $\mathbf{A}_3 = \begin{pmatrix} q & -1 \\ 1 & q \end{pmatrix}$, for $q \in \mathbb{R}$, so $\mathbf{A}_3$ is nonsingular, $\mathbf{A}_3 \cdot \mathbf{K}_5 \cdot \mathbf{A}_3^T = (q - i)^2 \cdot \mathbf{K}_5$, and

$$\mathbf{A}_3 \cdot \mathbf{K}_7 \cdot \mathbf{A}_3^T = \begin{pmatrix} (q\beta - 1)^2 & (\beta + q)(q\beta - 1) \\ (\beta + q)(q\beta - 1) & (\beta + q)^2 \end{pmatrix}.$$

If $\beta$ is real, then setting $q = -\beta$ gives a product equal to a multiple of $\mathbf{K}_3$. Otherwise, $q\beta - 1 \neq 0$, and the product is

$$\mathbf{A}_3 \cdot \mathbf{K}_7 \cdot \mathbf{A}_3^T = (q\beta - 1)^2 \cdot \begin{pmatrix} 1 & \frac{\beta+q}{q\beta-1} \\ \frac{\beta+q}{q\beta-1} & \left(\frac{\beta+q}{q\beta-1}\right)^2 \end{pmatrix},$$

and

$$\mathrm{Re}(\frac{\beta + q}{q\beta - 1}) = \frac{q^2 a + (a^2 + b^2 - 1)q - a}{|q\beta - 1|^2}.$$

The discriminant of the numerator of this real part is $(a^2+(b-1)^2)(a^2+(b+1)^2)$, which is nonnegative for any $a$, $b$, so there is a real root $q$. After the congruence transformation, the subspace is spanned by $\mathbf{K}_5$ and $\mathbf{K}_8(t) = \begin{pmatrix} 1 & i\cdot t \\ i\cdot t & -t^2 \end{pmatrix}$, for $t \in \mathbb{R}$, $t \neq 1$. Another congruence transformation, using $\mathbf{A}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, would give $\mathbf{A}_4 \cdot \mathbf{K}_5 \cdot \mathbf{A}_4^T = -\mathbf{K}_5$, and $\mathbf{K}_9(t) = \mathbf{A}_4 \cdot \mathbf{K}_8(t) \cdot \mathbf{A}_4^T = \begin{pmatrix} -t^2 & -i\cdot t \\ -i\cdot t & 1 \end{pmatrix}$. If $|t| > 1$, $\mathbf{K}_9(t) = -t^2 \cdot \mathbf{K}_8(\frac{1}{t})$. Since $\mathbf{K}_8(t)$ is congruent to a multiple of $\mathbf{K}_8(\frac{1}{t})$, $t$ can be chosen in the interval $[-1,1)$, as claimed in case (4).

It remains only to be checked that the representatives of case (4) are pairwise inequivalent, for different values of $t$. So, consider two subspaces of $S(2,\mathbb{C})$, the first spanned by $\mathbf{K}_5$ and $\mathbf{K}_8(t)$, as derived above, and the other spanned by $\mathbf{K}_5$ and $\mathbf{K}_8(s)$, for $-1 \leq s < 1$. If the subspaces are congruent, the pair of singular lines in one subspace must be transformed into the pair of singular lines in the other subspace (since congruence is a one-to-one, linear, rank-preserving transformation). There are two cases.

The first case is that the congruence transformation fixes each line: if $\mathbf{K}_5$ is equal to a scalar multiple of $\mathbf{A} \cdot \mathbf{K}_5 \cdot \mathbf{A}^T$ for some real matrix $\mathbf{A}$, it is not hard to check that $\mathbf{A}$ is of the form $\begin{pmatrix} p & q \\ -q & p \end{pmatrix}$ (this is left to the reader, who might use Equation (2.3) as a shortcut for some matrix calculations). Using this $\mathbf{A}$ to transform $\mathbf{K}_8(t)$ gives:

$$\mathbf{A} \cdot \mathbf{K}_8(t) \cdot \mathbf{A}^T = (p+iqt)^2 \cdot \begin{pmatrix} 1 & i\frac{pt+iq}{p+iqt} \\ i\frac{pt+iq}{p+iqt} & -\left(\frac{pt+iq}{p+iqt}\right)^2 \end{pmatrix},$$

and if this is equal to $\alpha \cdot \mathbf{K}_8(s)$ for some $\alpha \in \mathbb{C}$, then $\frac{pt+iq}{p+iqt} = s$. Equating real and imaginary parts, the only solutions of $pt = ps$ and $q = qts$ are $s = t$, or $p = 0$ and $st = 1$.

The second case is that the congruence transformation interchanges the two lines, so $\mathbf{A} \cdot \mathbf{K}_5 \cdot \mathbf{A}^T$ is equal to some scalar multiple of $\mathbf{K}_8(s)$. Recalling that the reader has already checked $\mathbf{K}_5$ is not diagonalizable, the $s = 0$ case can be excluded. $\mathbf{A}$ must be of the form $\begin{pmatrix} p & q \\ -qs & ps \end{pmatrix}$, so

$$\mathbf{A} \cdot \mathbf{K}_8(t) \cdot \mathbf{A}^T = (p+iqt)^2 \cdot \begin{pmatrix} 1 & i\frac{pst+iqs}{p+iqt} \\ i\frac{pst+iqs}{p+iqt} & -\left(\frac{pst+iqs}{p+iqt}\right)^2 \end{pmatrix}.$$

If $\mathbf{K}_8(t)$ is congruent to a multiple of $\mathbf{K}_5$, then $i\frac{pst+iqs}{p+iqt} = i$. Equating real and imaginary parts, the only solutions of $pst = p$ and $qs = qt$ are $s = t$, or $q = 0$ and $st = 1$. ∎

**Example 5.5.** Case (1) of the Theorem is similar to case (1) of Proposition 5.1. A representative coefficient matrix is $\mathbf{Q} = (0,0,1)$, and the map

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_1^2]$$

is undefined (with a double root) at $[1:0] \in \mathbb{R}P^1$.

**Example 5.6.** For case (2) of the Theorem, a representative coefficient matrix is $\mathbf{Q} = (0,1,0)$. Its kernel is the set $\{(\lambda,0,\mu)^T : \lambda,\mu \in \mathbb{C}\}$, whose image under $\mathbf{k}$ is the normal form derived in the above Proof, where the singular elements form the two lines, spanned by $(\mathbf{k}\circ\mathbf{v})((1,0)^T)$ and $(\mathbf{k}\circ\mathbf{v})((0,1)^T)$. The map

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0 u_1]$$

is undefined at two points in $\mathbb{R}P^1$, $[1:0]$ and $[0:1]$.

**Example 5.7.** For case (3) of the Theorem, a representative coefficient matrix is $\mathbf{Q} = (1,2i,-1)$. Its kernel is the set $\{(\mu - i\lambda, \lambda, \mu + i\lambda)^T : \lambda,\mu \in \mathbb{C}\}$, whose image under $\mathbf{k}$ is the normal form derived in the above Proof, where the singular elements form exactly one line, spanned by $(\mathbf{k}\circ\mathbf{v})((1,i)^T)$. The map

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [u_0^2 + 2iu_0u_1 - u_1^2] = [(u_0 + iu_1)^2]$$

is well-defined at every point in $\mathbb{R}P^1$, although $Q \circ v$ is undefined at $[1:i]$.

**Example 5.8.** For each value of $t$ in case (4) of the Theorem, a representative coefficient matrix is $\mathbf{Q} = (t, i(1+t), -1)$. $\mathbf{k}(\ker(Q))$ meets the complex affine Veronese variety in two lines, spanned by $(\mathbf{k}\circ\mathbf{v})((1,i)^T)$ and $(\mathbf{k}\circ\mathbf{v})((1,it)^T)$, $-1 \le t < 1$. At $t = 0$, one of these lines non-trivially meets the real affine Veronese variety. The map

$$Q \circ v \circ \delta : [u_0 : u_1] \mapsto [tu_0^2 + i(1+t)u_0u_1 - u_1^2] = [(tu_0 + iu_1)(u_0 + iu_1)]$$

is well-defined at every point in $\mathbb{R}P^1$ if $t \neq 0$. The map $Q \circ v$ is undefined at $[1:i]$ and $[1:it]$, the previously mentioned North pole and point on the meridian (so $t = 0$ is the intersection with the real equator, and $t = -1$ is the South pole).

Note that for $t \neq 0$, the maps from this Example, $Q \circ v \circ \delta : \mathbb{R}P^1 \to \{[1]\}$, are all equal to each other, even though the corresponding coefficient matrices $\mathbf{Q}$ are not r-equivalent. In fact, these maps are equal to the map from Example 5.7, even though the coefficient matrix $(1,2i,-1)$ is in a different c-equivalence class.

## 6. The real projective plane in $\mathbb{C}P^4$

In the case $m = 2$, $n = 4$, a $5 \times 6$ matrix $\mathbf{P}$ with rank 5 determines a map $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{C}P^4$. Although $P \circ v \circ \delta$ may not be defined on all of $\mathbb{R}P^2$, the image will be a real surface in the complex projective 4-space, with a homogeneous parametric equation of the form

$$[u_0 : u_1 : u_2] \mapsto P \cdot [u_0^2 : u_0u_1 : u_1^2 : u_0u_2 : u_1u_2 : u_2^2].$$

As in the previous two Sections, the c-equivalence problem is an easy place to start. In this case, $\mathbf{k}(\ker(\mathbf{P}))$ will be a one-dimensional pencil of $3 \times 3$ complex symmetric matrices, and as in Examples 4.1 and 4.2, the only invariant under congruence is the rank: 1, 2, or 3. Geometrically, there will be three types of projections of the complex Veronese variety from $\mathbb{C}P^5$ to $\mathbb{C}P^4$, a well-known fact in complex algebraic geometry ([H]).

**Example 6.1.** Given $\mathbf{P}$, if $\mathbf{k}(\ker(\mathbf{P}))$ is a line spanned by a matrix of rank 1, then this line is in the image of $\mathbf{v}$, and it is congruent to

$$\{\lambda \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C}\}.$$

**P** is c-equivalent to

$$\mathbf{Q}=\left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{array}\right).$$

The composite map from $\mathbb{C}P^2$ to $\mathbb{C}P^4$ is

$$Q\circ v:[z_0:z_1:z_2]\mapsto[z_0^2:z_0z_1:z_1^2:z_0z_2:z_1z_2],$$

which is undefined only at $[0:0:1]$, and nonsingular and one-to-one elsewhere. Its image is contained in the smooth complex surface

$$\{Z_0Z_2-Z_1^2=0,Z_4Z_0-Z_1Z_3=0,Z_4Z_1-Z_2Z_3=0\},$$

a "cubic scroll" ([H]).

**Example 6.2.** Given **P**, if $\mathbf{k}(\ker(\mathbf{P}))$ is a line spanned by a matrix of rank 2, then this line is congruent to

$$\{\lambda\cdot\left(\begin{array}{ccc}0&0&0\\0&1&0\\0&0&1\end{array}\right):\lambda\in\mathbb{C}\},$$

and **P** is c-equivalent to

$$\mathbf{Q}=\left(\begin{array}{cccccc}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&-1\\0&0&0&1&0&0\\0&0&0&0&1&0\end{array}\right).$$

The composite map from $\mathbb{C}P^2$ to $\mathbb{C}P^4$ is well-defined:

$$Q\circ v:[z_0:z_1:z_2]\mapsto[z_0^2:z_0z_1:z_1^2-z_2^2:z_0z_2:z_1z_2].$$

When restricted to the $\{z_0\neq 0\}$ affine neighborhood, it is a graph over the $Z_1,Z_3$-plane,

$$[1:z_1:z_2]\mapsto[1:z_1:z_1^2-z_2^2:z_2:z_1z_2],$$

so it is one-to-one and nonsingular. However, on the complement of this neighborhood, the map restricts to

$$[0:z_1:z_2]\mapsto[0:0:z_1^2-z_2^2:0:z_1z_2],$$

which is two-to-one, except for two singular points, at $[0:1:\pm i]$. For reference in later Examples, denote by $\mathcal{D}$ the projective line which is equal to the union of the two-to-one locus and the singular locus of a map $P\circ v$ in this c-equivalence class, and denote by $S_1$ and $S_2$ the two singular points. The image of $Q\circ v$ is equal to the complex surface

$$\{-Z_1^2+Z_0Z_2+Z_3^2=0,Z_4Z_0-Z_1Z_3=0\},$$

which is singular along the line $\{Z_0=Z_1=Z_3=0\}$ (the image of the line $\mathcal{D}$).

**Example 6.3.** Given $\mathbf{P}$, if $\mathbf{k}(\ker(\mathbf{P}))$ is a line spanned by a matrix of rank 3, then this line is congruent to

$$\{\lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C}\},$$

and $\mathbf{P}$ is c-equivalent to

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The composite map from $\mathbb{C}P^2$ to $\mathbb{C}P^4$ is well-defined, nonsingular, and one-to-one (an embedding):

$$Q \circ v : [z_0 : z_1 : z_2] \mapsto [z_0^2 - z_1^2 : z_0 z_1 : z_1^2 - z_2^2 : z_0 z_2 : z_1 z_2].$$

From these three c-equivalence classes, we can get some idea of how the r-equivalence will go. In the c-equivalence class from Example 6.1, the real projective plane $\delta(\mathbb{R}P^2)$ inside $\mathbb{C}P^2$ may intersect the point where $P \circ v$ is undefined, or it may miss it. In Example 6.2, the complex projective line $\mathcal{D}$ will intersect the real projective plane in a real projective line or just one point, and this intersection may contain none, one, or both of the singular points $S_1$, $S_2$. If $P$ is as in Example 6.3, then $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{C}P^4$ will be a composition of smooth embeddings, and since $P \circ v$ is a complex analytic diffeomorphism onto its image, the image $(P \circ v \circ \delta)(\mathbb{R}P^2)$ will be totally real in $\mathbb{C}P^4$. From the chart in Section 3, we expect $\mathcal{M}(2,4) = 2$.

The following classification of one-dimensional matrix pencils is recalled from [W], and the rank of each representative is also listed.

**Proposition 6.4.** *If $\mathbf{K}$ is a non-zero matrix in $S(3,\mathbb{C})$, then there is a matrix in the list below equal to $\lambda \cdot \mathbf{A} \cdot \mathbf{K} \cdot \mathbf{A}^T$ for some nonsingular real matrix $\mathbf{A}$ and non-zero complex scalar $\lambda$.*

(1) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *rank* 1;

(2) $\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *rank* 2;

(3) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha = e^{i\theta}$, $0 \le \theta \le \pi$, *rank* 2;

(4) $\begin{pmatrix} -it^2 & 1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $0 < t \le 1$, *rank* 1 *for* $t = 1$, *rank 2 for* $0 < t < 1$;

(5) $\begin{pmatrix} 0 & 0 & 1/2 \\ 0 & 0 & i/2 \\ 1/2 & i/2 & 0 \end{pmatrix}$, *rank* 2;

(6) $\begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & i/2 \\ 0 & i/2 & 1 \end{pmatrix}$, *rank* 3;

$$(7)\quad \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & i & 0 \\ 0 & 0 & e^{i\theta} \end{pmatrix},\ 0 \le \theta < 2\pi,\ \textit{rank } 3;$$

$$(8)\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 1 & i \end{pmatrix},\ \textit{rank } 2;$$

$$(9)\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -it^2\beta & \beta \\ 0 & \beta & i\beta \end{pmatrix},\ 0 < t < 1,\ \beta = e^{i\theta},\ 0 \le \theta < \pi,\ \textit{rank } 3;$$

$$(10)\quad \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & e^{i\psi} \end{pmatrix},\ 0 \le \theta \le \phi \le \psi < 2\pi,\ \textit{rank } 3.$$

*The cases are mutually inequivalent, except for the last case, where the three entries can be changed by an arbitrary rotation of the circle.* ∎

The last two cases show that there are, as expected, two real moduli for the r-equivalence subclasses of the rank 3 c-equivalence class. Since the embeddings in these rank 3 classes have no self-intersections, singularities, or complex tangents, any geometric interpretation of these invariants would be rather subtle, in analogy with the ellipses and their images from Example 4.10. So, we do not investigate the coefficient matrices corresponding to cases (6), (7), (9), (10), and instead look for more easily detectable geometric properties in the rank 1 and 2 cases.

Exactly as in Examples 4.4, 4.5, the rank 1 c-equivalence class, where $P \circ v$ is undefined at one point, will split into two r-equivalence classes, depending on whether this point is in the image of $\delta$.

**Example 6.5.** A parametric map $\mathbb{R}P^2 \to \mathbb{C}P^4$ which represents case (1) of the Proposition is

$$[u_0 : u_1 : u_2] \mapsto [u_0u_1 : u_1^2 : u_0u_2 : u_1u_2 : u_2^2].$$

As in Example 6.1, it is not defined at one point, but it is smooth and one-to-one everywhere else. The image is contained in a real projective space $\mathbb{R}P^4$ inside $\mathbb{C}P^4$.

**Example 6.6.** The other rank 1 matrix from Proposition 6.4 is in case (4), with $t = 1$. A representative coefficient matrix is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The induced map is a well-defined, totally real embedding of $\mathbb{R}P^2$ in $\mathbb{C}P^4$,

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 + u_1^2 : u_0u_1 + iu_1^2 : u_0u_2 : u_1u_2 : u_2^2],$$

although $Q \circ v$ is not defined at $[1 : i : 0]$.

The remaining Examples will be representatives of the rank 2 r-equivalence classes. The maps $\mathbb{R}P^2 \to \mathbb{C}P^4$ will all be well-defined, and the geometric property to watch will be the intersection of $\delta(\mathbb{R}P^2)$ with the locus $\mathcal{D}$ and the points $S_1$ and $S_2$.

**Example 6.7.** The first rank 2 matrix from the Proposition is in case (2), and a representative coefficient matrix is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The induced map is one-to-one:

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 : 2u_0u_1 + iu_1^2 : u_0u_2 : u_1u_2 : u_2^2].$$

The locus $\mathcal{D}$ of the map $Q \circ v$ is $\{[z_0 : z_1 : 0]\}$, and the singular points are $S_1 = [0 : 1 : 0]$ and $S_2 = [1 : i : 0]$. So, the image of $\delta$ meets $\mathcal{D}$ in a real projective line, which includes $S_1$, but not $S_2$. The restriction

$$Q \circ v \circ \delta : [u_0 : u_1 : 0] \mapsto [u_0^2 : 2u_0u_1 + iu_1^2 : 0 : 0 : 0]$$

is essentially the same as the map from Example 4.6, a real curve with a cusp at $(Q \circ v \circ \delta)([0 : 1 : 0]) = [0 : 1 : 0 : 0 : 0]$. The real Jacobian of $Q \circ v \circ \delta$ drops rank only at $[0 : 1 : 0]$, and at other points, the map is nonsingular and totally real.

**Example 6.8.** The next case of the Proposition is the family of rank 2 matrices in (3), with representative coefficient matrices of the form

$$\mathbf{Q} = \begin{pmatrix} \alpha & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The induced map is:

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [\alpha u_0^2 - u_1^2 : u_0u_1 : u_0u_2 : u_1u_2 : u_2^2].$$

The restriction to an affine neighborhood,

$$Q \circ v \circ \delta : [u_0 : u_1 : 1] \mapsto [\alpha u_0^2 - u_1^2 : u_0u_1 : u_0 : u_1 : 1],$$

is a graph over the $X_2, X_3$-plane, so it is one-to-one, nonsingular and totally real. The locus $\mathcal{D}$ of the map $Q \circ v$ is $\{[z_0 : z_1 : 0]\}$, and the two singular points are $[1 : \pm\sqrt{-\alpha} : 0]$. So, the image of $\delta$ meets $\mathcal{D}$ in a real projective line, which either misses both $S_1$ and $S_2$, or, if $\alpha = -1$, contains both $S_1 = [1 : 1 : 0]$ and $S_2 = [1 : -1 : 0]$. The restriction

$$Q \circ v \circ \delta : [u_0 : u_1 : 0] \mapsto [\alpha u_0^2 - u_1^2 : u_0u_1 : 0 : 0 : 0]$$

falls into the three cases from Examples 4.7, 4.8 and 4.9. If $\alpha \neq \pm 1$, the surface is a totally real immersion, with a single point of self-intersection (corresponding to the double point from Example 4.9). If $\alpha = -1$, the image $(P \circ v \circ \delta)(\mathbb{R}P^2)$ has a line segment of double points, connecting two differential-topological singularities, as in Example 4.8. If $\alpha = 1$, the image is a totally real immersion where the double points form a real projective line, as in Example 4.7. These two differently behaved double lines in the $\alpha = \pm 1$ cases resemble those in the real Steiner surfaces of types 7 and 8 in the classification of [CSS].

**Example 6.9.** The next case of the Proposition is the family of rank 2 matrices in (4), with $0 < t < 1$ and representative coefficient matrices of the form

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & t^2 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The induced map is one-to-one:

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 + t^2 u_1^2 : u_0 u_1 + i u_1^2 : u_0 u_2 : u_1 u_2 : u_2^2].$$

The locus $\mathcal{D}$ of the map $Q \circ v$ is $\{[z_0 : z_1 : 0]\}$, and the singular points are $[i(-1 \pm \sqrt{1-t^2}) : 1 : 0]$. So, the image of $\delta$ meets $\mathcal{D}$ in a real projective line, which, for each $t$, misses $S_1$ and $S_2$. The restriction

$$Q \circ v \circ \delta : [u_0 : u_1 : 0] \mapsto [u_0^2 + t^2 u_1^2 : u_0 u_1 + i u_1^2 : 0 : 0 : 0]$$

is essentially the same as a map from Example 4.10, a real ellipse. These maps $Q \circ v \circ \delta$ are totally real embeddings, and as $t \to 1^-$, they approach the embedding from Example 6.6.

In the above three Examples, $\delta(\mathbb{R}P^2)$ met $\mathcal{D}$ in a real projective line, and the r-equivalence class could be detected by the behavior of $P \circ v$ on $\mathcal{D}$. In the next two Examples, representing cases (5) and (8) of Proposition 6.4, the matrix representatives of the congruence classes don't contain copies of the $2 \times 2$ matrices from Proposition 4.3 as block submatrices. The intersection of the real projective plane with $\mathcal{D}$ will be just one point.

**Example 6.10.** A representative coefficient matrix for case (5) is

$$\mathbf{Q} = \begin{pmatrix} 0 & 0 & 0 & 1 & i & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The induced map is one-to-one:

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0 u_2 + i u_1 u_2 : u_0^2 : u_0 u_1 : u_1^2 : u_2^2].$$

The locus $\mathcal{D}$ of the map $Q \circ v$ is the line $\{z_1 = i z_0\}$ in $\mathbb{C}P^2$, where $[z_0 : i z_0 : z_2]$ and $[-z_0 : -i z_0 : z_2]$ are mapped to the same point. The singular points are $S_1 = [0 : 0 : 1]$ and $S_2 = [1 : i : 0]$. So, the image of $\delta$ meets $\mathcal{D}$ only at the point $S_1$. The map $Q \circ v \circ \delta$ is a smooth embedding, and its image has a CR singularity at $[0 : 0 : 0 : 0 : 1]$: a restriction to affine neighborhoods is

$$[u_0 : u_1 : 1] \mapsto [u_0 + i u_1 : u_0^2 : u_0 u_1 : u_1^2 : 1],$$

a real polynomial graph over its complex tangent space, the $Z_0$-axis. This CR singularity is unstable, in the sense that the embedding $Q \circ v \circ \delta$, which is totally real except for an isolated complex tangent, can be perturbed, by small changes of the entries of $Q$, to become totally real everywhere.

**Example 6.11.** A representative coefficient matrix for case (8) is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & -i & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The induced map is one-to-one:

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 - iu_1^2 : u_0^2 - u_1u_2 : u_1^2 + u_2^2 : u_0u_1 : u_0u_2].$$

The locus $\mathcal{D}$ of the map $Q \circ v$ is the line $\{z_2 = iz_1\}$ in $\mathbb{C}P^2$, where $[z_0 : z_1 : iz_1]$ and $[-iz_1 : z_0 : iz_0]$ are mapped to the same point. The singular points are $[\pm\sqrt{-i} : 1 : i]$. So, the image of $\delta$ meets $\mathcal{D}$ only at the point $[1 : 0 : 0]$, and misses both singular points. The map $Q \circ v \circ \delta$ is a totally real embedding.

## 7. Real projections of real Veronese varieties

This Section will briefly consider a construction related to that of Section 2, where we restrict our attention to real coefficient matrices. There are two closely related approaches to defining a "real projection," the first being to forget about $\mathbb{C}$, to consider $\underline{\mathbf{P}}$ as having only real number entries, and defining the Veronese map $\mathbf{v}_\mathbb{R}$ for real spaces as in Subsection 2.1. Then, the composition $\underline{\mathbf{P}} \circ \mathbf{v}_\mathbb{R}$ induces a map $\mathbb{R}P^m \to \mathbb{R}P^n$ that can be analyzed algebraically and geometrically.

The second approach would be to consider complex coefficient matrices $\mathbf{P}$ which satisfy the equality $\mathbf{P} = \mathbf{C}_1 \circ \mathbf{P} \circ \mathbf{C}_2$, where $\mathbf{C}_1$ and $\mathbf{C}_2$ are the conjugation operators on $\mathbb{C}^{n+1}$ and $\mathbb{C}^{(m+1)(m+2)/2}$, respectively. Then, the map $P \circ v \circ \delta : \mathbb{R}P^m \to \mathbb{C}P^n$ will have an image contained in the fixed point set of $C_1$, so it, too, could be considered as a map to $\mathbb{R}P^n$.

An interesting problem is then the classification of such maps, up to real automorphisms of both the domain and the target, which we'll call "R-equivalence" (the precise definition will be statement (1) of the Theorem below). For certain values of $m$, $n$, the classification of real projections has been studied, and is known to be somewhat different from the c-equivalence problem. Real projections of the real Veronese variety also appear in areas of mathematics related to algebraic geometry ([AF], [A], [Degen]).

Obviously, R-equivalence is not the same as r-equivalence, where complex coefficients of the parametrization, and complex transformations of the target space, are allowed, but there is a connection.

**Theorem 7.1.** *Given integers $m$, $n$, and two real coefficient matrices $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$, the following are equivalent.*

(1) *There exist matrices $\underline{\mathbf{A}}_1 \in GL(m+1,\mathbb{R})$, $\underline{\mathbf{B}} \in GL(n+1,\mathbb{R})$ such that for all $\mathbf{u} \in \mathbb{R}^{m+1} \setminus \{\mathbf{0}\}$,*

$$\underline{\mathbf{Q}} \cdot (\mathbf{v}_\mathbb{R}(\mathbf{u})) = \underline{\mathbf{B}} \cdot \underline{\mathbf{P}} \cdot (\mathbf{v}_\mathbb{R}(\underline{\mathbf{A}}_1 \cdot \mathbf{u})).$$

(2) *There exists $\underline{\mathbf{A}}_2 \in GL(m+1,\mathbb{R})$ such that the following $(m(m+3)/2-n)$-dimensional subspaces of $S(m+1,\mathbb{R})$ are equal:*

$$\mathbf{k}(\ker_\mathbb{R}(\underline{\mathbf{P}})) = \underline{\mathbf{A}}_2 \cdot (\mathbf{k}(\ker_\mathbb{R}(\underline{\mathbf{Q}}))) \cdot \underline{\mathbf{A}}_2^T.$$

(3) *$\underline{\mathbf{P}} + i\underline{\mathbf{0}}$ and $\underline{\mathbf{Q}} + i\underline{\mathbf{0}}$ are r-equivalent complex coefficient matrices.*

*Proof.* The equivalence of (1) and (2) is proved by copying the Proof of Theorem 2.6, changing $\mathbb{C}$ to $\mathbb{R}$. Without going into too much detail, the equivalence of (2) and (3) looks like a statement from Theorem 2.15, and the only technicality worth mentioning is that the notion of "kernel" has changed, so there is something to be checked. ∎

Another naïve count, as in Section 3, gives the difference between the dimension of the real parameter space and the dimension of the real group acting on it:

$$\mathcal{M}_R(m,n) = \frac{1}{2}(m^2n + 3mn - m^2 - 2n^2 - m - 2n).$$

Some easy applications of the Theorem occur when the dimension $n$ is just one less than the target dimension $(\frac{1}{2}(m+1)(m+2)-1)$ of the real Veronese map, so the kernel of the projection matrix is a real one-dimensional subspace. The real congruence classes of real symmetric matrices are characterized by rank and signature (Sylvester's Law of Inertia, [Searle]), and are represented by diagonal matrices with 1, $-1$, and 0 entries. The real congruence classes of one-dimensional pencils, where we can multiply by $-1$, are represented by (lines spanned by) non-zero matrices with at least as many $+1$ as $-1$ entries. So, for $n = \frac{1}{2}(m+1)(m+2)-2$, there are finitely many R-equivalence classes (and $\mathcal{M}_R \leq 0$ for all $m$).

**Example 7.2.** The congruence classes of one-dimensional subspaces of $S(2,\mathbb{R})$ are represented by lines spanned by one of these three normal forms:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So, there are three R-equivalence classes of real coefficient matrices, defining maps $\mathbb{R}P^1 \to \mathbb{R}P^1$ ($\mathcal{M}_R(1,1) = -1$). These three correspond to the representatives of r-equivalence classes from Section 4, which happened to have all real coefficients: Examples 4.4, 4.7, 4.8.

**Example 7.3.** The congruence classes of one-dimensional subspaces of $S(3,\mathbb{R})$ are represented by lines spanned by one of these normal forms:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

So, there are five R-equivalence classes of real coefficient matrices, defining maps $\mathbb{R}P^2 \to \mathbb{R}P^4$ ($\mathcal{M}_R(2,4) = -3$). The first three correspond to those representatives of r-equivalence classes from Section 6, where we were able to choose coefficient matrices with all real entries: the real cubic scroll of Example 6.5, and the $\alpha = \pm 1$ cases of Example 6.8. The last two of five represent Case (10) of Proposition 6.4. The fourth matrix appeared, with a representative real coefficient matrix, in Example 6.3. A representative real coefficient matrix for the last of these five is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and the parametric map

$$Q \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 + u_1^2 + 2u_2^2 : u_0^2 - u_1^2 : u_0u_1 : u_0u_2 : u_1u_2]$$

has an image which is contained in the $\{X_0 \neq 0\}$ real affine neighborhood of $\mathbb{R}P^4$, so it defines a smooth embedding $\mathbb{R}P^2 \to \mathbb{R}^4$. Projecting the real Veronese surface into $\mathbb{R}^4$ is a well-known way to construct embeddings of the projective plane ([A]).

**Example 7.4.** If $\mathbf{P}_{4\times 6}$ is a real coefficient matrix, then the image of $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{R}P^3$ is called a real Steiner surface. By finding the real congruence classes of two-dimensional subspaces of $S(3,\mathbb{R})$, the R-equivalence problem for $m = 2$, $n = 3$ was solved by [CSS], resulting in finitely many types of real Steiner surfaces ($\mathcal{M}_R(2,3) = 0$).

**Example 7.5.** If $\mathbf{P}_{3\times 6}$ is a real coefficient matrix, it defines a map $P\circ v\circ\delta : \mathbb{R}P^2 \to \mathbb{R}P^2$. The R-equivalence classification was found by [Degtyarev], and shown to have one continuous invariant (consistent with $\mathcal{M}_R(2,2) = 1$).

**Example 7.6.** If $\mathbf{P}_{2\times 6}$ is a real coefficient matrix, it defines a map $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{R}P^1$. Rather than considering the geometry of that map, we briefly change our point of view, and consider the two polynomials defined by the rows, $P_k(\mathbf{u}) = \sum p_k^{i,j} u_i u_j$, $k = 0, 1$, as the basis for a "pencil of quadratic forms." Evidently, matrices $\mathbf{P}$ and $\mathbf{Q}$ are R-equivalent if and only if $\lambda P_1 + \mu P_2$ and $\lambda Q_1 + \mu Q_2$ are real projectively equivalent pencils of forms, as described by ([L] §IV.11), which finds finitely many equivalence classes, consistent with $\mathcal{M}_R(2,1) = 0$. There are nine classes of pencils containing nondegenerate conics, and four classes of pencils with only degenerate conics.

## 8. A few examples in other dimensions

The r-equivalence problem in higher dimensions seems to be complicated, mostly by the large number of moduli, $\mathcal{M}(m,n)$. Instead of attempting any more congruence calculations, this Section will consider some Examples which demonstrate just some of the various possible geometric properties of quadratically parametrized maps $P \circ v \circ \delta : \mathbb{R}P^m \to \mathbb{C}P^n$.

### 8.1. **Images of real projective spaces.**

**Example 8.1.** The following construction of J. Vrabec appears in [$\mathrm{F}_2$] §3. It describes a real projective plane embedded in $\mathbb{C}^2$. When restricted to the unit sphere $S^2 \subseteq \mathbb{C}\times\mathbb{R}$, the map $F : \mathbb{C}\times\mathbb{R} \to \mathbb{C}^2 : (z,u) \mapsto (z^2, zu)$ identifies antipodal points, and its image is totally real except for one point with a complex tangent. The image $F(S^2)$ admits a parametrization of the form:

$$P \circ v \circ \delta : [u_0 : u_1 : u_2] \mapsto [u_0^2 + u_1^2 + u_2^2 : (u_0 + iu_1)^2 : (u_0 + iu_1)u_2].$$

The image of $P\circ v\circ\delta$ is exactly $F(S^2)$ because for any non-zero point $(u_0+iu_1, u_2) \in \mathbb{C}\times\mathbb{R}$, the point $(z,u) = (\frac{u_0+iu_1}{\sqrt{u_0^2+u_1^2+u_2^2}}, \frac{u_2}{\sqrt{u_0^2+u_1^2+u_2^2}})$ is on $S^2$, and $F(z,u)$ has the form of a pair of quadratic rational functions. As pointed out in [$\mathrm{F}_2$], generalizing this map to any parametrization of the form

$$[u_0^2 + u_1^2 + u_2^2 : P_1(u_0,u_1,u_2) : P_2(u_0,u_1,u_2)]$$

gives an image $(P \circ v \circ \delta)(\mathbb{R}P^2)$ which is contained in the affine neighborhood $\{Z_0 \neq 0\}$ of $\mathbb{C}P^2$.

**Example 8.2.** A construction of [F$_1$] gives an embedding of $\mathbb{R}P^3$ in the $\{Z_0 \neq 0\}$ neighborhood of $\mathbb{C}P^3$, which is the image of the unit sphere $S^3$ in $\mathbb{C}^2$, under the map $F : \mathbb{C}^2 \to \mathbb{C}^3 : (z_1, z_2)^T \mapsto (z_1^2, z_2^2, \sqrt{2}z_1z_2)^T$. Similarly to the previous Example, the submanifold can be parametrized by a map $P \circ v \circ \delta$ that takes $[u_0 : u_1 : u_2 : u_3]$ to

$$[u_0^2 + u_1^2 + u_2^2 + u_3^2 : (u_0 + iu_1)^2 : (u_2 + iu_3)^2 : \sqrt{2}(u_0 + iu_1)(u_2 + iu_3)].$$

The map $P \circ v : \mathbb{C}P^3 \to \mathbb{C}P^3$ is singular at every point; its image is contained in the hypersurface $\mathcal{H}^2 = \{2Z_1Z_2 - Z_3^2 = 0\}$. The image $(P \circ v \circ \delta)(\mathbb{R}P^3)$ is contained in $\mathcal{H}^2$, and in fact is equal to the intersection of $\mathcal{H}^2$ with the real unit sphere $S^5$ in the $\{Z_0 = 1\}$ affine neighborhood. So, this embedded $\mathbb{R}P^3$ is a real hypersurface in a complex surface, and with respect to the ambient space $\mathbb{C}P^3$, it is CR singular at every point: each tangent 3-plane contains a complex line. This is a topologically unstable phenomenon: the CR singular locus of a 3-manifold in three complex dimensions is expected to have codimension 2. In fact, [F$_1$] shows how this submanifold can be perturbed so that it becomes totally real.

8.2. **Images of complex projective spaces.** This Subsection will show that for any odd $m = 2k + 1$, there are some maps $P \circ v \circ \delta : \mathbb{R}P^{2k+1} \to \mathbb{C}P^n$ which are many-to-one, and which have an image homeomorphic to a complex projective space $\mathbb{C}P^k$. The basic construction will be a map $a_1 : \mathbb{R}P^{2k+1} \to \mathbb{C}P^k$ which is double covered by the well-known Hopf fibration $S^{2k+1} \to \mathbb{C}P^k$.

Once all the arrows have been defined, the following diagram will be commutative:

$$\begin{tikzcd}
\mathbb{C}^{2k+2} \arrow[dd, "\mathbf{v}_{\mathbb{C}}"] & \mathbb{R}^{2k+2} \arrow[l, "\boldsymbol{\delta}"'] \arrow[r, "\mathbf{a}_1"] \arrow[d, "\mathbf{v}_{\mathbb{R}}"] & \mathbb{C}^{k+1} \arrow[d, "\mathbf{\Delta}"] \\
 & \mathbb{R}^{(k+1)(2k+3)} \arrow[dl, "\boldsymbol{\delta}'"'] \arrow[dr, "\mathbf{a}_2"] & \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \arrow[d, "\mathbf{s}"] \\
\mathbb{C}^{(k+1)(2k+3)} \arrow[rr, "\mathbf{a}_3"] \arrow[dr, "\mathbf{P}\circ\mathbf{a}_3"] & & \mathbb{C}^{(k+1)^2} \arrow[dl, "\mathbf{P}"'] \\
 & \mathbb{C}^{n+1} &
\end{tikzcd}$$

The maps $\mathbf{v}_{\mathbb{C}} = \mathbf{v}$, $\mathbf{v}_{\mathbb{R}}$, $\boldsymbol{\delta}$, and $\boldsymbol{\delta}'$ are as in Section 2. The map $\mathbf{\Delta}$ is defined by $\mathbf{\Delta} : \mathbf{z} \mapsto (\mathbf{z}, \mathbf{C}(\mathbf{z}))$. The map $\mathbf{s}$ is defined on pairs of vectors by

$$\mathbf{s} : (\mathbf{z}, \mathbf{w}) \mapsto (z_0w_0, \dots, z_jw_\ell, \dots, z_kw_k)^T,$$

the entries of the matrix $\mathbf{z} \cdot \mathbf{w}^T$, put in some (unimportant) order. The set $\mathbb{C}^{k+1} \times \mathbb{C}^{k+1}$ is just considered as a set of ordered pairs, not a vector space, so strictly speaking, it does not make sense to check whether $\mathbf{\Delta}$ or $\mathbf{s}$ satisfy properties (2.1) or (2.2). However, they are related to the following well-known maps: the totally real diagonal embedding,

$$\Delta : \mathbb{C}P^k \to \mathbb{C}P^k \times \mathbb{C}P^k : z \mapsto (z, C(z)),$$

and the holomorphic Segre embedding,

$$s : \mathbb{C}P^k \times \mathbb{C}P^k \to \mathbb{C}P^{(k+1)^2-1} : (z, w) \mapsto [z_0w_0 : \ldots : z_jw_\ell : \ldots : z_kw_k].$$

The image of $\Delta$ is exactly the totally real submanifold

$$\Delta(\mathbb{C}P^k) = \{(z,w) : w = C(z)\}.$$

The composition

$$\mathbf{s} \circ \mathbf{\Delta} : \mathbf{z} \mapsto (z_0\bar{z}_0, \dots, z_j\bar{z}_\ell, \dots, z_k\bar{z}_k)^T$$

does satisfy (2.1) and (2.2), and the induced map $s \circ \Delta : \mathbb{C}P^k \to \mathbb{C}P^{k(k+2)}$ is a totally real embedding.

In terms of the coordinates on $\mathbb{R}^{2k+2}$, define the map $\mathbf{a}_1$ by

$$\mathbf{a}_1 : (u_0, u_1, \dots, u_{2k+1})^T \mapsto (u_0 + iu_1, \dots, u_{2j} + iu_{2j+1}, \dots, u_{2k} + iu_{2k+1})^T.$$

This is not particularly canonical, but it does satisfy (2.1) and (2.2), so it induces a map $a_1 : \mathbb{R}P^{2k+1} \to \mathbb{C}P^k$, so that if $z$ is some one-dimensional subspace of $\mathbb{C}^{k+1}$, then all the real lines in that complex line have the same image. In terms of the $u_j$ coordinates, each component of the composition $\mathbf{s} \circ \mathbf{\Delta} \circ \mathbf{a}_1$ is of the form

$$z_j\bar{z}_\ell = (u_{2j} + iu_{2j+1})(u_{2\ell} - iu_{2\ell+1}) = u_{2j}u_{2\ell} + u_{2j+1}u_{2\ell+1} + iu_{2j+1}u_{2\ell} - iu_{2j}u_{2\ell+1},$$

which is a complex linear combination of the components of $\mathbf{v}_{\mathbb{R}}(\mathbf{u})$. This defines the map $\mathbf{a}_2$, as a $(k+1)^2 \times (k+1)(2k+3)$ matrix whose entries are 0, 1, $i$, and $-i$. The map $\mathbf{a}_3$ is defined using the same matrix, just extending $\mathbf{a}_2$ to the domain $\mathbb{C}^{(k+1)(2k+3)}$ by complex linearity.

To show that $\mathbf{a}_3$ has full rank, it is enough, by the commutativity of the diagram, to show that the image of $\mathbf{s} \circ \mathbf{\Delta} = \mathbf{a}_2 \circ \mathbf{v}_{\mathbb{R}} \circ \mathbf{a}_1^{-1}$ spans $\mathbb{C}^{(k+1)^2}$. This is equivalent to checking that any $(k+1) \times (k+1)$ complex matrix is a complex linear combination of matrices of the form $\mathbf{z} \cdot \bar{\mathbf{z}}^T$. Let $\mathbf{e}_j$, $j = 0, \dots, k$, be the standard basis of $\mathbb{C}^{k+1}$, so that $\mathbf{e}_j \overline{\mathbf{e}_\ell}^T$ is a basis of the matrix space. The claim follows from the easily checked identity:

$$\mathbf{e}_j\overline{\mathbf{e}_\ell}^T = \frac{1}{2}(\mathbf{e}_j + \mathbf{e}_\ell)\overline{(\mathbf{e}_j + \mathbf{e}_\ell)}^T + \frac{i}{2}(\mathbf{e}_j + i\mathbf{e}_\ell)\overline{(\mathbf{e}_j + i\mathbf{e}_\ell)}^T - \frac{1+i}{2}(\mathbf{e}_j\overline{\mathbf{e}_j}^T + \mathbf{e}_\ell\overline{\mathbf{e}_\ell}^T).$$

A simple example is $k = 0$, where

$$\begin{aligned} (\mathbf{s} \circ \mathbf{\Delta} \circ \mathbf{a}_1)((u_0, u_1)^T) &= (u_0^2 + u_1^2), \\ \mathbf{v}_{\mathbb{R}}((u_0, u_1)^T) &= (u_0^2, u_0u_1, u_1^2)^T, \end{aligned}$$

so $\mathbf{a}_2$ and $\mathbf{a}_3$ are defined by the matrix $(1,0,1)_{1\times 3}$. When $\mathbf{a}_3$ is considered as a coefficient matrix, it falls into the $t = -1$ r-equivalence class from Theorem 5.4. When $k = 1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are represented by a $4 \times 10$ matrix.

Finally, the map $\mathbf{P}$ in the diagram can be any full rank complex matrix, with $n + 1 \leq (k+1)^2$, and it could be described as a coefficient matrix for maps of the form

$$P \circ s \circ \Delta : \mathbb{C}P^k \to \mathbb{C}P^n : z \mapsto [\dots : \sum_{j,\ell} p_i^{j,\ell} z_j\bar{z}_\ell : \dots]_{i=0\dots n}.$$

Such maps were considered by [C$_1$]. By the fact that the diagram commutes, any map $P \circ s \circ \Delta$ can be composed with $a_1$, giving a map $\mathbb{R}P^{2k+1} \to \mathbb{C}P^n$, so that the image $(P \circ s \circ \Delta)(\mathbb{C}P^k)$ is exactly equal to the image $(P \circ a_3 \circ v \circ \delta)(\mathbb{R}P^{2k+1})$.

This shows that among all maps $Q \circ v \circ \delta : \mathbb{R}P^{2k+1} \to \mathbb{C}P^n$, as defined in Section 2 with $n + 1 \leq (k+1)^2$, there are always some such that $\mathbf{Q}_{(n+1)\times(k+1)(2k+3)}$ is of the form $\mathbf{P}_{(n+1)\times(k+1)^2} \circ \mathbf{a}_3$, so the image of $Q \circ v \circ \delta$ is equal to the image of some complex projective space under the map $P \circ s \circ \Delta$. The simple case $k = n = 0$ was considered in Section 5. The immersions of $\mathbb{C}P^1$ and $\mathbb{C}P^2$ constructed in [C$_1$] showed that maps $P \circ s \circ \Delta$ can have many interesting geometric properties, including self-intersections, differential-topological singularities, and CR singularities. The above construction shows that the classification of maps $\mathbb{R}P^{2k+1} \to \mathbb{C}P^n$ will be at least as complicated as the equivalence problem for maps $\mathbb{C}P^k \to \mathbb{C}P^n$ stated in [$\mathrm{C}_1$].

8.3. **Normal forms for CR singularities.** The local geometry of a real $m$-submanifold $M$ in $\mathbb{C}^n$ near CR singular points has been studied by several authors, mostly when $m \leq n$, by finding a holomorphic coordinate system in which the defining equations of $M$ in some neighborhood are in a simple "normal form." For some real analytic submanifolds $M$, it can be proved that there is a holomorphic coordinate system where the definining equations are real polynomials, and for some other real analytic submanifolds, $M$ can be put into a polynomial normal form only by a formal coordinate change.

Some of these normal form polynomials define real varieties in $\mathbb{C}^n$ which admit parametrizations of the form $P \circ v \circ \delta$. The image of $P \circ v$ will be contained in some complex variety, which could be considered a "complexification" of $M$.

**Example 8.3.** The surface $M^2 = \{(Z_1, Z_2) : Z_2 = Z_1 \bar{Z}_1\} \subseteq \mathbb{C}^2$ is a paraboloid, contained in the real hyperplane $\mathrm{Im}(Z_2) = 0$. It is one of the normal forms considered by [M], for a surface which is totally real in $\mathbb{C}^2$ except for a complex tangent at the origin, with Bishop invariant equal to 0.

There are several ways to parametrize this surface by rational functions, all of which add one "point at infinity," so $M \cup \{[0 : 0 : 1]\}$ is a smooth 2-sphere in $\mathbb{C}P^2$.

The first parametrization uses a complex coefficient matrix, $\mathbf{P}_{3\times 6}$, to define a map $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{C}P^2$:

$$[u_0 : u_1 : u_2] \mapsto [u_0^2 : u_0(u_1 + iu_2) : (u_1 + iu_2)(u_1 - iu_2)]. \tag{8.1}$$

The idea is simply to use $u_1$, $u_2$ as parameters for the real and imaginary parts of the $Z_1$-axis. The real line at infinity in the domain, $[0 : u_1 : u_2]$, is mapped to a single point, $[0 : 0 : 1]$, in the complex line at infinity.

The second will use a coefficient matrix $\mathbf{P}_{3\times 4}$, to define a map $P \circ s \circ \Delta : \mathbb{C}P^1 \to \mathbb{C}P^2$, as in Subsection 8.2:

$$[z_0 : z_1] \mapsto [z_0 \bar{z}_0 : \bar{z}_0 z_1 : z_1 \bar{z}_1].$$

The image is projectively equivalent to the stereographic projection of the sphere, described in [$\mathrm{C}_1$] §2, which has two complex tangents, so $P \circ s \circ \Delta : \mathbb{C}P^1 \to M \cup \{[0 : 0 : 1]\}$ is a diffeomorphism, with CR singularities at $[1 : 0 : 0]$ and $[0 : 0 : 1]$.

The third method is to take, as in Subsection 8.2, the real and imaginary parts of the coordinates $z_0$, $z_1$ in the previous map to get a map $\mathbb{R}P^3 \to \mathbb{C}P^2$, using a coefficient matrix $\mathbf{P}_{3\times 10}$:

$$[u_0 : u_1 : u_2 : u_3] \mapsto [u_0^2 + u_1^2 : (u_0 - iu_1)(u_2 + iu_3) : u_2^2 + u_3^2].$$

Yet another method uses the fact that $M$ is contained in a real 3-plane, so that it can be parametrized as a real Steiner surface (Example 7.4). The real coefficient matrix $\mathbf{P}_{4\times 6}$ defines a map $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{R}P^3 \subseteq \mathbb{C}P^3$:

$$[u_0 : u_1 : u_2] \mapsto [u_0^2 : u_0 u_1 : u_0 u_2 : u_1^2 + u_2^2].$$

Again, the real line at infinity in the domain is mapped to a single point, $[0 : 0 : 0 : 1]$. The image is still a sphere, and equal to a paraboloid in the real affine neighborhood $\{[1 : X : Y : Z]\}$. However the target space has changed: $\mathbb{R}P^3$ is totally real in $\mathbb{C}P^3$, and the hyperplane $\{\mathrm{Im}(Z_2) = 0\}$ is not totally real in $\mathbb{C}P^2$.

The CR geometry of the sphere is also different: it is totally real with respect to the complex structure of the ambient space $\mathbb{C}P^3$. Composing this map with another projection, $\mathbb{C}P^3 \to \mathbb{C}P^2 : [Z_0 : Z_1 : Z_2 : Z_3] \mapsto [Z_0 : Z_1 + iZ_2 : Z_3]$, will recover the map (8.1), taking the totally real sphere to the CR singular sphere in $\mathbb{C}P^2$.

**Example 8.4.** [C$_2$] considers the following real affine variety, a smooth real surface in $\mathbb{C}^3$ with a CR singularity at the origin:

$$M^2 = \{(Z_1, Z_2, Z_3) : Z_2 = \bar{Z}_1^2, Z_3 = Z_1\bar{Z}_1\}.$$

This subset of $\mathbb{C}^3$ is a graph over the $Z_1$-axis, and it is equal to the image of the parametrization

$$[u_0 : u_1 : u_2] \mapsto [u_0^2 : u_0(u_1 + iu_2) : (u_1 - iu_2)^2 : (u_1 + iu_2)(u_1 - iu_2)],$$

restricted to the $\{u_0 = 1\}$, $\{Z_0 = 1\}$ neighborhoods. In fact, even including the points at infinity, $P \circ v \circ \delta : \mathbb{R}P^2 \to \mathbb{C}P^3$ is a smooth embedding. The map

$$P \circ v : [z_0 : z_1 : z_2] \mapsto [z_0^2 : z_0z_1 + iz_0z_2 : z_1^2 - 2iz_1z_2 - z_2^2 : z_1^2 + z_2^2]$$

is not defined at the point $[0 : i : 1]$, and its Jacobian drops rank at $[1 : 0 : 0]$ and $[0 : -i : 1]$. It is a type of complex Steiner surface (Example 2.8). The image of $P \circ v$ in $\mathbb{C}P^3$ is contained in the hypersurface $\mathcal{H}^2 = \{Z_0Z_3^2 - Z_1^2Z_2 = 0\}$. The intersection of $\mathcal{H}^2$ with the neighborhood $\{Z_0 = 1\}$ is the complex Whitney umbrella, a ruled cubic surface, and the smallest complex affine variety containing $M^2$. The singular locus of $\mathcal{H}^2$ is the double line $\{[Z_0 : 0 : Z_2 : 0]\}$, whose intersection with the image of $P \circ v \circ \delta$ contains only the origin: $(P \circ v)([1 : 0 : 0]) = [1 : 0 : 0 : 0] \in M^2$. So, the CR singular point of $M^2$ coincides with the "pinch point" of its complexification in $\mathbb{C}^3$. (The real Whitney umbrella in $\mathbb{R}^3$, with its double line and pinch point, is illustrated in [A].)

**Example 8.5.** [C$_2$] considers the following real affine variety, a smooth real 4-manifold in $\mathbb{C}^5$ with a CR singularity at the origin:

$$M^4 = \{Y_2 = 0, Y_3 = 0, Z_4 = (\bar{Z}_1 + X_2 + iX_3)^2, Z_5 = Z_1(\bar{Z}_1 + X_2 + iX_3)\}.$$

This subset of $\mathbb{C}^5$ is a graph over its tangent space at the origin, the real 4-plane $T_0$ with coordinates $Z_1$, $X_2$, $X_3$ (where $Z_k = X_k + iY_k$ for $k = 1, \ldots, 5$), and it is equal to the image of the parametrization

$$\begin{array}{rcl} P \circ v \circ \delta : \mathbb{R}P^4 & \to & \mathbb{C}P^5 : \\ {[u_0 : u_1 : u_2 : u_3 : u_4]} & \mapsto & [u_0^2 : u_0(u_1 + iu_2) : u_0u_3 : u_0u_4 : \\ & & \quad (u_1 - iu_2 + u_3 + iu_4)^2 : \\ & & \quad (u_1 + iu_2)(u_1 - iu_2 + u_3 + iu_4)], \end{array}$$

restricted to the $\{u_0 = 1\}$, $\{Z_0 = 1\}$ neighborhoods. The map $P \circ v \circ \delta$ is not defined on the set

$$\{u_0 = 0, u_1 + u_3 = 0, u_2 - u_4 = 0\},$$

a real projective line in the $\mathbb{R}P^3$ at infinity. The map $P \circ v : \mathbb{C}P^4 \to \mathbb{C}P^5$ is undefined on the set

$$L = \{z_0 = 0, z_1 - iz_2 + z_3 + iz_4 = 0\},$$

a complex projective plane in the $\mathbb{C}P^3$ at infinity. The singular locus of the map $P \circ v$ is the following subset of $\mathbb{C}P^4$:

$$(\{z_0 = 0\} \cup \{z_1 + iz_2 = 0, z_1 - iz_2 + z_3 + iz_4 = 0\}) \setminus L. \tag{8.2}$$

The hyperplane at infinity, $\{z_0 = 0\}$, is mapped to the line $\{[0:0:0:0:Z_4:Z_5]\}$ by $P \circ v$ (where it is defined), and the complex affine 3-space $\{z_0 \neq 0, z_1 + iz_2 = 0, z_1 - iz_2 + z_3 + iz_4 = 0\}$ is mapped to the complex affine plane $\mathcal{H}_p = \{[1:0:Z_2:Z_3:0:0]\}$. The image of $P \circ v$ in $\mathbb{C}P^5$ is contained in the hypersurface $\mathcal{H}^4 = \{Z_0 Z_5^2 - Z_1^2 Z_4 = 0\}$. The singular locus of $\mathcal{H}^4$ is the projective 3-space $\mathcal{H}_s = \{Z_1 = 0, Z_5 = 0\}$.

Since $P \circ v$ behaves strangely on the hyperplane at infinity, for the rest of this Example it will be restricted to the $\{z_0 = 1\}$, $\{Z_0 = 1\}$ affine neighborhoods. $P \circ v : \mathbb{C}^4 \to \mathbb{C}^5$ is well-defined but singular, and in affine space it will be easier to see some of the geometry of $M^4$, $P \circ v$, and an interesting multi-valued reflection.

The intersection of $\mathcal{H}^4$ with the affine neighborhood $\{Z_0 = 1\}$ is the product of a complex Whitney umbrella (from the previous Example) and an affine $\mathbb{C}^2$, and it is the smallest complex affine variety containing $M^4$.

The variety $\mathcal{H}^4$ is "ruled" in the sense that its intersection with each hyperplane $Z_4 = k \in \mathbb{C}$ is a pair of intersecting complex 3-planes $\{Z_4 = k, Z_5 = \pm\sqrt{k} Z_1\}$. When $k = 0$, these two 3-planes coincide, and are equal to $T_0 + iT_0$, the smallest complex subspace containing the real 4-plane $T_0$ (this is the tangent space of $M^4$ that contains a complex line). The inverse image under $P \circ v$ of these two planes is a pair of parallel affine 3-planes in $\mathbb{C}^4$, disjoint for $k \neq 0$: $\{z_1 - iz_2 + z_3 + iz_4 = \pm\sqrt{k}\}$. The real variety $M^4$ is similarly ruled by totally real 2-planes parametrized by constant $Z_4$-value $k \in \mathbb{C}$: the two planes $\{\bar{Z}_1 + X_2 + iX_3 = \pm\sqrt{k}, Y_2 = Y_3 = 0, Z_4 = k, Z_5 = \pm\sqrt{k} Z_1\}$ are parallel and disjoint if $k \neq 0$. Their inverse image under $P \circ v \circ \delta : \mathbb{R}^4 \to M^4$ is the pair of planes $\{u_1 + u_3 = \mathrm{Re}(\pm\sqrt{k}), u_2 + u_4 = \mathrm{Im}(\pm\sqrt{k})\}$.

$P \circ v : \mathbb{C}^4 \to \mathcal{H}^4$ is one-to-one, except for a double point locus. A given point $(Z_1, Z_2, Z_3, Z_4, Z_5)^T$ in the image of $P \circ v$ is of the form:

$$\begin{aligned}
Z_1 &= z_1 + iz_2 \\
Z_2 &= z_3 \\
Z_3 &= z_4 \\
Z_4 &= (z_1 - iz_2 + z_3 + iz_4)^2 \\
Z_5 &= (z_1 + iz_2)(z_1 - iz_2 + z_3 + iz_4),
\end{aligned}$$

which uniquely determines $z_3$ and $z_4$. If $Z_1 \neq 0$, then the following system of equations has exactly one solution for $(z_1, z_2)$:

$$\begin{aligned}
z_1 + iz_2 &= Z_1 \\
z_1 - iz_2 &= \frac{Z_5}{Z_1} - Z_2 - iZ_3.
\end{aligned}$$

If $Z_1 = 0$, the quantity $z_1 - iz_2$ can be found using $Z_4$, but the system:

$$\begin{aligned}
z_1 + iz_2 &= 0 \\
z_1 - iz_2 &= \pm\sqrt{Z_4} - Z_2 - iZ_3
\end{aligned}$$

has two solutions if $Z_4 \neq 0$, and one solution if $Z_4 = 0$. Define $\mathcal{D}^3 = \{z_1 + iz_2 = 0\} \subseteq \mathbb{C}^4$, so $(P \circ v)(\mathcal{D}^3)$ is exactly $\mathcal{H}_s$ in $\mathbb{C}^5$, which can be seen as the product of $\mathbb{C}^2$ with the umbrella's double line. If $w_1 = -(z_1 + z_3 + iz_4)$, then $(z_1, iz_1, z_3, z_4)^T$ and $(w_1, iw_1, z_3, z_4)^T \in \mathcal{D}^3$ have the same image in $\mathcal{H}_s$. The map

$$\tau : (z_1, iz_1, z_3, z_4)^T \mapsto (-(z_1 + z_3 + iz_4), -i(z_1 + z_3 + iz_4), z_3, z_4)^T$$

is an involution of $\mathcal{D}^3$ which interchanges the two inverse images. The fixed point set of $\tau$ is exactly the singular locus (8.2) of $P \circ v : \mathbb{C}^4 \to \mathbb{C}^5$, on which $P \circ v$ is one-to-one, with image $\mathcal{H}_p$, the product of $\mathbb{C}^2$ with the “pinch point” of the umbrella.

The affine 3-space $\mathcal{H}_s$ meets $M^4$ in the totally real surface $\mathcal{T}^2 = \{Z_1 = Y_2 = Y_3 = 0,\ Z_4 = (X_2 + iX_3)^2,\ Z_5 = 0\}$. The intersection of the complex plane $\mathcal{H}_p = \{Z_1 = 0, Z_4 = 0, Z_5 = 0\}$ and the real manifold $M^4$ is a set containing exactly one point, the CR singular point.

The conjugation map, $C$, an involution on the domain of $P \circ v : \mathbb{C}^4 \to \mathbb{C}^5$, can be used to induce an “antiholomorphic correspondence,” $\rho = (P \circ v) \circ C \circ (P \circ v)^{-1}$ on the image, $\mathcal{H}^4$. For $Z = (Z_1, \ldots, Z_5)^T \in \mathcal{H} \setminus \mathcal{H}_s$,

$$\begin{aligned} \rho : Z \quad \mapsto \quad & (\overline{\frac{Z_5}{Z_1} - Z_2 - iZ_3}, \bar{Z}_2, \bar{Z}_3, \overline{(Z_1 + Z_2 - iZ_3)}^2, \\ & \overline{(\frac{Z_5}{Z_1} - Z_2 - iZ_3)(Z_1 + Z_2 - iZ_3)})^T. \end{aligned}$$

For $Z = (0, Z_2, Z_3, Z_4, 0)^T \in \mathcal{H}_s$, $\rho$ is double-valued outside $\mathcal{H}_p$:

$$\begin{aligned} \rho : Z \quad \mapsto \quad & (\overline{\pm\sqrt{Z_4} - Z_2 - iZ_3}, \bar{Z}_2, \bar{Z}_3, \overline{(Z_2 - iZ_3)}^2, \\ & \overline{(\pm\sqrt{Z_4} - Z_2 - iZ_3)(Z_2 - iZ_3)})^T. \end{aligned}$$

For $Z \in \mathcal{H}_p \subseteq \mathcal{H}_s$ $(Z_4 = 0)$, the two reflections coincide. The real manifold $M^4$ is contained in $\rho(M^4)$, with points outside $\mathcal{T}^2 = M^4 \cap \mathcal{H}_s$ fixed. Also the origin is a fixed point. The image of $Z = (0, X_2, X_3, (X_2 + iX_3)^2, 0)^T \in \mathcal{T}^2$ is the pair $Z$ (fixed) and $(-2(X_2 - iX_3), X_2, X_3, (X_2 + iX_3)^2, -2(X_2^2 + X_3^2))^T$. So $\rho(\mathcal{T}^2)$ is the union of $\mathcal{T}^2$ with another surface $\tilde{\mathcal{T}}^2$ which is also totally real, meets $\mathcal{H}_s^3$, and $M^4$, only at the origin, and whose (single-valued) image under $\rho$ is $\mathcal{T}^2$.

Some of the unpublished papers in the references are available from the author’s web site, `www.ipfw.edu/math/Coffman/`.

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