Complexification of the CR Cross-Cap

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1 Introduction

These notes are based on research conducted at the University of Chicago. They represent a more self-contained version of a chapter from my dissertation [C_1].[1]

The main result of [C_1] established a normal form for a generic complex tangent of a real m-submanifold in a complex n-manifold, in the case $m < n$. "Generic" refers to a point where the real tangent space contains exactly one complex line, and so that the defining functions satisfy certain non-degeneracy conditions. The "formal stability" theorem says that for a generic complex tangent of a submanifold defined by real analytic functions in a neighborhood of the origin of $\mathbb{C}^n$, there exists a formal holomorphic coordinate change transforming the submanifold into a fixed real algebraic variety M^m. This variety depends only on m and n (there are no continuous invariants, unlike the $m = n$ case) and its defining functions are linear and quadratic polynomials. In these notes, the geometry of these real varieties and their complexifications is considered.

There is some choice in the algebraic normal form. The following two are formally equivalent equations for M^4 in $\mathbb{C}^5$; these dimensions are the smallest where the non-degeneracy conditions are satisfied.

The non-degenerate normal form

$$\begin{aligned} y_2 &= y_3 = 0 \\ z_4 &= x_2(\bar{z}_1 + z_1) + ix_3(\bar{z}_1 - z_1) + z_1^2 + \bar{z}_1^2 \\ z_5 &= z_1\bar{z}_1 \end{aligned}$$

has the advantage of using real-valued functions, and the $z_5 = z_1\bar{z}_1$ expression is both simple and geometrically familiar. Another normal form,

$$\begin{aligned} y_2 &= y_3 = 0 \\ z_4 &= (\bar{z}_1 + x_2 + ix_3)^2 \\ z_5 &= z_1(\bar{z}_1 + x_2 + ix_3), \end{aligned}$$

has complex-valued defining functions and more monomials, but has the advantage of having the same "antiholomorphic variable" $\bar{z}_1 + x_2 + ix_3$ in both

[1] Note added in 2016: Some of these ideas more recently appeared in [C_2], [C_3], [C_4], and [C_5].

expressions. These were the equations used in the proof of the formal stability theorem, and this normal form will be preferred in Section 3 notes to simplify calculations.

Section 2 considers some degenerate cases, and normal forms for complex tangents in higher dimensions. Section 3 of the paper will consider an approach to complexification of the real varieties, and an interesting multi-valued reflection. Section 4 is a purely algebraic computation showing M^4 is indeed totally real away from the origin.

2 Examples of Normal Forms

The coordinates $z_1, \ldots z_5$ of $\mathbb{C}^5$ define real coordinates $x_1, y_1, \ldots x_5, y_5$, with the usual complex structure. The 4-plane T_0 spanned by x_1, y_1, x_2, x_3 contains the complex line with coordinate z_1 and two real directions.

Example 2.1 The normal form varieties mentioned in the Introduction are graphs over the plane T_0, which is the real tangent space at the origin in $\mathbb{C}^5$. They are smooth affine varieties, and it can be shown (Section 4) that they are totally real except for the isolated point at the origin.

Example 2.2 Complex conjugation on $\mathbb{C}^5$ reverses orientation. If an orientation on T_0 space is chosen, complex conjugation maps $z_1 \mapsto \bar{z}_1$, and preserves x_2, x_3, reversing the chosen orientation. The first normal form for M^4 is not conjugation invariant— its image under conjugation is

$$\begin{aligned} y_2 &= y_3 = 0 \\ z_4 &= x_2(\bar{z}_1 + z_1) - ix_3(\bar{z}_1 - z_1) + z_1^2 + \bar{z}_1^2 \\ z_5 &= z_1\bar{z}_1, \end{aligned}$$

This variety is related to M^4 by the holomorphic transformation $z_3 \mapsto -z_3$, but when restricted to T_0, this transformation also reverses orientation. The second normal form is also not invariant under complex conjugation of all five coordinates. In fact, if the formal coordinate changes of the stability theorem are also required to preserve an orientation given to T_0, there are two cases of the non-degenerate normal form, corresponding to a choice of $+i$ or $-i$ in either of the equations for z_4.

In [C_1], it was shown that the locus of complex tangents, for generically immersed, oriented, compact, real submanifolds of complex submanifolds, can be described by characteristic class formulas. When the complex tangents are generically isolated, there are formulas in chern and pontrjagin numbers counting an "index sum" of complex tangents. These indices represent an oriented intersection number, and are generically $+1$ or -1; the two cases represent the inequivalent normal forms under orientation-preserving transformations.

Example 2.3 The following variety, defined by quadratic and linear polynomials, does not satisfy the second non-degeneracy condition of [C$_1$]:

$$\begin{aligned} y_2 &= y_3 = 0 \\ z_4 &= x_2(\bar{z}_1 + z_1) + z_1^2 + \bar{z}_1^2 \\ z_5 &= z_1\bar{z}_1. \end{aligned}$$

This manifold has complex tangent locus N_1 equal to the x_3-axis. It is the product of a 3-manifold with an isolated complex tangent in complex (z_1, z_2, z_4, z_5) space with a real line in the z_3 space. Both phenomena, complex tangents of $M^3 \subseteq \mathbb{C}^4$, and non-isolated complex tangents of $M^4 \subseteq \mathbb{C}^5$, are topologically unstable.

Since the hypotheses of the formal stability theorem are not satisfied, there are submanifolds whose defining functions have the above quadratic terms, but also higher-order terms which cannot be eliminated by formal holomorphic coordinate changes.

Example 2.4 The following variety also has a topologically unstable, but isolated, complex tangent; it is a surface in $\mathbb{C}^3$:

$$\begin{aligned} z_2 &= \bar{z}_1^2 \\ z_3 &= z_1\bar{z}_1. \end{aligned}$$

Example 2.5 A generic 6-manifold in $\mathbb{C}^8$ has an isolated complex tangent, as modeled by the following graph over the z_1, x_2, $\ldots$, x_5 plane:

$$\begin{aligned} y_2 &= y_3 = y_4 = y_5 = 0 \\ z_6 &= x_4(\bar{z}_1 + z_1) + ix_5(\bar{z}_1 - z_1) \\ z_7 &= x_2(\bar{z}_1 + z_1) + ix_3(\bar{z}_1 - z_1) + z_1^2 + \bar{z}_1^2 \\ z_8 &= z_1\bar{z}_1. \end{aligned}$$

The origin is the only complex tangent.

Example 2.6 A higher-dimensional example, but where $n - m$, and therefore the codimension of N_1, is the same as the previous example, is a generic 8-manifold M^8 in $\mathbb{C}^{10}$. N_1 is expected to be a codimension 6 surface, and a model for M^8 is a product of a totally real $\mathbb{R}^2$ with the above 6-manifold:

$$\begin{aligned} y_2 &= y_3 = y_4 = y_5 = y_6 = y_7 = 0 \\ z_8 &= x_4(\bar{z}_1 + z_1) + ix_5(\bar{z}_1 - z_1) \\ z_9 &= x_2(\bar{z}_1 + z_1) + ix_3(\bar{z}_1 - z_1) + z_1^2 + \bar{z}_1^2 \\ z_{10} &= z_1\bar{z}_1. \end{aligned}$$

The complex tangent locus N_1 is the plane with coordinates x_6, x_7. The tangent plane to M^8 is constant along this locus and always contains a complex line parallel to the z_1-axis. If the defining equations for the manifold have these quadratic parts, but also higher-order terms, the locus N_1 is a surface tangent to the x_6, x_7-plane.

3 Complexification and Reflection

The second choice of the normal form for the non-degenerate complex tangent of a 4-manifold in $\mathbb{C}^5$ is the real variety M^4 defined by

$$\begin{aligned} y_2 &= y_3 = 0 \\ z_4 &= (\bar{z}_1 + x_2 + ix_3)^2 \\ z_5 &= z_1(\bar{z}_1 + x_2 + ix_3). \end{aligned}$$

It is easy to see that M^4 is contained in the complex hypersurface $\mathcal{H}^4$ defined by

$$z_1^2 z_4 - z_5^2 = 0.$$

As in Example 2.2, M^4 is not conjugation-invariant, but its conjugate is contained in the same "complex envelope" $\mathcal{H}^4$.

$\mathcal{H}^4$ is singular along the coordinate plane $\mathcal{H}_s^3 = \{z_1 = z_5 = 0\}$, which meets M^4 in the totally real surface $T^2 = \{z_1 = y_2 = y_3 = 0,\ z_4 = (x_2 + ix_3)^2,\ z_5 = 0\}$.

If the defining equations are rewritten in terms of only z and $\bar{z}$, and then $\bar{z}$ are replaced by independent holomorphic variables w, the six real defining polynomials become six complex polynomials in z, w, defining the "complexification" $M_{\mathbb{C}}^4$, a smooth complex 4-manifold in $\mathbb{C}^{10}$:

$$\begin{aligned} z_2 - w_2 &= z_3 - w_3 = 0 \\ z_4 &= (w_1 + z_2 + iz_3)^2 \\ w_4 &= (z_1 + z_2 - iz_3)^2 \\ z_5 &= z_1(w_1 + z_2 + iz_3) \\ w_5 &= w_1(z_1 + z_2 - iz_3). \end{aligned}$$

These implicit equations could also be considered as a holomorphic parametric map $\mathbb{C}^4 \to \mathbb{C}^{10}$ with image $M_{\mathbb{C}}^4$, with global parameters z_1, w_1, z_2, z_3.

The map $\rho : (z, w) \mapsto (\bar{w}, \bar{z})$ is a real structure on $\mathbb{C}^{10}$ (a complex-antilinear involution). Its fixed point set is the totally real 10-plane $\{w = \bar{z}\}$, and it restricts to an antiholomorphic involution on $M_{\mathbb{C}}^4$. The intersection $M_{\mathbb{C}}^4 \cap \{w = \bar{z}\}$ is the "real part" $M_{\mathbb{R}}^4$ of the complex manifold $M_{\mathbb{C}}^4$. $M_{\mathbb{R}}^4$ is a totally real submanifold of $\mathbb{C}^{10}$, and a graph over M^4. It could be considered a "totally real resolution" of the CR-singular submanifold M^4.

The image under π, the projection $\mathbb{C}^{10} \to \mathbb{C}^5$ onto the original z variables, of $M_{\mathbb{C}}^4$ in $\mathbb{C}^5$ is the singular variety $\mathcal{H}^4$; such projections are considered in [W_{84}]. An algebraic interpretation of the projection geometry is an "elimination" of the w variables, as in classical elimination theory; see [CLO]. The projection $M_{\mathbb{C}}^4 \to \mathcal{H}^4$ is one-to-one over most points— for a fixed $z \in \mathcal{H}^4$, solving the complexified equations for w_1 gives the solution $w_1 = \frac{z_5}{z_1} - z_2 - iz_3$ when $z_1 \neq 0$. Since $z_1^2 z_4 = z_5^2$, the z_4 equation is satisfied and $w_2, \ldots, w_5$ are determined uniquely. Points with $z_1 = 0$ project to the singular locus $\mathcal{H}_s^3$, and there may be two inverse images, with $w_1 = \pm\sqrt{z_4} - z_2 - iz_3$ unique only if $z_4 = 0$. Let $\mathcal{D}^3$ denote the two-to-one locus $M_{\mathbb{C}}^4 \cap \{z_1 = 0\} = \pi^{-1}\mathcal{H}_s^3$. $\mathcal{D}^3$ is a complex manifold and admits a holomorphic involution τ which exchanges the two inverse images of the projection:

$$\begin{aligned}\tau : w_1 &\mapsto -(w_1 + 2z_2 + 2iz_3),\\ w_5 &\mapsto -(w_5 + 2z_2^2 + 2z_3^2)\end{aligned}$$

Considering $M^4_{\mathbb{C}}$ as parametrized by z_1, w_1, z_2, z_3 and composing these functions with the projection π, gives a map $\mathbb{C}^4 \to \mathbb{C}^5$. This map has rank 4 except along the subspace $z_1 = 0$, $w_1 + z_2 + iz_3 = 0$, where it has rank 3. The image of this subspace in $M^4_{\mathbb{C}}$ is the fixed-point set of τ, and its (one-to-one) projection by π is the complex plane $\mathcal{H}^4_p = \{z_1 = z_4 = z_5 = 0\}$ inside $\mathcal{H}^3_s$. If $\mathcal{H}^4$ is considered as the "Whitney umbrella" variety $\times\, \mathbb{C}^2$, then $\mathcal{H}^3_s$ is its double line $\times\, \mathbb{C}^2$, and $\mathcal{H}^4_p$ is the pinch point $\times\, \mathbb{C}^2$.

Points $(0, x_2, x_3, (x_2 + ix_3)^2, 0)$ in $T^2 = M^4 \cap \mathcal{H}^3_s$ have inverse image given by $(z_1, w_1, \ldots, z_5, w_5) =$

$$(0, 0, x_2, x_2, x_3, x_3, (x_2 + ix_3)^2, (x_2 - ix_3)^2, 0, 0), \text{ or,}$$

$$(0, -2(x_2+ix_3), x_2, x_2, x_3, x_3, (x_2+ix_3)^2, (x_2-ix_3)^2, 0, -2(x_2+ix_3)(x_2-ix_3)).$$

Both these surfaces are totally real; the first is contained in the $w = \bar{z}$ plane, but the second intersects this plane only at the origin of $\mathbb{C}^{10}$.

The variety $\mathcal{H}^4$ is "ruled" in the sense that its intersection with each hyperplane $z_4 = k \in \mathbb{C}$ is a pair of intersecting complex 3-planes $\{z_4 = k, z_5 = \pm\sqrt{k} z_1\}$. When $k = 0$, these two 3-planes coincide, and are equal to $T_0 + iT_0$, the smallest complex subspace containing the real tangent plane T_0. The inverse image under π of these two planes is a pair of varieties, disjoint for $k \neq 0$:

$$\begin{aligned} w_1 &= \pm\sqrt{k} - z_2 - iz_3\\ z_2 - w_2 &= z_3 - w_3 = 0\\ z_4 &= k\\ w_4 &= (z_1 + z_2 - iz_3)^2\\ z_5 &= \pm\sqrt{k} z_1\\ w_5 &= (\pm\sqrt{k} - z_2 - iz_3)(z_1 + z_2 - iz_3).\end{aligned}$$

The real variety M^4 is similarly ruled by totally real 2-planes parametrized by constant z_4-value $k \in \mathbb{C}$: the two planes $\{\bar{z}_1 + x_2 + ix_3 = \pm\sqrt{k}, y_2 = y_3 = 0, z_4 = k, z_5 = \pm\sqrt{k} z_1\}$ are parallel and disjoint if $k \neq 0$.

The reflection ρ, restricted to $M^4_{\mathbb{C}}$, induces an antiholomorphic correspondence (cf [W94], [W96]) $Q = \pi\rho\pi^{-1}$ on $\mathcal{H}^4$. For $z = (z_1, \ldots, z_5) \in \mathcal{H} \setminus \mathcal{H}_s$,

$$Q(z) = (\overline{\frac{z_5}{z_1} - z_2 - iz_3}, \bar{z}_2, \bar{z}_3, \overline{(z_1 + z_2 - iz_3)}^2, \overline{(\frac{z_5}{z_1} - z_2 - iz_3)(z_1 + z_2 - iz_3)}).$$

For $z \in \mathcal{H}_s$, Q is double-valued outside $\mathcal{H}_p$:

$$Q(0, z_2, z_3, z_4, 0) = (\overline{\pm\sqrt{z_4} - z_2 - iz_3}, \bar{z}_2, \bar{z}_3, \overline{(z_2 - iz_3)}^2, \overline{(\pm\sqrt{z_4} - z_2 - iz_3)(z_2 - iz_3)}).$$

For $z \in \mathcal{H}_p \subseteq \mathcal{H}_s$ ($z_4 = 0$), the two reflections coincide. The real manifold M^4 is contained in $Q(M^4)$, with points outside $T^2 = M^4 \cap \mathcal{H}_s$ fixed. Also the origin is a fixed point. The image of $z = (0, x_2, x_3, (x_2 + ix_3)^2, 0) \in T^2$ is the pair z (fixed) and $(-2(x_2 - ix_3), x_2, x_3, (x_2 + ix_3)^2, -2(x_2^2 + x_3^2))$. So $Q(T^2)$ is the union of T^2 with another surface $\tilde{T}^2$ which is also totally real, meets $\mathcal{H}_s^3$ (and M^4) only at the origin, and whose (single-valued) image under Q is T^2.

The set $\mathcal{D}$ is not ρ-invariant, and in fact $\rho\mathcal{D}$ is the two-to-one locus of the projection π_2 onto the w-space $\mathbb{C}^5$. The intersection $\mathcal{D} \cap \rho\mathcal{D}$ is the complex surface $\{z_1 = w_1 = z_2 - w_2 = z_3 - w_3 = 0, z_4 = (z_2 + iz_3)^2, w_4 = (z_2 - iz_3)^2, z_5 = w_5 = 0\}$, and equal to the complexification $T^2_{\mathbb{C}}$ of T^2. Its projection by π is the complex surface $\{z_1 = 0, z_4 = (z_2 + iz_3)^2, z_5 = 0\} \subseteq \mathcal{H}_s^3$. $\mathcal{D} \cap \rho\mathcal{D}$ is not τ-invariant; the image $\tau(\mathcal{D} \cap \rho\mathcal{D}) = \mathcal{D} \cap \tau\rho\mathcal{D}$ is the complex surface

$$\begin{aligned} z_1 &= z_5 = 0 \\ w_1 &= -2(z_2 + iz_3) \\ z_2 - w_2 &= z_3 - w_3 = 0 \\ z_4 &= (z_2 + iz_3)^2 \\ w_4 &= (z_2 - iz_3)^2 \\ w_5 &= -2(z_2^2 + z_3^2). \end{aligned}$$

The reflection of this surface, $\rho\tau(\mathcal{D} \cap \rho\mathcal{D}) = \rho\mathcal{D} \cap \rho\tau\rho\mathcal{D}$, is the complexification of $\tilde{T}^2$, and projects by π to the surface $\{z_1 = -2(z_2 - iz_3), z_4 = (z_2 + iz_3)^2, z_5 = -2(z_2^2 + z_3^2)\}$.

4 A Standard Basis Calculation

The first normal form for M^4 is a graph over the plane T_0, and a smooth affine subvariety of $\mathbb{R}^{10}$:

$$\begin{aligned} y_2 &= 0 \\ y_3 &= 0 \\ z_4 &= x_2(\bar{z}_1 + z_1) + ix_3(\bar{z}_1 - z_1) + z_1^2 + \bar{z}_1^2 \\ z_5 &= z_1\bar{z}_1, \end{aligned}$$

or, in terms of the real coordinates, $y_2 = y_3 = y_4 = y_5 = 0$ and

$$\begin{aligned} x_4 &= 2(x_1^2 - y_1^2 + x_1x_2 + x_3y_1) \\ x_5 &= x_1^2 + y_1^2. \end{aligned}$$

This intersection of two quadric hypersurfaces and four real hyperplanes defines a codimension six submanifold of $\mathbb{C}^5$, with a complex tangent at the origin by construction. N_1 is not only isolated, as expected, but a singleton. If $r_1, \ldots, r_{10}$ are the coordinates for the tangent bundle of $\mathbb{C}^5$ in the $x_1, y_1, \ldots, x_5, y_5$ directions, the six defining equations for the 4-plane tangent to M^4 at the point

$(x_1, \ldots, y_5)$ are $r_4 = r_6 = r_8 = r_{10} = 0$ and

$$\begin{aligned} 0 &= -4x_1r_1 - 2x_2r_1 + 4y_1r_2 - 2x_3r_2 - 2x_1r_3 - 2y_1r_5 + r_7 \\ 0 &= -2x_1r_1 - 2y_1r_2 + r_9, \end{aligned}$$

and for the rotation of the tangent bundle by i are $r_3 = r_5 = r_7 = r_9 = 0$ and

$$\begin{aligned} 0 &= 4y_1r_1 - 2x_3r_1 + 4x_1r_2 + 2x_2r_2 + 2x_1r_4 + 2y_1r_6 - r_8 \\ 0 &= -2y_1r_1 + 2x_1r_2 - r_{10}. \end{aligned}$$

Weighting x_4 and x_5 to make the graphing functions homogeneous, and computing a Gröbner basis (see [CLO]) of the ideal f given by eighteen functions defining the manifold and its complex tangents, gives twenty-five basis elements, of which twelve are $(y_2, y_3, y_4, y_5, r_3, \ldots, r_{10})$ and the remaining are

$$\begin{aligned} f_{13} &= x_5r_2 \\ f_{14} &= x_5r_1 \\ f_{15} &= x_4r_1 + y_1x_2r_2 - x_1x_3r_2 \\ f_{16} &= x_3r_1 - 4x_1r_2 - x_2r_2 \\ f_{17} &= x_2r_1 - 4y_1r_2 + x_3r_2 \\ f_{18} &= x_2^2r_2 - 8y_1x_3r_2 + x_3^2r_2 + 4x_4r_2 \\ f_{19} &= y_1r_1 - x_1r_2 \\ f_{20} &= y_1x_2x_3r_2 - x_1x_3^2r_2 + 4x_1x_4r_2 + x_2x_4r_2 \\ f_{21} &= y_1^2 - 1/2x_1x_2 - 1/2y_1x_3 + 1/4x_4 - 1/2x_5 \\ f_{22} &= x_1r_1 + y_1r_2 \\ f_{23} &= x_1x_2r_2 + y_1x_3r_2 - x_4r_2 \\ f_{24} &= x_1y_1r_2 + 1/4y_1x_2r_2 - 1/4x_1x_3r_2 \\ f_{25} &= x_1^2 + 1/2x_1x_2 + 1/2y_1x_3 - 1/4x_4 - 1/2x_5. \end{aligned}$$

The polynomials x_5r_2, x_5r_1 indicate a zero-dimensional intersection of the tangent space with its rotation by i unless $x_5 = 0$, which, since $x_5 = x_1^2 + y^2$, happens only when x_1 and y_1 are both zero. Concatenating the ideal f with the ideal (x_1, y_1) gives a new ideal with basis $(x_1, y_1, (x_2^2 + x_3^2)r_2, x_2r_1 + x_3r_2, x_3r_1 - x_2r_2, y_2, y_3, x_4, y_4, x_5, y_5, r_3, \ldots r_{10})$. Now, by inspection, the only complex tangent is at the origin.

References

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