# Trace, Metric, and Reality:
# Notes on Abstract Linear Algebra

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Abstract. Elementary properties of the trace operator, and of some natural vector valued generalizations, are given basis-free statements and proofs, using canonical maps from abstract linear algebra. Properties of tensor contraction with respect to a non-degenerate (but possibly indefinite) metric are similarly analyzed.

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# Preface

These notes are a mostly self-contained collection of some theorems of linear algebra that arise in geometry, particularly results about the trace and bilinear forms. Many results are stated with complete proofs, the main method of proof being the use of canonical maps from abstract linear algebra.

So, the content of these notes is highly dependent on the notation for these maps developed in Chapter 1. This notation will be used in all the subsequent Chapters, which appear in a logical order, but for $1 < m < n$, it is possible to follow Chapter 1 immediately by Chapter $n$, with only a few citations of Chapter $m$. To review the elementary prerequisites, some foundational material appears in Chapter 0 and the Appendices.

In such a collection of results, there will be several statements which will not be needed in later Lemmas, Theorems, or Examples, and can be skipped without losing any logical steps. Such statements will be labeled "Proposition" or "Exercise," with a short proof following from a "Hint" or left to the reader entirely. There are a few statements which are needed in later steps but whose proofs do not fit the basis-free theme; they are labeled "Claim," with proofs left to the references.

Adam Coffman

# Overview

The goal of these notes is to present the subject of linear algebra in a way that is both natural as its own area of mathematics, and applicable, particularly to geometry. The unifying theme is the trace operator on spaces of linear maps, and generalizations of the trace, including vector valued traces, and traces with respect to non-degenerate inner products. The emphasis is on the canonical nature of the objects and maps, and on the basis-free methods of proof. The first definition of the trace (Definition 2.3) is essentially the "conceptual" approach of Mac Lane-Birkhoff ([**MB**] §IX.10) and Bourbaki ([**B**]). This approach also is taken in disciplines using linear algebra as a tool, for example, representation theory and mathematical physics ([**FH**] §13.1, [**Geroch**] Chapter 14, [**K**]). Some of the subsequent formulas for the trace (Theorem 2.10, and in Section 2.4) could be used as alternate but equivalent definitions. In most cases, it is not difficult to translate the results into the usual statements about matrices and tensors, and in some cases, the proofs are more economical than choosing a basis and using matrices. In particular, no unexpected deviations from matrix theory arise.

Part of the motivation for this approach is a study of vector valued Hermitian forms, with respect to abstractly defined complex and real structures. The conjugate linear nature of these objects necessitates careful treatment of scalar multiplication, duality of vector spaces and maps, and tensor products of vector spaces and maps ([**GM**], [**P**]). The study of Hermitian forms seems to require a preliminary investigation into the fundamentals of the theory of bilinear forms, which now forms the first half of these notes. The payoff from the detailed treatment of bilinear forms will be the natural way in which the Hermitian case follows, in the second half.

Chapter 0 gives a brief review of elementary facts about vector spaces, as in a first college course; this should be prerequisite knowledge for most readers. Chapter 1 then sketches a review of notions of spaces of maps $\mathrm{Hom}(U, V)$, tensor products $U \otimes V$, and direct sums $U \oplus V$, and introduces some canonical linear maps, with the notation and basic concepts which will be used in all the subsequent Chapters. Chapter 2 starts with a definition of the usual trace of a map $V \to V$, and then states definitions for the generalized trace of maps $V \otimes U \to V \otimes W$, or $V \to V \otimes W$, whose output is an element of $\mathrm{Hom}(U, W)$, or $W$, respectively. Many of the theorems can be viewed as linear algebra versions of more general statements in category theory, as considered by [**JSV**], [**Maltsiniotis**], [**K**], [**PS**], [**Stolz-Teichner**], [**S**].

Chapter 3 offers a similar basis-free approach to definitions, properties, and examples of a metric on a vector space, and the trace, or contraction, with respect to a metric. The metrics are assumed to be non-degenerate, and finite-dimensionality is a consequence. The main construction is a generalization of the well-known inner product $Tr(A^T \cdot B)$ on the space of matrices; the construction of Theorem 3.41 shows how a metric on $\mathrm{Hom}(U, V)$ is induced by arbitrary metrics on $U$ and $V$, so that $\mathrm{Hom}(U, V)$ is isometric to $U^* \otimes V$ with the induced tensor product metric. Chapter 4 develops the $W$-valued case of the trace with respect to a metric.

The basis-free approach is motivated in part by its usefulness in the geometry of vector bundles and structures on them, including bilinear and Hermitian forms, and almost complex structures. Important geometric applications include real vector bundles with Riemannian metrics, pseudo-Riemannian metrics (since definiteness is not assumed), or symplectic forms. The linear algebra results can be restated geometrically, with linear maps directly replaced by bundle morphisms, "distinguished non-zero element" by "nonvanishing section," and in some cases, "$\mathbb{K}$" by "trivial line bundle."

The plan is to proceed at an elementary pace, so that if the first few Lemmas in Chapter 1 make sense to the reader, then nothing more advanced will be encountered after that. In particular, the relationships with differential geometry and category theory can be ignored entirely by the uninterested reader and are mentioned here only in optional "Remarks." It will be pointed out when the finite-dimensionality is used— for example, in the Theorems in Chapter 2 about the vector valued trace $Tr_{V;U,W}$, $V$ must be finite-dimensional, but $U$ and $W$ need not be.

CHAPTER 0

# Review of Elementary Linear Algebra

## 0.1. Vector spaces

Definition 0.1. Given a set $V$, a field $\mathbb{K}$, a binary operation $+ : V \times V \to V$ (addition), and a function $\cdot : \mathbb{K} \times V \to V$ (scalar multiplication), $(V, +, \cdot)$ is a <u>vector space</u> means that the operations have all of the following properties:

(1) Associative Law for Addition: For any $u \in V$ and $v \in V$ and $w \in V$, $(u + v) + w = u + (v + w)$.
(2) Existence of a Zero Element: There exists an element $0_V \in V$ such that for any $v \in V$, $v + 0_V = v$.
(3) Existence of an Opposite: For each $v \in V$, there exists an element of $V$, called $-v \in V$, such that $v + (-v) = 0_V$.
(4) Associative Law for Scalar Multiplication: For any $\rho, \sigma \in \mathbb{K}$ and $v \in V$, $(\rho\sigma) \cdot v = \rho \cdot (\sigma \cdot v)$.
(5) Scalar Multiplication Identity: For any $v \in V$, $1 \cdot v = v$.
(6) Distributive Law: For all $\rho, \sigma \in \mathbb{K}$ and $v \in V$, $(\rho + \sigma) \cdot v = (\rho \cdot v) + (\sigma \cdot v)$.
(7) Distributive Law: For all $\rho \in \mathbb{K}$ and $u, v \in V$, $\rho \cdot (u + v) = (\rho \cdot u) + (\rho \cdot v)$.

It is a convenient abbreviation to refer to a vector space $(V, +, \cdot)$ as just $V$, as in all of the following Exercises.

Exercise 0.2 (Right Cancellation). Given $u, v, w \in V$, if $u + w = v + w$, then $u = v$.

Hint. These first several Exercises, 0.2 through 0.11, can be proved using only the first three axioms about addition. $\blacksquare$

Exercise 0.3. Given $u, w \in V$, if $u + w = w$, then $u = 0_V$. $\blacksquare$

Exercise 0.4. For any $v \in V$, $(-v) + v = 0_V$. $\blacksquare$

Exercise 0.5. For any $v \in V$, $0_V + v = v$. $\blacksquare$

Exercise 0.6 (Left Cancellation). Given $u, v, w \in V$, if $w + u = w + v$, then $u = v$. $\blacksquare$

Exercise 0.7 (Uniqueness of Zero Element). Given $u, w \in V$, if $w + u = w$, then $u = 0_V$. $\blacksquare$

Exercise 0.8 (Uniqueness of Additive Inverse). Given $v, w \in V$, if $v + w = 0_V$ then $v = -w$ and $w = -v$. $\blacksquare$

Exercise 0.9. $-0_V = 0_V$. $\blacksquare$

Exercise 0.10. For any $v \in V$, $-(-v) = v$. $\blacksquare$

Exercise 0.11. Given $u, x \in V$, $-(u+x) = (-x) + (-u)$. $\blacksquare$

The previous results only used the properties of "+," but the next result, even though its statement refers only to +, uses a scalar multiplication trick, together with the distributive axioms, which relate scalar multiplication to addition.

Theorem 0.12 (Commutative Property of Addition). *For any $v, w \in V$,*

$$v + w = w + v.$$

Proof. We start with this element of $V$, $(1+1)\cdot(v+w)$, and then set LHS=RHS, and use both distributive laws:

$$\begin{array}{rcl} (1+1)\cdot(v+w) & = & (1+1)\cdot(v+w) \\ ((1+1)\cdot v) + ((1+1)\cdot w) & = & (1\cdot(v+w)) + (1\cdot(v+w)) \\ ((1\cdot v)+(1\cdot v)) + ((1\cdot w)+(1\cdot w)) & = & (v+w)+(v+w) \\ (v+v)+(w+w) & = & (v+w)+(v+w). \end{array}$$

Then, the associative law gives $v + (v + (w + w)) = v + (w + (v + w))$, and Left Cancellation leaves $v + (w + w) = w + (v + w)$. Using the associative law again, $(v+w)+w = (w+v)+w$, and Right Cancellation gives the result $v+w = w+v$. $\blacksquare$

Exercise 0.13. For any $v \in V$, $0 \cdot v = 0_V$. $\blacksquare$

Exercise 0.14. For any $v \in V$, $(-1)\cdot v = -v$. $\blacksquare$

Exercise 0.15. For any $\rho \in \mathbb{K}$, $\rho \cdot 0_V = 0_V$. $\blacksquare$

Exercise 0.16. For any $\rho \in \mathbb{K}$ and $u \in V$, $(-\rho)\cdot u = -(\rho \cdot u)$. $\blacksquare$

Exercise 0.17. Given $\rho \in \mathbb{K}$ and $u \in V$, if $\rho \cdot u = 0_V$, then $\rho = 0$ or $u = 0_V$. $\blacksquare$

Exercise 0.18. If $\frac{1}{2} \in \mathbb{K}$, then for any $v \in V$, the following are equivalent. (1) $v + v = 0_V$, (2) $v = -v$, (3) $v = 0_V$.

Hint. Only the implication (1) $\implies$ (3) requires $\frac{1}{2} \in \mathbb{K}$; the others can be proved using only properties of +. $\blacksquare$

Definition 0.19. It is convenient to abbreviate the sum $v + (-w)$ as $v - w$. This defines vector subtraction, so that <u>$v$ minus $w$</u> is defined to be the sum of $v$ and the opposite (or "additive inverse") of $w$.

Notation 0.20. Considering the associative law for addition, it is convenient to write the sum of more than two terms without all the parentheses: $u+v+w$ can mean either $(u+v)+w$, or $u+(v+w)$, since we get the same result either way. In light of Exercise 0.16, we can write $-\rho \cdot v$ to mean either $(-\rho)\cdot v$ or $-(\rho \cdot v)$, since these are the same. The multiplication "dot" can be used, or omitted, for both scalar times scalar and scalar times vector, when it is clear which symbols are scalars and which are vectors: instead of $3 \cdot u$, just write $3u$. It is also convenient to establish an "order of operations," so that scalar multiplication is done before addition or subtraction. So, $4v+u-3w$ is a short way to write $(4\cdot v)+(u+(-(3\cdot w)))$.

## 0.2. Subspaces

The general idea of the statement "$W$ is a subspace of $V$" is that $W$ is a vector space contained in a bigger vector space $V$ with the same field $\mathbb{K}$, and the $+$ and $\cdot$ operations are the same in $W$ as they are in $V$.

Definition 0.21. Let $(V, +_V, \cdot_V)$ be a vector space with field of scalars $\mathbb{K}$. A set $W$ is a <u>subspace</u> of $V$ means:

- $W \subseteq V$, and
- There are operations $+_W : W \times W \to W$ and $\cdot_W : \mathbb{K} \times W \to W$ such that $(W, +_W, \cdot_W)$ is a vector space, and
- For all $x$, $y \in W$, $x +_V y = x +_W y$, and
- For all $x \in W$, $\rho \in \mathbb{K}$, $\rho \cdot_V x = \rho \cdot_W x$.

Theorem 0.22. *If $W$ is a subspace of $V$, where $V$ has zero element $0_V$, then $0_V$ is an element of $W$, and is equal to the zero element of $W$.*

Proof. By the second part of Definition 0.21, $W$ is a vector space, so by Definition 0.1 applied to $W$, $W$ contains a zero element $0_W \in W$. By the first part of Definition 0.21, $W \subseteq V$, which implies $0_W \in V$. By Definition 0.1 applied to $W$, $0_W +_W 0_W = 0_W$, and by Definition 0.21, $0_W +_V 0_W = 0_W +_W 0_W$. It follows that $0_W +_V 0_W = 0_W \in V$, and then Exercise 0.3 implies $0_W = 0_V$. ∎

Theorem 0.22 can be used in this way: if $W$ is a set that does <u>not</u> contain $0_V$ as one of its elements, then $W$ is <u>not</u> a subspace of $V$.

Theorem 0.23. *If $W$ is a subspace of $V$, then for every $w \in W$, the opposite of $w$ in $W$ is the same as the opposite of $w$ in $V$.*

Proof. Let $w$ be an element of $W$; then $w \in V$ because $W \subseteq V$.

First, we show that an additive inverse of $w$ in $W$ is also an additive inverse of $w$ in $V$. Let $y$ be any additive inverse of $w$ in $W$, meaning $y \in W$ and $w +_W y = 0_W$. (There exists at least one such $y$, by Definition 0.1 applied to $W$.) $W \subseteq V$ implies $y \in V$. From Theorem 0.22, $0_W = 0_V$, and $w +_W y = w +_V y$ by Definition 0.21, so $w +_V y = 0_V$, which means $y$ is an additive inverse of $w$ in $V$.

Second, we show that an additive inverse of $w$ in $V$ is also an additive inverse of $w$ in $W$. Let $z$ be any additive inverse of $w$ in $V$, meaning $z \in V$ and $w +_V z = 0_V$. (There exists at least one such $z$, by Definition 0.1 applied to $V$.) Then $w +_V z = 0_V = w +_V y$, so by Left Cancellation in $V$, $z = y$ and $y \in W$, which imply $z \in W$ and $w +_W z = w +_W y = 0_W$, meaning $z$ is an additive inverse of $w$ in $W$.

By uniqueness of opposites (Exercise 0.8 applied to either $V$ or $W$), we can refer to $y = z$ as "the" opposite of $w$, and denote it $y = -w$. ∎

Theorem 0.23 also implies that subtraction in $W$ is the same as subtraction in $V$: by Definition 0.19, for $v$, $w \in W$, $v -_W w = v +_W y = v +_V y = v -_V w$.

Theorem 0.23 can be used in this way: if $W$ is a subset of a vector space $V$ and there is an element $w \in W$, where the opposite of $w$ in $V$ is <u>not</u> an element of $W$, then $W$ is <u>not</u> a subspace of $V$.

THEOREM 0.24. *Let $(V, +_V, \cdot_V)$ be a vector space, and let $W$ be a subset of $V$. The following are equivalent.*

- *$W$ satisfies all of these three properties:*
  *(1) $x \in W$, $y \in W$ imply $x +_V y \in W$ (closure under $+_V$ addition), and*
  *(2) $\rho \in \mathbb{K}$, $x \in W$ imply $\rho \cdot_V x \in W$ (closure under $\cdot_V$ scalar multiplication), and*
  *(3) $W \neq \emptyset$.*
- *$W$ is a subspace of $V$.*

PROOF. Let $V$ have zero element $0_V$.

First suppose $W$ is a subspace, so that by Definition 0.21, $W$ is a vector space and contains a zero element $0_W$, which shows $W \neq \emptyset$, and (3) is true. From Definition 0.1 the vector space $W$ has an operation $+_W$ so that $x \in W$, $y \in W$ imply $x +_W y \in W$, and from the definition of subspace, $x +_W y = x +_V y$, so $x +_V y \in W$, establishing (1). Similarly, from Definition 0.1, $W$ has a scalar multiplication so that $\rho \in \mathbb{K}$ implies $\rho \cdot_W x \in W$, and from the definition of subspace, $\rho \cdot_W x = \rho \cdot_V x$, so $\rho \cdot_V x \in W$, establishing (2).

Conversely, it follows from (1), (2), and (3) that $W$ is a subspace of $V$, as follows: $W$ is a subset of $V$ by hypothesis. Define $+_W$ and $\cdot_W$ for $x, y \in W$ and $\rho \in \mathbb{K}$ by $x +_W y = x +_V y$, and $\rho \cdot_W x = \rho \cdot_V x$ — these define operations on $W$ by (1) and (2), and the last two properties from Definition 0.21 are satisfied by construction. It remains to check the other properties from Definition 0.1 to show that $(W, +_W, \cdot_W)$ is a vector space. Since $W \neq \emptyset$ by (3), there is some $x \in W$, and by (2), $0 \cdot_V x \in W$. By Exercise 0.13, $0 \cdot_V x = 0_V$, so $0_V \in W$, and it satisfies $x +_W 0_V = x +_V 0_V = x$ for all $x \in W$, so $0_V$ is a zero element for $W$. The scalar multiple identity also works: $1 \cdot_W x = 1 \cdot_V x = x$. Also by (2), for any $x \in W$, $(-1) \cdot_V x \in W$, and it is easy to check $(-1) \cdot_V x$ is an additive inverse of $x$ in $W$: $x +_W ((-1) \cdot_V x) = (1 \cdot_V x) +_V ((-1) \cdot_V x) = (1 + (-1)) \cdot_V x = 0 \cdot_V x = 0_V$. The other vector space properties, (1), (4), (6), (7) from Definition 0.1, follow immediately from the facts that these properties hold in $V$ and the operations in $W$ give the same sums and scalar multiples. ∎

EXERCISE 0.25. Given a vector space $V$, if $W$ is a subspace of $V$ and $U$ is a subspace of $W$, then $U$ is a subspace of $V$. ∎

EXERCISE 0.26. Given a vector space $V$, if $W$ is a subspace of $V$, $U$ is a subspace of $V$, and $U \subseteq W$, then $U$ is a subspace of $W$. ∎

EXERCISE 0.27. Given a vector space $V$, if $U$ is a subspace of $V$ and $W$ is a subspace of $V$ then the intersection $U \cap W$ is a subspace of $V$. ∎

DEFINITION 0.28. A vector space $V$ is finite-dimensional means that there does not exist an infinite ordered list of subspaces of $V$,

$$(V_0, V_1, V_2, \ldots, V_\nu, \ldots),$$

with $V_\nu \subsetneq V_{\nu+1}$ for all $\nu$.

EXERCISE 0.29. Given a vector space $V$, if $V$ is finite-dimensional and $W$ is a subspace of $V$, then $W$ is finite-dimensional. ∎

Definition 0.30. Given a vector space $V$ and any subset $S \subseteq V$, the span of $S$ is the subset of $V$ defined as the set of all (finite) sums of elements of $S$ with coefficients in $\mathbb{K}$:

$$\{\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_\nu s_\nu : \alpha_1, \ldots, \alpha_\nu \in \mathbb{K}, s_1, \ldots, s_\nu \in S\}.$$

This is always a subspace of $V$. (We define the span of $\emptyset$ to be $\{0_V\}$.)

Exercise 0.31. Given a vector space $V$, if $V$ is finite-dimensional then there exists a finite subset $S \subseteq V$ such that the span of $S$ is equal to $V$. ∎

Definition 0.32. Given an ordered list (possibly with repeats) of elements of a vector space $V$, $(u_1, u_2, \ldots, u_\nu)$, the list is linearly independent means that the following implication holds for scalars $\alpha_1, \ldots, \alpha_\nu \in \mathbb{K}$:

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_\nu u_\nu = 0_V \implies \alpha_1 = \alpha_2 = \ldots = \alpha_\nu = 0.$$

The empty list is linearly independent. We will not have occasion to consider infinite lists.

Claim 0.33. *If $V$ is a vector space equal to the span of a finite set $S \subseteq V$ with $\nu$ elements, then $V$ is finite-dimensional and there is no linearly independent list of elements of $V$ of length $> \nu$.* ∎

Definition 0.34. Given an ordered list of elements of a vector space $V$, $(u_1, u_2, \ldots, u_\nu)$, the list is an ordered basis of $V$ means that the list is linearly independent and the span of the set $\{u_1, u_2, \ldots, u_\nu\}$ is equal to $V$.

## 0.3. Additive functions and linear functions

Let $U$, $V$, and $W$ be vector spaces with the same field of scalars $\mathbb{K}$.

Definition 0.35. A function $F : U \to V$ is additive means: $F$ has the property that $F(u + w) = F(u) + F(w)$ for all $u, w \in U$.

Definition 0.36. A function $F : U \to V$ is linear means: $F$ is additive and also has the scaling property: $F(\rho \cdot u) = \rho \cdot F(u)$ for all $\rho \in \mathbb{K}$ and all $u \in U$.

Exercise 0.37. If $F : U \to V$ is additive, then: $F(0_U) = 0_V$, and for all $u \in U$, $F(-u) = -F(u)$. ∎

Exercise 0.38. If $\mathbb{K}$ contains the rational numbers $\mathbb{Q}$ as a subrng, and $F : U \to V$ is additive, then for every rational number $\rho \in \mathbb{Q}$, $F(\rho \cdot u) = \rho \cdot F(u)$.

Hint. Start by showing that for integers $\nu \in \mathbb{Z}$, $F(\nu \cdot u) = \nu \cdot F(u)$. ∎

Exercise 0.39. Give an example of a field, vector spaces $V$ and $W$, and a function $F : V \to W$, such that $F$ has the scaling property, but which is not linear because it is not additive. ∎

Exercise 0.40. Give an example of a field, vector spaces $V$ and $W$, and a function $F : V \to W$ such that $F$ has the additive property, but which is not linear because it does not have the scaling property.

Hint. Try this for $\mathbb{K} = \mathbb{C}$ and then for $\mathbb{K} = \mathbb{R}$ (harder). ∎

NOTATION 0.41. At this point, the arrow notation $F : V \to W$ will only be used for functions that are linear from a vector space $V$ (the domain) to another vector space $W$ (the target). A linear function will also be called a map, linear map, or arrow, or when it is convenient to emphasize the scalar field, a $\mathbb{K}$-linear map. A function which is not necessarily linear will be denoted with a variant $\rightsquigarrow$ arrow symbol.

EXAMPLE 0.42. Given a vector space $V$ and a subspace $W$, the canonical subspace inclusion function $Q : W \to V$ defined by $Q(w) = w$ is linear.

NOTATION 0.43. As a special case of the above inclusion map (and as in Example 6.15), the identity map $Id_V : V \to V$, defined by the formula $Id_V(v) = v$, is linear.

EXERCISE 0.44. Given vector spaces $U$, $V$, $W$, and functions $F : W \to V$, $G : V \to U$, if $F$ and $G$ are linear, then the composite $G \circ F : W \to U$ is linear. ∎

The $\circ$ notation for composites is as in Notation 6.16.

EXAMPLE 0.45. For a linear function $F : W \to V$, the image of $F$ is the set $F(W) = \{F(w) \in V : w \in W\}$. The image is always a subspace of the target $V$.

EXAMPLE 0.46. For a linear function $F : W \to V$, the kernel of $F$ is the set $\ker(F) = \{w \in W : F(w) = 0_V\}$. The kernel is always a subspace of the domain $W$.

DEFINITION 0.47. A linear map $C : X \to Y$ is a linear monomorphism means: $C$ has the following cancellation property for any compositions with linear maps $A$ and $B$ (which are well-defined in the sense that $X$ is the target space of both $A$ and $B$),

$$C \circ A = C \circ B \implies A = B.$$

EXERCISE 0.48. Given a linear map $F : U \to V$, the following are equivalent.

(1) $F$ is one-to-one.
(2) $F$ is left cancellable.
(3) $F$ has a left inverse: there exists a function $H : V \rightsquigarrow U$ such that $H \circ F = Id_U$.
(4) $F$ is a linear monomorphism.
(5) $\ker(F) = \{0_U\}$.

HINT. The first three properties are considered in Exercise 6.17, and their equivalence is a matter of set theory only, without using the linearity of $F$. In particular, the function $H$ is not necessarily linear. The implication (2) $\implies$ (4) is trivial, and (1) $\implies$ (5) uses only $F(0_U) = 0_V$ from Exercise 0.37. The implications (5) $\implies$ (1) and (4) $\implies$ (5) follow from the linearity of $F$. ∎

EXERCISE 0.49. Given a list $(u_1, u_2, \ldots, u_\nu)$ of elements of $V$, if $A : V \to W$ is linear and the list $(A(u_1), \ldots, A(u_\nu))$ is independent in $W$, then $(u_1, u_2, \ldots, u_\nu)$ is independent. If $A : V \to W$ is linear and one-to-one, and $(u_1, u_2, \ldots, u_\nu)$ is independent, then $(A(u_1), \ldots, A(u_\nu))$ is independent. ∎

EXERCISE 0.50. If $V$ is finite-dimensional and $F : U \to V$ is linear and one-to-one, then $U$ is finite-dimensional. ∎

EXERCISE 0.51. Given linear maps $A : W \to V$ and $F : U \to V$, if $A(W) \subseteq F(U)$ and $H : V \rightsquigarrow U$ is any left inverse of $F$, then $H \circ A : W \to U$ is linear.

HINT. For any two elements $v_1$, $v_2 \in F(U)$, $v_1 = F(u_1)$ for some unique $u_1$ as in Exercise 0.48, and similarly $v_2 = F(u_2)$. Using the linearity of $F$ and any scalar $\lambda$,

$$\begin{aligned} H(v_1 + v_2) &= H(F(u_1) + F(u_2)) = H(F(u_1 + u_2)) \\ &= u_1 + u_2 = H(F(u_1)) + H(F(u_2)) = H(v_1) + H(v_2), \\ H(\lambda \cdot v_1) &= H(\lambda \cdot F(u_1)) = H(F(\lambda \cdot u_1)) \\ &= \lambda \cdot u_1 = \lambda \cdot H(F(u_1)) = \lambda \cdot H(v_1). \end{aligned} \tag{0.1}$$

In particular, for $w_1$, $w_2 \in W$, let $A(w_1) = F(u_1)$ and $A(w_2) = F(u_2)$. Then, using the additivity of $A$ and (0.1),

$$(H \circ A)(w_1 + w_2) = H(A(w_1) + A(w_2)) = (H \circ A)(w_1) + (H \circ A)(w_2).$$

The scaling property for $H \circ A$ follows similarly. ∎

DEFINITION 0.52. A map $C : X \to Y$ is a <u>linear epimorphism</u> means: $C$ has the following cancellation property for any compositions with linear maps $A$ and $B$,

$$A \circ C = B \circ C \implies A = B.$$

EXERCISE 0.53. If the linear map $F : U \to V$ has a right inverse, meaning that there exists $H : V \rightsquigarrow U$ so that $F \circ H = Id_V$, then $F$ is onto and right cancellable as in Exercise 6.18, and the right cancellable property implies that $F$ is a linear epimorphism. ∎

EXERCISE 0.54. Given vector spaces $V$, $W$, and a linear function $F : W \to V$, the following are equivalent.

(1) $F$ is both one-to-one and onto.
(2) $F$ has a right inverse $H_1 : V \rightsquigarrow W$ and a left inverse $H_2 : V \rightsquigarrow W$.
(3) There exists a linear map $G : V \to W$ such that $G$ is a left inverse of $F$ and $G$ is a right inverse of $F$.

HINT. The (3) $\implies$ (2) direction is trivial. The equivalence (1) $\iff$ (2) does not use the linearity of $F$ — see Exercise 6.20, which also shows that (2) implies $H_1 = H_2$, so $G : V \rightsquigarrow W$ in (3) can be chosen to equal $H_1 = H_2$. The linearity of $F$ then implies the linearity of $G$ by Exercise 0.51. ∎

NOTATION 0.55. Given vector spaces $V$, $W$, and a linear function $F : W \to V$, $F$ is <u>invertible</u> means that $F$ satisfies any of the three equivalent properties (1), (2), or (3) from Exercise 0.54, as in Definition 6.21. However, we usually use property (3), so that as in Notation 6.22, $G$ is the unique <u>inverse</u> of $F$, denoted $G = F^{-1}$. It follows that $F^{-1}$ is also invertible, with inverse $F$.

CLAIM 0.56. *If $V$ is finite-dimensional and $F : V \to V$ is linear and $F$ is either a linear monomorphism or a linear epimorphism, then $F$ is invertible.* ∎

CLAIM 0.57. *Given a finite-dimensional vector space $V$ and a linear map $F : V \to V$, the following are equivalent.*

(1) *For all linear maps $A : V \to V$, $A \circ F = F \circ A : V \to V$.*
(2) *There exists a scalar $\lambda \in \mathbb{K}$ so that for all $v \in V$, $F(v) = \lambda \cdot v$.*

PROOF. The second property can be denoted $F = \lambda \cdot Id_V$. See [B] Exercise II.1.26 or [J] §3.11. $\blacksquare$

CHAPTER 1

# Abstract Linear Algebra

## 1.1. Spaces of linear maps

For this Chapter, we fix an arbitrary field $(\mathbb{K}, +, \cdot, 0, 1)$. All vector spaces use the same scalar field $\mathbb{K}$, and $+$ and $\cdot$ also refer to the vector space addition and scalar multiplication. A $\mathbb{K}$-linear map $A$ with domain $U$ and target $V$, such that $A(u) = v$, will be written as $A : U \to V : u \mapsto v$, or the $A$ may appear near the arrow when several maps are combined in a diagram.

Notation 1.1. The set of all $\mathbb{K}$-linear maps from $U$ to $V$ is denoted $\mathrm{Hom}(U, V)$.

Claim 1.2. $\mathrm{Hom}(U, V)$ *is itself a vector space over* $\mathbb{K}$*. If* $U$ *and* $V$ *are finite-dimensional, then* $\mathrm{Hom}(U, V)$ *is finite-dimensional.* ∎

Notation 1.3. $\mathrm{Hom}(V, V)$ is abbreviated $\mathrm{End}(V)$, the space of endomorphisms of $V$. The space $\mathrm{End}(V)$ has a distinguished element, the identity map denoted $Id_V : v \mapsto v$ from Notation 0.43.

Notation 1.4. $\mathrm{Hom}(V, \mathbb{K})$ is abbreviated $V^*$, the <u>dual space</u> of $V$.

Definition 1.5. For maps $A : U' \to U$ and $B : V \to V'$, define

$$\mathrm{Hom}(A, B) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U', V')$$

so that for $F : U \to V$,

$$\mathrm{Hom}(A, B)(F) = B \circ F \circ A : U' \to V'.$$

Lemma 1.6. ([B] §II.1.2) *If* $A : U \to V$*,* $B : V \to W$*,* $C : X \to Y$*,* $D : Y \to Z$*, then*

$$\mathrm{Hom}(A, D) \circ \mathrm{Hom}(B, C) = \mathrm{Hom}(B \circ A, D \circ C) : \mathrm{Hom}(W, X) \to \mathrm{Hom}(U, Z).$$

∎

Definition 1.7. For any vector spaces $U$, $V$, $W$, define a <u>generalized transpose</u> map,

$$t_{UV}^{W} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(U, W)),$$

so that for $A : U \to V$, $B : V \to W$,

$$t_{UV}^{W}(A) = \mathrm{Hom}(A, Id_W) : B \mapsto B \circ A.$$

LEMMA 1.8. *For any vector spaces $U$, $V$, $W$, $U'$, $V'$, $W'$, and any maps $E : U' \to U$, $F : V \to V'$, $G : W \to W'$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U,V) \arrow[r, "t_{UV}^{W}"] \arrow[dd, "\mathrm{Hom}(E,F)"'] & \mathrm{Hom}(\mathrm{Hom}(V,W),\mathrm{Hom}(U,W)) \arrow[d, "\mathrm{Hom}(\mathrm{Hom}(F,Id_W),\mathrm{Hom}(E,G))"] \\
& \mathrm{Hom}(\mathrm{Hom}(V',W),\mathrm{Hom}(U',W')) \\
\mathrm{Hom}(U',V') \arrow[r, "t_{U'V'}^{W'}"] & \mathrm{Hom}(\mathrm{Hom}(V',W'),\mathrm{Hom}(U',W')) \arrow[u, "\mathrm{Hom}(\mathrm{Hom}(Id_{V'},G),\mathrm{Hom}(Id_{U'},Id_{W'}))"']
\end{tikzcd}$$

PROOF. For any $A : U \to V$,

$$\begin{array}{rcl}
& & \mathrm{Hom}(\mathrm{Hom}(Id_{V'},G),\mathrm{Hom}(Id_{U'},Id_{W'})) \circ t_{U'V'}^{W'} \circ \mathrm{Hom}(E,F) : \\
A & \mapsto & (t_{U'V'}^{W'}(F \circ A \circ E)) \circ \mathrm{Hom}(Id_{V'},G) \\
& = & \mathrm{Hom}(F \circ A \circ E, Id_{W'}) \circ \mathrm{Hom}(Id_{V'},G), \\
& & \mathrm{Hom}(\mathrm{Hom}(F,Id_W),\mathrm{Hom}(E,G)) \circ t_{UV}^{W} : \\
A & \mapsto & \mathrm{Hom}(E,G) \circ \mathrm{Hom}(A,Id_W) \circ \mathrm{Hom}(F,Id_W).
\end{array}$$

The claimed equality follows from Lemma 1.6. ∎

NOTATION 1.9. In the special case $W = \mathbb{K}$, $t_{UV}^{\mathbb{K}}$ is a canonical transpose map from $\mathrm{Hom}(U,V)$ to $\mathrm{Hom}(V^*,U^*)$, and it is abbreviated $t_{UV}^{\mathbb{K}} = t_{UV}$.

NOTATION 1.10. $t_{UV}(A) = \mathrm{Hom}(A, Id_{\mathbb{K}})$ is abbreviated by $A^* : V^* \to U^*$, so that for $\phi \in V^*$, $A^*(\phi)$ is the map $\phi \circ A : U \to \mathbb{K}$, i.e., $(A^*(\phi))(u) = \phi(A(u))$.

CLAIM 1.11. *For any vector spaces $U$, $V$, the map*

$$t_{UV} : \mathrm{Hom}(U,V) \to \mathrm{Hom}(V^*,U^*) : A \mapsto A^*$$

*is one-to-one. If $V$ is finite-dimensional then $t_{UV}$ is invertible.* ∎

LEMMA 1.12. *For $A : U \to V$ and $F : V \to V'$, $(F \circ A)^* = A^* \circ F^*$, and if $A$ is invertible, then so is $A^*$, with $(A^{-1})^* = (A^*)^{-1}$. Also, $Id_{V^*} = Id_V^*$.*

PROOF. The claim about $F \circ A$ follows from Lemma 1.8 (with $E = Id_U$, $G = Id_{\mathbb{K}}$), or by applying Lemma 1.6 directly. Note that $t_{VV} : \mathrm{End}(V) \to \mathrm{End}(V^*)$ takes the distinguished element $Id_V \in \mathrm{End}(V)$ to the distinguished element $t_{VV}(Id_V) = Id_V^* = Id_{V^*} \in \mathrm{End}(V^*)$. ∎

DEFINITION 1.13. For any vector spaces $V$, $W$, define

$$d_{VW} : V \to \mathrm{Hom}(\mathrm{Hom}(V,W),W)$$

so that for $v \in V$, $H \in \mathrm{Hom}(V,W)$,

$$(d_{VW}(v)) : H \mapsto H(v).$$

LEMMA 1.14. *For any vector spaces $U$, $V$, $W$, $X$ and any maps $H : U \to V$, $G : W \to X$, the following diagram is commutative.*

$$\begin{tikzcd}
U \arrow[rr, "d_{UW}"] \arrow[dr, "d_{UX}"] \arrow[dd, "H"'] & & \mathrm{Hom}(\mathrm{Hom}(U,W),W) \arrow[d, "\mathrm{Hom}(\mathrm{Hom}(H,Id_W),G)"] \\
& \mathrm{Hom}(\mathrm{Hom}(U,X),X) \arrow[r, "\mathrm{Hom}(\mathrm{Hom}(H,G),Id_X)"'] & \mathrm{Hom}(\mathrm{Hom}(V,W),X) \\
V \arrow[rr, "d_{VX}"] & & \mathrm{Hom}(\mathrm{Hom}(V,X),X) \arrow[u, "\mathrm{Hom}(\mathrm{Hom}(Id_V,G),Id_X)"']
\end{tikzcd}$$

PROOF. For $u \in U$, $A : V \to W$,

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Hom}(Id_V,G),Id_X) \circ d_{VX} \circ H : u &\mapsto (d_{VX}(H(u))) \circ \mathrm{Hom}(Id_V,G) : \\
A &\mapsto (G \circ A)(H(u)), \\
\mathrm{Hom}(\mathrm{Hom}(H,G),Id_X) \circ d_{UX} : u &\mapsto (d_{UX}(u)) \circ \mathrm{Hom}(H,G) : \\
A &\mapsto (G \circ A \circ H)(u), \\
\mathrm{Hom}(\mathrm{Hom}(H,Id_W),G) \circ d_{UW} : u &\mapsto G \circ (d_{UW}(u)) \circ \mathrm{Hom}(H,Id_W) : \\
A &\mapsto G((d_{UW}(u))(A \circ H)) \\
&= G((A \circ H)(u)).
\end{aligned}$$

∎

As a special case of Lemma 1.14 with $W = X$, $G = Id_W$, for any $H : U \to V$,

$$(t^W_{\mathrm{Hom}(V,W),\mathrm{Hom}(U,W)}(t^W_{UV}(H))) \circ d_{UW} = d_{VW} \circ H, \tag{1.1}$$

where $t^W_{\mathrm{Hom}(V,W),\mathrm{Hom}(U,W)}(t^W_{UV}(H)) = \mathrm{Hom}(\mathrm{Hom}(H,Id_W),Id_W)$ by Definition 1.7 of the $t$ maps.

NOTATION 1.15. In the special case $W = \mathbb{K}$, $d_{V\mathbb{K}}$ is abbreviated $d_V$. It is the canonical double duality map $d_V : V \to V^{**}$, defined by $(d_V(v))(\phi) = \phi(v)$.

The case (1.1) of Lemma 1.14 then gives the equation ([**B**] §II.2.7, [**AF**] §20)

$$d_V \circ A = A^{**} \circ d_U, \tag{1.2}$$

where for $A : U \to V$, $A^{**}$ abbreviates $t_{V^*U^*}(t_{UV}(A))$.

CLAIM 1.16. *The canonical map $d_V$ is one-to-one. $d_V$ is invertible if and only if $V$ is finite-dimensional.*

PROOF. The one-to-one property is easily checked. See [**B**] §II.7.5. ∎

LEMMA 1.17. $\mathrm{Hom}(d_{VW}, Id_W) \circ d_{\mathrm{Hom}(V,W),W} = Id_{\mathrm{Hom}(V,W)}$.

PROOF. For $v \in V$, $K : V \to W$,

$$\begin{aligned}
((\mathrm{Hom}(d_{VW}, Id_W) \circ d_{\mathrm{Hom}(V,W),W})(K))(v) &= (d_{\mathrm{Hom}(V,W),W}(K))(d_{VW}(v)) \\
&= (d_{VW}(v))(K) = K(v).
\end{aligned}$$

∎

In the $W = \mathbb{K}$ case, this one-sided inverse relation gives ([**AF**] §20)

$$d_V^* \circ d_{V^*} = Id_{V^*}.$$

REMARK 1.18. In some applications, the vector space $V$ is often identified with a subspace of $V^{**}$, or the map $d_V$ is ignored, but it is less trouble than might be expected to keep $V$ and $V^{**}$ distinct, and always accounting for $d_V$ turns out to be convenient bookkeeping.

LEMMA 1.19. *Suppose $U$ is finite-dimensional. For $A : U \to V$, if $A^*$ has a linear right inverse $F : U^* \to V^*$, so that $A^* \circ F = Id_{U^*}$, then $A$ has a linear left inverse. If $A^*$ has a linear left inverse $E : U^* \to V^*$, so that $E \circ A^* = Id_{V^*}$, then $A$ has a linear right inverse.*

PROOF. Using Lemma 1.12, case (1.2) of Lemma 1.14, and Claim 1.16,

$$d_U^{-1} \circ F^* \circ d_V \circ A = d_U^{-1} \circ F^* \circ A^{**} \circ d_U = d_U^{-1} \circ (A^* \circ F)^* \circ d_U = Id_U.$$

Using the one-to-one property of $d_V$,

$$\begin{aligned} d_V \circ A \circ d_U^{-1} \circ E^* \circ d_V &= A^{**} \circ E^* \circ d_V = (E \circ A^*)^* \circ d_V = d_V \\ \implies A \circ d_U^{-1} \circ E^* \circ d_V &= Id_V. \end{aligned}$$

∎

DEFINITION 1.20. For any vector space $W$, define $m : W \to \operatorname{Hom}(\mathbb{K}, W)$ so that for $w \in W$,

$$m(w) : \lambda \mapsto \lambda \cdot w.$$

LEMMA 1.21. *For any vector spaces $W$, $W'$, with two maps $m$, $m'$ as indicated, and any maps $\phi \in \mathbb{K}^*$, $G : W \to W'$, the following diagram is commutative.*

$$\begin{tikzcd} W \arrow[r, "m"] \arrow[d, "\phi(1)\cdot G"'] & \operatorname{Hom}(\mathbb{K}, W) \arrow[d, "\operatorname{Hom}(\phi, G)"] \\ W' \arrow[r, "m'"] & \operatorname{Hom}(\mathbb{K}, W'). \end{tikzcd}$$

PROOF. For $\lambda \in \mathbb{K}$, $w \in W$,

$$\begin{aligned} \operatorname{Hom}(\phi, G) \circ m : w &\mapsto G \circ (m(w)) \circ \phi : \lambda \mapsto G(\phi(\lambda) \cdot w), \\ m' \circ (\phi(1) \cdot G) : w &\mapsto m'(\phi(1) \cdot G(w)) : \lambda \mapsto \lambda \cdot \phi(1) \cdot G(w). \end{aligned}$$

∎

LEMMA 1.22. *For any vector space $W$, $m : W \to \operatorname{Hom}(\mathbb{K}, W)$ is invertible.*

PROOF. An inverse is

$$m^{-1} = d_{\mathbb{K}W}(1) : \operatorname{Hom}(\mathbb{K}, W) \to W : A \mapsto A(1).$$

Checking both composites, $((m \circ m^{-1})(A))(\lambda) = \lambda \cdot A(1) = A(\lambda)$, and $(m^{-1} \circ m)(w) = (m(w))(1) = 1 \cdot w = w$. ∎

## 1.2. Tensor products

DEFINITION 1.23. For vector spaces $U$, $V$, $W$, a function $A : U \times V \rightsquigarrow W$ is a bilinear function means: for any $\lambda \in \mathbb{K}$, $u_1$, $u_2 \in U$, $v_1$, $v_2 \in V$,

$$\begin{aligned} A(u_1 + u_2, v_1) &= A(u_1, v_1) + A(u_2, v_1), \\ A(u_1, v_1 + v_2) &= A(u_1, v_1) + A(u_1, v_2), \text{ and} \\ A(\lambda \cdot u_1, v_1) &= A(u_1, \lambda \cdot v_1) = \lambda \cdot A(u_1, v_1). \end{aligned}$$

REMARK 1.24. We remark that the above Definition is different from the notion of bilinear form, from Definition 3.1 in Chapter 3. The way in which these types of functions are related is described in Example 1.55.

For any vector spaces $U$ and $V$, there exists a tensor product vector space $U \otimes V$, which we informally define as the set of formal finite sums

$$\rho_1 \cdot u_1 \otimes v_1 + \rho_2 \cdot u_2 \otimes v_2 + \cdots + \rho_\nu \cdot u_\nu \otimes v_\nu, \tag{1.3}$$

for scalars $\rho_1, \ldots, \rho_\nu \in \mathbb{K}$, elements $u_1, \ldots, u_\nu \in U$, and elements $v_1, \ldots, v_\nu \in V$, with addition and scalar multiplication carried out in the usual way, subject to the relations:

$$\begin{aligned} (u_1 + u_2) \otimes v &= u_1 \otimes v + u_2 \otimes v, \\ u \otimes (v_1 + v_2) &= u \otimes v_1 + u \otimes v_2, \\ (\rho \cdot u) \otimes v &= u \otimes (\rho \cdot v) = \rho \cdot (u \otimes v). \end{aligned}$$

A term $u \otimes v$ is also called a tensor product of the vectors $u$ and $v$. The operation taking a pair of vectors to their tensor product is a bilinear function, denoted

$$\boldsymbol{\tau} : U \times V \rightsquigarrow U \otimes V : (u, v) \mapsto u \otimes v.$$

$U \otimes V$ and $\boldsymbol{\tau}$ have the property that for any bilinear function $A : U \times V \rightsquigarrow W$ as in Definition 1.23, there is a unique linear map $a : U \otimes V \to W$ such that $A = a \circ \boldsymbol{\tau}$, that is, for any $u \in U$ and $v \in V$,

$$A(u, v) = a(u \otimes v). \tag{1.4}$$

So bilinear functions can be converted to linear maps, by replacing the domain $U \times V$ with the tensor product $U \otimes V$.

A more formal proof of the existence of $U \otimes V$ and $\boldsymbol{\tau}$ as above uses methods different from the main stream of this Chapter, so we refer to Appendix 6.3, where (1.4) is stated as Theorem 6.35. The following re-statement of the description (1.3) also follows from the construction of Appendix 6.3:

LEMMA 1.25. *For any $U$, $V$, there exists a space $U \otimes V$ and a bilinear function $\boldsymbol{\tau} : U \times V \rightsquigarrow U \otimes V$ such that $U \otimes V$ is equal to the span of elements of the form $\boldsymbol{\tau}(u, v) = u \otimes v$. Any linear map $B : U \otimes V \to W$ is uniquely determined by its values on the set of elements $u \otimes v$.* ∎

REMARK 1.26. In general, although the set of tensor products of vectors, $\{u \otimes v : u \in U, v \in V\}$ spans the space $U \otimes V$, not every element of $U \otimes V$ is of the form $u \otimes v$ — generally elements are finite sums as in (1.3).

LEMMA 1.27. *If $U$ and $V$ are finite-dimensional, then so is $U \otimes V$.* ∎

EXAMPLE 1.28. The scalar multiplication operation $\cdot_U : \mathbb{K}\times U \rightsquigarrow U$ is a bilinear function, and induces a map

$$l_U : \mathbb{K}\otimes U \to U : \lambda\otimes u \mapsto \lambda\cdot u.$$

The map $l_U$ is invertible, with inverse $l_U^{-1}(u) = 1\otimes u$, and sometimes $l_U$ is abbreviated $l$, with the same $l$ notation used for maps $U\otimes\mathbb{K}\to U : u\otimes\lambda\mapsto\lambda\cdot u$.

EXAMPLE 1.29. The switching function

$$\mathbf{S} : U\times V \rightsquigarrow V\times U : (u,v)\mapsto (v,u)$$

composes with $\boldsymbol{\tau} : V\times U \rightsquigarrow V\otimes U$:

$$\boldsymbol{\tau}\circ\mathbf{S} : U\times V \rightsquigarrow V\otimes U : (u,v)\mapsto v\otimes u.$$

This composite is a bilinear function and induces an invertible, $\mathbb{K}$-linear switching map

$$s : U\otimes V\to V\otimes U : u\otimes v\mapsto v\otimes u.$$

CLAIM 1.30. *The spaces $(U\otimes V)\otimes W$ and $U\otimes(V\otimes W)$ are related by a canonical, invertible, $\mathbb{K}$-linear map.* ■

REMARK 1.31. In addition to the discussion in Appendix 6.3, the above statements about tensor products follow standard constructions as in references [**MB**] §IX.8, [**AF**] §19, [**K**] §II.1. From this point, product sets $U\times V$ and the function $\boldsymbol{\tau}$ will not appear often, but could be used by the reader to verify certain maps are well-defined. Lemma 1.25 will be used frequently and without comment when defining maps (as already done in Examples 1.28 and 1.29) or proving equality of two maps (as in Lemma 1.36, below).

REMARK 1.32. The notion of a multilinear map is also standard, and the spaces from Claim 1.30 can be identified with each other and yet another space, a triple tensor product $U\otimes V\otimes W$. This is spanned by elements of the form $u\otimes v\otimes w$, and it is a convenient abbreviation for our purposes to also identify elements $(u\otimes v)\otimes w = u\otimes(v\otimes w) = u\otimes v\otimes w$. We leave the justification for this to the references; some applications or generalizations of linear algebra keep track of this associativity and do not use such abbreviations, but we will make these identifications without comment. Similarly, notions of associativity for tensor products of more than three spaces or vectors will be implicitly assumed as needed.

A tensor product of $\mathbb{K}$-linear maps canonically induces a $\mathbb{K}$-linear map between tensor product spaces, as follows.

DEFINITION 1.33. For any vector spaces $U_1$, $U_2$, $V_1$, $V_2$, define

$$j : \mathrm{Hom}(U_1,V_1)\otimes\mathrm{Hom}(U_2,V_2)\to\mathrm{Hom}(U_1\otimes U_2, V_1\otimes V_2),$$

so that for $A : U_1\to V_1$, $B : U_2\to V_2$, $u\in U_1$, and $v\in U_2$, the map $j(A\otimes B)$ acts as:

$$(j(A\otimes B)) : u\otimes v\mapsto (A(u))\otimes(B(v)).$$

CLAIM 1.34. *The canonical map $j$ is one-to-one. If one of the ordered pairs $(U_1,U_2)$, $(U_1,V_1)$, or $(U_2,V_2)$ consists of finite-dimensional spaces, then $j$ is invertible.*

PROOF. See [**B**] §II.7.7 or [**K**] §II.2. ■

Although it frequently occurs in the literature that $j(A \otimes B)$ and $A \otimes B$ are identified, we will maintain the distinction. In this Section and later (many times in Section 2.2), the canonical map $j$ will appear in diagrams as a function in its own right, and it is useful to keep track of it everywhere, instead of only ignoring it sometimes. However, the notation $j(A \otimes B)$ is not always as convenient as the following abbreviation.

NOTATION 1.35. Let $[A \otimes B]$ denote $j(A \otimes B)$, so that, for example, the equation

$$(j(A \otimes B))(u \otimes v) = (A(u)) \otimes (B(v))$$

appears as

$$[A \otimes B](u \otimes v) = (A(u)) \otimes (B(v)).$$

LEMMA 1.36. ([B] §II.3.2, [K] §II.6)

$$[A \otimes B] \circ [E \otimes F] = [(A \circ E) \otimes (B \circ F)].$$

∎

Note that the brackets conveniently establish an order of operations, and appear three times in the Lemma, but may stand for three distinct canonical $j$ maps, depending on the domains of $A$, $B$, $E$, and $F$. When it is necessary or convenient to keep track of different maps, the $j$ symbols are used instead of the brackets, and are sometimes labeled with subscripts, primes, etc., as in the following Lemma.

LEMMA 1.37. *For any vector spaces $U_1$, $U_2$, $U_3$, $U_4$, $V_1$, $V_2$, $V_3$, $V_4$, with maps $j'$, $j''$ as indicated, and any maps $A_1 : U_3 \to U_1$, $A_2 : U_4 \to U_2$, $B_1 : V_1 \to V_3$, $B_2 : V_2 \to V_4$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2) \arrow[r, "j'"] \arrow[d, "{[\mathrm{Hom}(A_1,B_1) \otimes \mathrm{Hom}(A_2,B_2)]}"] & \mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2) \arrow[d, "{\mathrm{Hom}([A_1 \otimes A_2],[B_1 \otimes B_2])}"] \\
\mathrm{Hom}(U_3, V_3) \otimes \mathrm{Hom}(U_4, V_4) \arrow[r, "j''"] & \mathrm{Hom}(U_3 \otimes U_4, V_3 \otimes V_4)
\end{tikzcd}$$

PROOF. For $E : U_1 \to V_1$ and $F : U_2 \to V_2$,

$$\begin{aligned}
E \otimes F &\mapsto (j'' \circ [\mathrm{Hom}(A_1, B_1) \otimes \mathrm{Hom}(A_2, B_2)])(E \otimes F) \\
&= j''((\mathrm{Hom}(A_1, B_1)(E)) \otimes (\mathrm{Hom}(A_2, B_2)(F))) \\
&= j''((B_1 \circ E \circ A_1) \otimes (B_2 \circ F \circ A_2)), \\
E \otimes F &\mapsto (\mathrm{Hom}([A_1 \otimes A_2], [B_1 \otimes B_2]) \circ j')(E \otimes F) \\
&= [B_1 \otimes B_2] \circ (j'(E \otimes F)) \circ [A_1 \otimes A_2] \\
&= j''((B_1 \circ E \circ A_1) \otimes (B_2 \circ F \circ A_2)).
\end{aligned}$$

The last step uses Lemma 1.36. ∎

LEMMA 1.38. *For any vector spaces $U$, $W$, and any maps $\phi \in \mathbb{K}^*$, $F \in \mathrm{Hom}(U, W)$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathbb{K} \otimes U \arrow[r, "l_U"] \arrow[d, "{[\phi \otimes F]}"'] & U \arrow[d, "\phi(1) \cdot F"] \\
\mathbb{K} \otimes W \arrow[r, "l_W"] & W
\end{tikzcd}$$

PROOF. For $\lambda \in \mathbb{K}$, $u \in U$,

$$\begin{aligned} l_W \circ [\phi \otimes F] : \lambda \otimes u &\mapsto l_W((\phi(\lambda)) \otimes (F(u))) = \phi(\lambda) \cdot F(u), \\ (\phi(1) \cdot F) \circ l_U : \lambda \otimes u &\mapsto \phi(1) \cdot F(\lambda \cdot u). \end{aligned}$$

∎

LEMMA 1.39. *For any vector spaces $U$, $U'$, $V$, $V'$, any maps $A : U \to U'$, $B : V \to V'$, and switching maps $s$ and $s'$, the following diagram is commutative.*

$$\begin{tikzcd} U \otimes V \arrow[r, "s"] \arrow[d, "{[A\otimes B]}"'] & V \otimes U \arrow[d, "{[B\otimes A]}"] \\ U' \otimes V' \arrow[r, "s'"] & V' \otimes U' \end{tikzcd}$$

PROOF. For $u \in U$, $v \in V$,

$$([B \otimes A] \circ s)(u \otimes v) = (B(v)) \otimes (A(u)) = (s' \circ [A \otimes B])(u \otimes v).$$

∎

DEFINITION 1.40. For arbitrary vector spaces $U$, $V$, $W$, define

$$n : \mathrm{Hom}(U, V) \otimes W \to \mathrm{Hom}(U, V \otimes W)$$

so that for $A : U \to V$, $w \in W$, $u \in U$,

$$n(A \otimes w) : u \mapsto (A(u)) \otimes w.$$

NOTATION 1.41. The ordering of the spaces from Definition 1.40 is not canonical; the "$n$" label (with various subscripts) is used for analogously defined maps:

$$\begin{aligned} n_0 : \mathrm{Hom}(U, V) \otimes W &\to \mathrm{Hom}(U, V \otimes W) : A \otimes w : (u \mapsto (A(u)) \otimes w) \\ n_1 : \mathrm{Hom}(U, V) \otimes W &\to \mathrm{Hom}(U, W \otimes V) : A \otimes w : (u \mapsto w \otimes (A(u))) \\ n_2 : W \otimes \mathrm{Hom}(U, V) &\to \mathrm{Hom}(U, V \otimes W) : w \otimes A : (u \mapsto (A(u)) \otimes w) \\ n_3 : W \otimes \mathrm{Hom}(U, V) &\to \mathrm{Hom}(U, W \otimes V) : w \otimes A : (u \mapsto w \otimes (A(u))). \end{aligned}$$

For map with the $n$ label appearing in a diagram or equation, its type from the above list of four formulas can usually be determined by context. The above variants are related to each other by compositions with switching maps.

LEMMA 1.42. *For any vector spaces $U$, $U'$, $V$, $V'$, $W$, $W'$, with maps $n$, $n'$ as indicated, and any maps $F : U' \to U$, $B : V \to V'$, $C : W \to W'$, the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Hom}(U, V) \otimes W \arrow[r, "n"] \arrow[d, "{[\mathrm{Hom}(F,B)\otimes C]}"'] & \mathrm{Hom}(U, V \otimes W) \arrow[d, "{\mathrm{Hom}(F,[B\otimes C])}"] \\ \mathrm{Hom}(U', V') \otimes W' \arrow[r, "n'"] & \mathrm{Hom}(U', V' \otimes W') \end{tikzcd}$$

PROOF. For $A : U \to V$, $w \in W$, $u' \in U'$,

$$\begin{aligned} A \otimes w &\mapsto (\mathrm{Hom}(F, [B \otimes C]) \circ n)(A \otimes w) = [B \otimes C] \circ (n(A \otimes w)) \circ F : \\ u' &\mapsto [B \otimes C]((A(F(u'))) \otimes w) = ((B \circ A \circ F)(u')) \otimes (C(w)), \\ A \otimes w &\mapsto (n' \circ [\mathrm{Hom}(F, B) \otimes C])(A \otimes w) = n'((B \circ A \circ F) \otimes (C(w))) : \\ u' &\mapsto ((B \circ A \circ F)(u')) \otimes (C(w)). \end{aligned}$$ ∎

The $n$ map is related to a canonical $j$ map.

LEMMA 1.43. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U,V)\otimes W \arrow[r, "n"] \arrow[d, "{[Id_{\mathrm{Hom}(U,V)}\otimes m]}"'] & \mathrm{Hom}(U,V\otimes W) \arrow[d, "{\mathrm{Hom}(l_U, Id_{V\otimes W})}"] \\
\mathrm{Hom}(U,V)\otimes \mathrm{Hom}(\mathbb{K},W) \arrow[r, "j"] & \mathrm{Hom}(U\otimes\mathbb{K}, V\otimes W)
\end{tikzcd}$$

PROOF. Starting with $A\otimes w\in \mathrm{Hom}(U,V)\otimes W$,

$$\begin{aligned}
A\otimes w &\mapsto (j\circ[Id_{\mathrm{Hom}(U,V)}\otimes m])(A\otimes w) = j(A\otimes(m(w))):\\
u\otimes\lambda &\mapsto (A(u))\otimes(\lambda\cdot w) = \lambda\cdot(A(u))\otimes w,\\
A\otimes w &\mapsto (\mathrm{Hom}(l_U, Id_{V\otimes W})\circ n)(A\otimes w) = (n(A\otimes w))\circ l_U:\\
u\otimes\lambda &\mapsto (A(\lambda\cdot u))\otimes w = \lambda\cdot(A(u))\otimes w.
\end{aligned}$$

∎

LEMMA 1.44. *The canonical map $n:\mathrm{Hom}(U,V)\otimes W\to \mathrm{Hom}(U,V\otimes W)$ is one-to-one, and if $U$ or $W$ is finite-dimensional, then $n$ is invertible.*

PROOF. This follows from Lemma 1.22, Claim 1.34, Lemma 1.43, and the invertibility of the $l_U$ map as in Example 1.28. See also [B] §II.7.7 or [AF] §20. ∎

Lemma 1.42, Lemma 1.43, and Lemma 1.44 all generalize in straightforward ways to re-ordered variants of $n$ maps as in Notation 1.41.

LEMMA 1.45. *For any $U$, $V$, $W$, $W'$, and $n$ maps as indicated, the following diagram is commutative.*

$$\begin{tikzcd}
W\otimes\mathrm{Hom}(U,V)\otimes W' \arrow[r, "{[n_3\otimes Id_{W'}]}"] \arrow[d, "{[Id_W\otimes n_0]}"] & \mathrm{Hom}(U,W\otimes V)\otimes W' \arrow[d, "n_0'"] \\
W\otimes\mathrm{Hom}(U,V\otimes W') \arrow[r, "n_3'"] & \mathrm{Hom}(U,W\otimes V\otimes W')
\end{tikzcd}$$

PROOF. For $A\in\mathrm{Hom}(U,V)$, $w\in W$, $w'\in W'$, $u\in U$,

$$\begin{aligned}
n_0'\circ[n_3\otimes Id_{W'}]: w\otimes A\otimes w' &\mapsto n_0'((n_3(w\otimes A))\otimes w'):\\
u &\mapsto (n_3(w\otimes A))(u)\otimes w' = w\otimes(A(u))\otimes w',\\
n_3'\circ[Id_W\otimes n_0]: w\otimes A\otimes w' &\mapsto n_3'(w\otimes(n_0(A\otimes w'))):\\
u &\mapsto w\otimes(n_0(A\otimes w'))(u) = w\otimes(A(u))\otimes w'.
\end{aligned}$$

∎

DEFINITION 1.46. For arbitrary vector spaces $U$, $V$, $W$ define

$$q:\mathrm{Hom}(V,\mathrm{Hom}(U,W))\to\mathrm{Hom}(V\otimes U,W)$$

so that for $K:V\to\mathrm{Hom}(U,W)$, $v\in V$, $u\in U$,

$$(q(K))(v\otimes u) = (K(v))(u).$$

LEMMA 1.47. *For any vector spaces $U$, $V$, $W$, $q$ is invertible.*

PROOF. For $D \in \mathrm{Hom}(V \otimes U, W)$, check that

$$q^{-1}(D) : v \mapsto (u \mapsto D(v \otimes u))$$

defines an inverse. See also [**AF**] §19, [**B**] §II.4.1, [**MB**] §IX.11, or [**K**] §II.1. ∎

REMARK 1.48. In some applications, the map $q^{-1}$ is called a "currying" transformation, and so $q$ is an "uncurrying" map.

NOTATION 1.49. In the same way as Notation 1.41, there are different orderings of the spaces from Definition 1.46, with the "$q$" label being used in either of these two cases:

$$\begin{aligned} q_1 : \mathrm{Hom}(V, \mathrm{Hom}(U, W)) &\to \mathrm{Hom}(V \otimes U, W) : K \mapsto (v \otimes u \mapsto (K(v))(u)), \\ q_2 : \mathrm{Hom}(V, \mathrm{Hom}(U, W)) &\to \mathrm{Hom}(U \otimes V, W) : K \mapsto (u \otimes v \mapsto (K(v))(u)). \end{aligned} \tag{1.5}$$

For map with a $q$ label appearing in a diagram or equation, its type from the above list can usually be determined by context. The above variants are related to each other by composition; for a switching map $s : U \otimes V \to V \otimes U$,

$$q_2 = \mathrm{Hom}(s, Id_W) \circ q_1. \tag{1.6}$$

Both maps from (1.5) are invertible as in Lemma 1.47 and satisfy suitably re-ordered versions of Lemma 1.50, Lemma 1.51, and Lemma 1.52.

LEMMA 1.50. ([**AF**] §20) *For any vector spaces $U_1$, $V_1$, $W_1$, $U_2$, $V_2$, $W_2$, with maps $q_1$, $q_2$ as indicated, and any maps $D : V_2 \to V_1$, $E : U_2 \to U_1$, $F : W_1 \to W_2$, the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Hom}(V_1, \mathrm{Hom}(U_1, W_1)) \arrow[r, "q_1"] \arrow[d, "\mathrm{Hom}(D, \mathrm{Hom}(E,F))"'] & \mathrm{Hom}(U_1 \otimes V_1, W_1) \arrow[d, "\mathrm{Hom}({[E \otimes D]}, F)"] \\ \mathrm{Hom}(V_2, \mathrm{Hom}(U_2, W_2)) \arrow[r, "q_2"] & \mathrm{Hom}(U_2 \otimes V_2, W_2) \end{tikzcd}$$

PROOF. Starting with any $G : V_1 \to \mathrm{Hom}(U_1, W_1)$,

$$\begin{aligned} (\mathrm{Hom}([E \otimes D], F) \circ q_1)(G) : u \otimes v &\mapsto (F \circ (q_1(G)) \circ [E \otimes D])(u \otimes v) \\ &= F((G(D(v)))(E(u))), \\ (q_2 \circ \mathrm{Hom}(D, \mathrm{Hom}(E, F)))(G) : u \otimes v &\mapsto (q_2(\mathrm{Hom}(E, F) \circ G \circ D))(u \otimes v) \\ &= ((\mathrm{Hom}(E, F) \circ G \circ D)(v))(u) \\ &= (F \circ (G(D(v))) \circ E)(u) \\ &= F((G(D(v)))(E(u))). \end{aligned}$$

∎

LEMMA 1.51. *The following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, U))) \arrow[r, "q_1"] \arrow[d, "\mathrm{Hom}(Id_X, q_3)"'] & \mathrm{Hom}(X \otimes Y, \mathrm{Hom}(Z, U)) \arrow[d, "q_2"] \\ \mathrm{Hom}(X, \mathrm{Hom}(Y \otimes Z, U)) \arrow[r, "q_4"] & \mathrm{Hom}(X \otimes Y \otimes Z, U) \end{tikzcd}$$

PROOF. For $G \in \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, U)))$, $x \in X$, $y \in Y$, $z \in Z$,

$$\begin{aligned}
(q_2 \circ q_1)(G) : x \otimes y \otimes z &\mapsto (q_2(q_1(G)))(x \otimes y \otimes z) \\
&= ((q_1(G))(x \otimes y))(z) \\
&= ((G(x))(y))(z), \\
(q_4 \circ \mathrm{Hom}(Id_X, q_3))(G) : x \otimes y \otimes z &\mapsto (q_4(q_3 \circ G))(x \otimes y \otimes z) \\
&= ((q_3 \circ G)(x))(y \otimes z) = (q_3(G(x)))(y \otimes z) \\
&= ((G(x))(y))(z). \quad \blacksquare
\end{aligned}$$

LEMMA 1.52. *For any vector spaces $U_1$, $U_2$, $V_1$, $V_2$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2) \arrow[r, "n"'] \arrow[d, "j"'] & \mathrm{Hom}(U_1, V_1 \otimes \mathrm{Hom}(U_2, V_2)) \arrow[d, "\mathrm{Hom}(Id_{U_1}, n')"'] \\
\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2) & \mathrm{Hom}(U_1, \mathrm{Hom}(U_2, V_1 \otimes V_2)) \arrow[l, "q"]
\end{tikzcd}$$

PROOF. For $A \in \mathrm{Hom}(U_1, V_1)$, $B \in \mathrm{Hom}(U_2, V_2)$, $u \in U_1$, $v \in U_2$,

$$\begin{aligned}
(q \circ \mathrm{Hom}(Id_{U_1}, n') \circ n) : A \otimes B &\mapsto q(n' \circ (n(A \otimes B))) : \\
u \otimes v &\mapsto ((n' \circ (n(A \otimes B)))(u))(v) \\
&= (n'((A(u)) \otimes B))(v) = (A(u)) \otimes (B(v)) \\
&= (j(A \otimes B))(u \otimes v).
\end{aligned}$$

$\blacksquare$

EXAMPLE 1.53. The generalized transpose map from Definition 1.7 is a distinguished element in the following vector space:

$$t_{UV}^W \in \mathrm{Hom}(\mathrm{Hom}(U, V), \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(U, W))),$$

and its image under this $q$ map,

$$\begin{aligned}
q \quad : \quad & \mathrm{Hom}(\mathrm{Hom}(U, V), \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(U, W))) \\
\to \quad & \mathrm{Hom}(\mathrm{Hom}(U, V) \otimes \mathrm{Hom}(V, W), \mathrm{Hom}(U, W))
\end{aligned}$$

is the following map:

$$\begin{aligned}
q(t_{UV}^W) : \mathrm{Hom}(U, V) \otimes \mathrm{Hom}(V, W) &\to \mathrm{Hom}(U, W) \\
A \otimes B &\mapsto (q(t_{UV}^W))(A \otimes B) = (t_{UV}^W(A))(B) \\
&= \mathrm{Hom}(A, Id_W)(B) = B \circ A. \qquad (1.7)
\end{aligned}$$

The operation of composition of linear maps is a bilinear function, as in Definition 1.23:

$$\begin{aligned}
\circ : \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, W) &\rightsquigarrow \mathrm{Hom}(U, W) \\
(A, B) &\mapsto B \circ A. \qquad (1.8)
\end{aligned}$$

The agreement of (1.7) and (1.8) shows that $q(t_{UV}^W)$ is the unique linear map corresponding to composition as a bilinear function as in (1.4) and Theorem 6.35:

$$\circ = q(t_{UV}^W) \circ \boldsymbol{\tau}.$$

Remark 1.54. The conclusion of Example 1.53 is that the generalized transpose map is a linear, curried version of composition. Considering

$$q(t_{UV}^W) \in \mathrm{Hom}(\mathrm{Hom}(U,V) \otimes \mathrm{Hom}(V,W), \mathrm{Hom}(U,W))$$

as a distinguished element in a vector space has an analogue in matrix algebra (see, for example, [**CHL**]), where matrix multiplication can be viewed as an element of a space of tensors.

Example 1.55. Let $\mathbf{B} : V \times V \rightsquigarrow \mathbb{K}$ be a bilinear function as in Definition 1.23, so that there exists a unique linear map $B : V \otimes V \to \mathbb{K}$ satisfying $B \circ \boldsymbol{\tau} = \mathbf{B}$ as in (1.4). The invertible map $q^{-1} : \mathrm{Hom}(V \otimes V, \mathbb{K}) \to \mathrm{Hom}(V, \mathrm{Hom}(V, \mathbb{K}))$ transforms $B \in (V \otimes V)^*$ to $q^{-1}(B) \in \mathrm{Hom}(V, V^*)$. So, every bilinear function $\mathbf{B} : V \times V \rightsquigarrow \mathbb{K}$ has a linearized, curried form which is an element of $\mathrm{Hom}(V, V^*)$, called a bilinear form — such maps are the main topic of Chapter 3.

Definition 1.56. For any vector spaces $U$, $V$, $W$ define

$$e_{UV}^W : \mathrm{Hom}(U,V) \to \mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W)$$

so that for $A : U \to V$, $B : V \to W$, and $u \in U$,

$$e_{UV}^W(A) : B \otimes u \mapsto B(A(u)) \in W.$$

Lemma 1.57. *For any vector spaces $U$, $V$, $W$, $U'$, $V'$, $W'$, and any maps $E : U' \to U$, $F : V \to V'$, $G : W \to W'$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U,V) \arrow[r, "e_{UV}^W"] \arrow[dd, "\mathrm{Hom}(E,F)"'] & \mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W) \arrow[d, "\mathrm{Hom}([\mathrm{Hom}(F,Id_W)\otimes E],G)"] \\
& \mathrm{Hom}(\mathrm{Hom}(V',W) \otimes U', W') \\
\mathrm{Hom}(U',V') \arrow[r, "e_{U'V'}^{W'}"] & \mathrm{Hom}(\mathrm{Hom}(V',W') \otimes U', W') \arrow[u, "\mathrm{Hom}([\mathrm{Hom}(Id_{V'},G)\otimes Id_{U'}],Id_{W'})"']
\end{tikzcd}$$

Proof. For any $A : U \to V$, $C : V' \to W$, $u \in U'$,

$$\begin{array}{rcl}
& & \mathrm{Hom}([\mathrm{Hom}(Id_{V'}, G) \otimes Id_{U'}], Id_{W'}) \circ e_{U'V'}^{W'} \circ \mathrm{Hom}(E,F) : \\
A & \mapsto & (e_{U'V'}^{W'}(F \circ A \circ E)) \circ [\mathrm{Hom}(Id_{V'}, G) \otimes Id_{U'}] : \\
C \otimes u & \mapsto & (e_{U'V'}^{W'}(F \circ A \circ E))((G \circ C) \otimes u) \\
& = & (G \circ C)((F \circ A \circ E)(u)), \\
& & \mathrm{Hom}([\mathrm{Hom}(F, Id_W) \otimes E], G) \circ e_{UV}^W : \\
A & \mapsto & G \circ (e_{UV}^W(A)) \circ [\mathrm{Hom}(F, Id_W) \otimes E] : \\
C \otimes u & \mapsto & G((e_{UV}^W(A))((C \circ F) \otimes (E(u)))) \\
& = & G((C \circ F)(A(E(u)))). \quad \blacksquare
\end{array}$$

Lemma 1.58. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(U,V) \arrow[rr, "e_{UV}^W"] \arrow[d, "t_{UV}^W"'] & & \mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W) \\
\mathrm{Hom}(\mathrm{Hom}(V,W), \mathrm{Hom}(U,W)) \arrow[urr, "q"] & &
\end{tikzcd}$$

PROOF.

$$q \circ t_{UV}^{W} : A \mapsto q(\mathrm{Hom}(A, Id_W)) : B \otimes u \mapsto (B \circ A)(u) = (e_{UV}^{W}(A))(B \otimes u).$$

∎

NOTATION 1.59. For the special case $W = \mathbb{K}$, abbreviate $e_{UV}^{\mathbb{K}} = e_{UV}$ (or sometimes just $e$):

$$e_{UV} : \mathrm{Hom}(U, V) \to (V^* \otimes U)^*$$

so that for $A : U \to V$, $\phi \in V^*$, and $u \in U$,

$$e_{UV}(A) : \phi \otimes u \mapsto \phi(A(u)) \in \mathbb{K}.$$

LEMMA 1.60. *If $V$ is finite-dimensional, then $e_{UV}$ is invertible.*

PROOF. This follows from Lemma 1.58, Claim 1.11, and Lemma 1.47. ∎

DEFINITION 1.61. For any vector spaces $U$, $V$, define

$$k_{UV} : U^* \otimes V \to \mathrm{Hom}(U, V)$$

so that for $\xi \in U^*$, $v \in V$, and $u \in U$,

$$(k_{UV}(\xi \otimes v)) : u \mapsto \xi(u) \cdot v \in V.$$

LEMMA 1.62. *For maps $A : U' \to U$, $B : V \to V'$, the following diagram is commutative.*

$$\begin{tikzcd} U^* \otimes V \arrow[r, "k_{UV}"] \arrow[d, "{[A^* \otimes B]}"'] & \mathrm{Hom}(U, V) \arrow[d, "{\mathrm{Hom}(A,B)}"] \\ U'^* \otimes V' \arrow[r, "k_{U'V'}"] & \mathrm{Hom}(U', V') \end{tikzcd}$$

PROOF. For $\phi \otimes v \in U^* \otimes V$, $u \in U'$,

$$\begin{array}{rcl} (\mathrm{Hom}(A, B) \circ k_{UV})(\phi \otimes v) & = & B \circ (k_{UV}(\phi \otimes v)) \circ A : \\ u & \mapsto & B(\phi(A(u)) \cdot v) = \phi(A(u)) \cdot B(v), \\ (k_{U'V'} \circ [A^* \otimes B])(\phi \otimes v) & = & k_{U'V'}((A^*(\phi)) \otimes (B(v))) : \\ u & \mapsto & (A^*(\phi))(u) \cdot B(v) = \phi(A(u)) \cdot B(v). \end{array}$$

∎

LEMMA 1.63. *For any vector space $V$, $k_{V\mathbb{K}} = l_{V^*} : V^* \otimes \mathbb{K} \to V^*$.*

PROOF. For $v \in V$, $\varphi \in V^*$, $\lambda \in \mathbb{K}$,

$$\begin{array}{rcl} k_{V\mathbb{K}} : \varphi \otimes \lambda & \mapsto & k_{V\mathbb{K}}(\varphi \otimes \lambda) : v \mapsto \varphi(v) \cdot \lambda, \\ l_{V^*} : \varphi \otimes \lambda & \mapsto & \lambda \cdot \varphi : v \mapsto (\lambda \cdot \varphi)(v). \end{array}$$

∎

Lemma 1.64. *The canonical map $k_{UV}$ is one-to-one, and if $U$ or $V$ is finite-dimensional then $k_{UV}$ is invertible.*

Proof. The $k_{UV}$ map (sometimes abbreviated $k$) is related to a canonical $n$ map. The following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(U,\mathbb{K})\otimes V \arrow[r, "n"] \arrow[dr, "k_{UV}"'] & \mathrm{Hom}(U,\mathbb{K}\otimes V) \arrow[d, "\mathrm{Hom}(Id_U,l_V)"] \\
& \mathrm{Hom}(U,V)
\end{tikzcd}$$

$$l_V\circ(n(\phi\otimes v)) : u\mapsto \phi(u)\cdot v = (k_{UV}(\phi\otimes v))(u).$$

The $n$ map is one-to-one, and, if $U$ or $V$ is finite-dimensional, then $n$ is invertible by Lemma 1.44, which used Claim 1.34. The $l_V$ map is invertible as in Example 1.28. See also [B] §II.7.7 or [K] §II.2. ∎

Lemma 1.65. *For any vector spaces $U$, $V$, $W$, this diagram is commutative*

$$\begin{tikzcd}
& \mathrm{Hom}(U,V)\otimes W \arrow[d, "n"] \\
U^*\otimes V\otimes W \arrow[ur, "{[k_{UV}\otimes Id_W]}"] \arrow[r, "k_{U,V\otimes W}"'] & \mathrm{Hom}(U,V\otimes W)
\end{tikzcd}$$

Proof. For $\phi\otimes v\otimes w\in U^*\otimes V\otimes W$ and $u\in U$,

$$\begin{array}{rcl}
\phi\otimes v\otimes w & \mapsto & n((k_{UV}(\phi\otimes v))\otimes w) : u\mapsto(\phi(u)\cdot v)\otimes w \\
\phi\otimes v\otimes w & \mapsto & k_{V,V\otimes W}(\phi\otimes v\otimes w) : u\mapsto \phi(u)\cdot(v\otimes w).
\end{array}$$

∎

The diagram from Lemma 1.65 can be modified by replacing the $n$ map with a re-ordered variant as in Notation 1.41, or adding one or more arrows to compose with switching maps as in Example 1.29, to get analogous statements about commutativity by a similar calculation.

Remark 1.66. Unlike the canonical maps $t^W_{UV}$, $d_{VW}$, $s$, $j$, $n$, $q$, and $e^W_{UV}$, the $k$ maps explicitly refer to the set of scalars $\mathbb{K}$, in both the dual space $U^*$ and the scalar multiplication $\cdot$ in $V$. The maps $l$ and $m$ also refer to scalar multiplication.

Remark 1.67. The canonical maps appearing in this Chapter are well-known in abstract linear algebra. Lemma 1.8, Lemma 1.14, Lemma 1.21, Lemma 1.37, Lemma 1.38, Lemma 1.39, Lemma 1.42, Lemma 1.50, Lemma 1.57, and Lemma 1.62 can be interpreted as statements about the naturality of the $t$, $d$, $m$, $j$, $l$, $s$, $n$, $q$, $e$, and $k$ maps, in a technical sense of category theory. In geometry, these same lemmas also are enough to show that these maps transform in the right way under (pointwise linear, invertible) changes from one local trivialization to another in a vector bundle, so that these basis-free constructions on vector spaces extend to well-defined maps of vector bundles.

Remark 1.68. These canonical maps also appear in concrete matrix algebra and applications: see [**Magnus**], [**G$_2$**] (particularly §I.8 and §VI.3), [**Graham**]; for historical references, see [**HS**] and [**HJ**] (Chapter 4). For example, the map $k^{-1}_{UV}:\mathrm{Hom}(U,V)\to U^*\otimes V$ is a "vectorization," or "*vec*" map. The $t_{UV}$ map, of course, is analogous to the transpose operation ($A \mapsto A'$, in the notation of [HS] and [Magnus]) and an analogue of Lemma 1.62 is the equality:

$$vec(ABC) = (C' \otimes A)vecB,$$

attributed by [HS] to [R], see also [Nissen], [HJ], [Magnus] (§1.10).

NOTATION 1.69. The composite of the $e$ and $k$ maps is denoted

$$f_{UV} = e_{UV} \circ k_{UV} : U^* \otimes V \to (V^* \otimes U)^*.$$

Sometimes $f_{UV}$ is abbreviated $f$.

The output $f_{UV}(\phi \otimes v)$ acts on $\xi \otimes u$ to give $\phi(u) \cdot \xi(v) \in \mathbb{K}$:

$$(e_{UV}(k_{UV}(\phi \otimes v)))(\xi \otimes u) = \xi((k_{UV}(\phi \otimes v))(u)) = \xi(\phi(u) \cdot v) = \phi(u) \cdot \xi(v).$$

EXERCISE 1.70. For maps $A : U' \to U$, $B : V \to V'$, the following diagram is commutative.

$$\begin{tikzcd}
U^* \otimes V \arrow[r, "f_{UV}"] \arrow[d, "{[A^* \otimes B]}"'] & (V^* \otimes U)^* \arrow[d, "{[B^* \otimes A]^*}"] \\
U'^* \otimes V' \arrow[r, "f_{U'V'}"] & (V'^* \otimes U')^*
\end{tikzcd}$$

HINT. This can be checked directly as in the Proof of Lemma 1.62; it also follows as a corollary of Lemma 1.57 and Lemma 1.62. ∎

LEMMA 1.71. $f_{UV}^* \circ d_{V^* \otimes U} = f_{VU} : V^* \otimes U \to (U^* \otimes V)^*$.

PROOF. For $\xi \otimes u \in V^* \otimes U$, and $\phi \otimes v \in U^* \otimes V$,

$$\begin{aligned}
(f_{UV}^*(d_{V^* \otimes U}(\xi \otimes u)))(\phi \otimes v) &= (d_{V^* \otimes U}(\xi \otimes u))(f_{UV}(\phi \otimes v)) \\
&= (f_{UV}(\phi \otimes v))(\xi \otimes u) \\
&= \phi(u) \cdot \xi(v) \\
&= (f_{VU}(\xi \otimes u))(\phi \otimes v).
\end{aligned}$$

∎

NOTATION 1.72. For any vector spaces $U$, $V$, the following composite is denoted:

$$p_{UV} = [d_V \otimes Id_{U^*}] \circ s : U^* \otimes V \to V^{**} \otimes U^*.$$

So, for $\phi \in U^*$, $v \in V$, $p_{UV}(\phi \otimes v) = (d_V(v)) \otimes \phi$. Sometimes $p_{UV}$ is abbreviated $p$.

LEMMA 1.73. *For maps $B : V \to V'$, $C : U^* \to U'^*$, the following diagram is commutative.*

$$\begin{tikzcd}
U^* \otimes V \arrow[r, "p_{UV}"] \arrow[d, "{[C \otimes B]}"'] & V^{**} \otimes U^* \arrow[d, "{[B^{**} \otimes C]}"] \\
U'^* \otimes V' \arrow[r, "p_{U'V'}"] & V'^{**} \otimes U'^*
\end{tikzcd}$$

HINT. This can be checked using case (1.2) of Lemma 1.14. ∎

LEMMA 1.74. *The following diagram is commutative. If $V$ is finite-dimensional then all the maps are invertible.*

$$\begin{tikzcd}
U^*\otimes V \arrow[r, "f_{UV}"] \arrow[d, "p_{UV}"] & (V^*\otimes U)^* \\
V^{**}\otimes U^* \arrow[r, "j"] & \mathrm{Hom}(V^*\otimes U, \mathbb{K}\otimes\mathbb{K}) \arrow[u, "\mathrm{Hom}(Id_{V^*\otimes U}, l)"']
\end{tikzcd}$$

PROOF.

$$\begin{array}{rcl}
\phi\otimes v & \mapsto & (\mathrm{Hom}(Id_{V^*\otimes U}, l)\circ j\circ p_{UV})(\phi\otimes v) = l\circ[(d_V(v))\otimes\phi]: \\
\xi\otimes u & \mapsto & \xi(v)\cdot\phi(u) \\
 & = & (f(\phi\otimes v))(\xi\otimes u).
\end{array}$$

■

LEMMA 1.75. *The following diagram is commutative. If $V$ is finite-dimensional then all the maps are invertible.*

$$\begin{tikzcd}
U^*\otimes V \arrow[r, "k_{UV}"] \arrow[d, "p_{UV}"'] & \mathrm{Hom}(U,V) \arrow[d, "t_{UV}"] \\
V^{**}\otimes U^* \arrow[r, "k_{V^*U^*}"] & \mathrm{Hom}(V^*, U^*)
\end{tikzcd}$$

PROOF.

$$\begin{array}{rcl}
\phi\otimes v & \mapsto & (k_{V^*U^*}\circ p_{UV})(\phi\otimes v) = k_{V^*U^*}((d_V(v))\otimes\phi): \\
\xi & \mapsto & (k_{V^*U^*}((d_V(v))\otimes\phi))(\xi) = (d_V(v))(\xi)\cdot\phi = \xi(v)\cdot\phi: \\
u & \mapsto & \xi(v)\cdot\phi(u), \\
\phi\otimes v & \mapsto & (t_{UV}\circ k_{UV})(\phi\otimes v): \\
\xi & \mapsto & (t_{UV}(k_{UV}(\phi\otimes v)))(\xi) = \xi\circ(k_{UV}(\phi\otimes v)): \\
u & \mapsto & \xi((k_{UV}(\phi\otimes v))(u)) = \xi(\phi(u)\cdot v) = \phi(u)\cdot\xi(v).
\end{array}$$

The diagram shows how Claim 1.11, Claim 1.16, and Lemma 1.64 are related. ■

REMARK 1.76. The map $p$ corresponds to another well-known object in matrix algebra, denoted the "vec-permutation matrix" $\mathbf{I}_{m,n}$ by [**HS**]. The equality $vecA = \mathbf{I}vecA'$ (in the notation from Remark 1.68) corresponds to the equality $k\circ p = t\circ k$ from Lemma 1.75, and [**HS**] states a matrix analogue of Lemma 1.73. This matrix has also been called the "commutation matrix" $K_{mn}$ by [**Magnus**] (§3.1) (with the equivalent property $KvecA = vecA'$) or the "shuffle matrix" by [**L**], [**HJ**].

## 1.3. Direct sums

The following definition applies to any integer $\nu \geq 2$ (see also [**AF**] §6).

DEFINITION 1.77. Given vector spaces $V$, $V_1$, $V_2$, ..., $V_\nu$ and ordered $\nu$-tuples of maps $(P_1, P_2, \dots, P_\nu)$ and $(Q_1, Q_2, \dots, Q_\nu)$, where $P_i : V \to V_i$, $Q_i : V_i \to V$ for $i = 1, 2, \dots, \nu$, $V$ is a direct sum of $(V_1, V_2, \dots, V_\nu)$ means:

$$Q_1 \circ P_1 + Q_2 \circ P_2 + \dots + Q_\nu \circ P_\nu = Id_V$$

and

$$P_i \circ Q_I = \begin{cases} Id_{V_i} & \text{if } i = I \\ 0_{\mathrm{Hom}(V_I, V_i)} & \text{if } i \neq I \end{cases} .$$

This data is sometimes abbreviated $V = V_1 \oplus V_2 \oplus \dots \oplus V_\nu$, when the maps $P_i$ (called projections) and $Q_i$ (inclusions) are understood. Note that the ordering is part of the notation and that each map $Q_i$ is one-to-one (having a left inverse as in Exercise 0.48) and that if $V$ is finite-dimensional then each $V_i$ is also finite-dimensional (by Exercise 0.50 — this fact may not be mentioned at every subsequent occurrence).

We will most frequently consider direct sums of two spaces; in this $\nu = 2$ case, the above Definition requires that five equations are satisfied by $(P_1, P_2)$, $(Q_1, Q_2)$, but it is enough to check only three of these equations.

LEMMA 1.78. *Given vector spaces $V$, $V_1$, $V_2$, if there exist $P_1 : V \to V_1$, $P_2 : V \to V_2$, $Q_1 : V_1 \to V$, $Q_2 : V_2 \to V$ such that:*

$$\begin{aligned} Q_1 \circ P_1 + Q_2 \circ P_2 &= Id_V \\ P_1 \circ Q_1 &= Id_{V_1} \\ P_2 \circ Q_2 &= Id_{V_2} \end{aligned}$$

*then $V = V_1 \oplus V_2$.*

PROOF. The conclusion is that the pairs $(P_1, P_2)$ and $(Q_1, Q_2)$ satisfy Definition 1.77, and the only two equations remaining to be checked are $P_2 \circ Q_1 = 0_{\mathrm{Hom}(V_1, V_2)}$ and $P_1 \circ Q_2 = 0_{\mathrm{Hom}(V_2, V_1)}$.

$$\begin{aligned} P_2 \circ Q_1 &= P_2 \circ Id_V \circ Q_1 = P_2 \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \circ Q_1 \\ &= P_2 \circ Q_1 \circ P_1 \circ Q_1 + P_2 \circ Q_2 \circ P_2 \circ Q_1 \\ &= P_2 \circ Q_1 \circ Id_{V_1} + Id_{V_2} \circ P_2 \circ Q_1 \\ &= P_2 \circ Q_1 + P_2 \circ Q_1 \\ &= 0_{\mathrm{Hom}(V_1, V_2)}, \end{aligned}$$

the last step using Theorem 0.3. The other equation is similarly checked. ∎

THEOREM 1.79. *Given vector spaces $V$, $V_1$, $V_2$, $V_3$, $V_4$, if $V = V_1 \oplus V_2$ and $V_2 = V_3 \oplus V_4$, then $V = V_1 \oplus V_3 \oplus V_4$.*

PROOF. Let $(P_1, P_2)$, $(Q_1, Q_2)$ be the projections and inclusions for $V = V_1 \oplus V_2$, and let $(P_3, P_4)$, $(Q_3, Q_4)$ be the direct sum data for $V_2 = V_3 \oplus V_4$. The projections $(P_1, P_3 \circ P_2, P_4 \circ P_2)$ and inclusions $(Q_1, Q_2 \circ Q_3, Q_2 \circ Q_4)$ give a canonical construction for the claimed direct sum. The first equation from Definition 1.77 is:

$$\begin{aligned} & Q_1 \circ P_1 + (Q_2 \circ Q_3) \circ (P_3 \circ P_2) + (Q_2 \circ Q_4) \circ (P_4 \circ P_2) \\ = \quad & Q_1 \circ P_1 + Q_2 \circ (Q_3 \circ P_3 + Q_4 \circ P_4) \circ P_2 \\ = \quad & Q_1 \circ P_1 + Q_2 \circ Id_{V_2} \circ P_2 \\ = \quad & Id_V. \end{aligned}$$

The remaining nine equations are also easily checked. ∎

EXAMPLE 1.80. If $H : U \to V$ is an invertible map between arbitrary vector spaces, and $V = V_1 \oplus V_2$, then $U$ is a direct sum of $V_1$ and $V_2$, with projections $P_i \circ H : U \to V_i$ for $i = 1, 2$, and inclusions $H^{-1} \circ Q_i : V_i \to U$.

EXAMPLE 1.81. If $V = V_1 \oplus V_2$, and $U$ is any vector space, then $V \otimes U$ is a direct sum of $V_1 \otimes U$ and $V_2 \otimes U$. The projections and inclusions are $[P_i \otimes Id_U] : V \otimes U \to V_i \otimes U$, and $[Q_i \otimes Id_U] : V_i \otimes U \to V \otimes U$. There is an analogous direct sum $U \otimes V = U \otimes V_1 \oplus U \otimes V_2$.

EXAMPLE 1.82. If $V = V_1 \oplus V_2$, and $U$ is any vector space, then $\mathrm{Hom}(U, V)$ is a direct sum of $\mathrm{Hom}(U, V_1)$ and $\mathrm{Hom}(U, V_2)$. The projections and inclusions are $\mathrm{Hom}(Id_U, P_i) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, V_i)$, and $\mathrm{Hom}(Id_U, Q_i) : \mathrm{Hom}(U, V_i) \to \mathrm{Hom}(U, V)$.

EXAMPLE 1.83. If $V = V_1 \oplus V_2$, and $U$ is any vector space, then $\mathrm{Hom}(V, U)$ is a direct sum of $\mathrm{Hom}(V_1, U)$ and $\mathrm{Hom}(V_2, U)$. The projections are $\mathrm{Hom}(Q_i, Id_U) : \mathrm{Hom}(V, U) \to \mathrm{Hom}(V_i, U)$, and the inclusions are

$$\mathrm{Hom}(P_i, Id_U) : \mathrm{Hom}(V_i, U) \to \mathrm{Hom}(V, U).$$

EXAMPLE 1.84. As the $U = \mathbb{K}$ special case of the previous Example, if $V = V_1 \oplus V_2$, then $V^* = V_1^* \oplus V_2^*$, with projections $Q_i^*$ and inclusions $P_i^*$.

LEMMA 1.85. *Given $V = V_1 \oplus V_2$, the image of $Q_2$, i.e., the subspace $Q_2(V_2)$ of $V$, is equal to the subspace $\ker(P_1)$, the kernel of $P_1$, which is also equal to $\ker(Q_1 \circ P_1)$.*

PROOF. The second equality follows from $\ker(P_1) \subseteq \ker(Q_1 \circ P_1) \subseteq \ker(P_1 \circ Q_1 \circ P_1) = \ker(P_1)$. It follows from $P_1 \circ Q_2 = 0_{\mathrm{Hom}(V_2, V_1)}$ that $Q_2(V_2) \subseteq \ker(P_1)$, and if $P_1(v) = 0_{V_1}$, then $v = (Q_1 \circ P_1 + Q_2 \circ P_2)(v) = Q_2(P_2(v))$, so $\ker(P_1) \subseteq Q_2(V_2)$. ∎

LEMMA 1.86. *Given $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$, with projections and inclusions $P_i'$, $Q_i'$, $P_i$, $Q_i$, respectively, if the maps $A_1 : U_1 \to V_1$ and $A_2 : U_2 \to V_2$ are both invertible, then the map $Q_1 \circ A_1 \circ P_1' + Q_2 \circ A_2 \circ P_2' : U \to V$ is invertible.*

PROOF. The inverse is $Q_1' \circ A_1^{-1} \circ P_1 + Q_2' \circ A_2^{-1} \circ P_2$. ∎

LEMMA 1.87. *Given a direct sum $V = V_1 \oplus V_2$ as in Definition 1.77, another direct sum $U = U_1 \oplus U_2$, with projections and inclusions $P_i'$ and $Q_i'$, and $H : U \to V$, the following are equivalent.*

(1) $Q_1 \circ P_1 \circ H = H \circ Q_1' \circ P_1'$.
(2) $Q_2 \circ P_2 \circ H = H \circ Q_2' \circ P_2'$.
(3) $P_1 \circ H \circ Q_2' = 0_{\mathrm{Hom}(U_2,V_1)}$ *and* $P_2 \circ H \circ Q_1' = 0_{\mathrm{Hom}(U_1,V_2)}$.
(4) *There exist maps* $H_1 : U_1 \to V_1$ *and* $H_2 : U_2 \to V_2$ *such that* $H = Q_1 \circ H_1 \circ P_1' + Q_2 \circ H_2 \circ P_2'$.

PROOF. First, for (1) $\iff$ (2),

$$\begin{array}{rcl} H = (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H & = & H \circ (Q_1' \circ P_1' + Q_2' \circ P_2') \\ = Q_1 \circ P_1 \circ H + Q_2 \circ P_2 \circ H & = & H \circ Q_1' \circ P_1' + H \circ Q_2' \circ P_2'. \end{array}$$

Applying either equality (1) or (2), then subtracting, gives the other equality. For (1) $\implies$ (3),

$$\begin{array}{rcl} P_1 \circ H \circ Q_2' & = & P_1 \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H \circ Q_2' \\ & = & P_1 \circ H \circ Q_1' \circ P_1' \circ Q_2' = 0_{\mathrm{Hom}(U_2,V_1)}, \\ P_2 \circ H \circ Q_1' & = & P_2 \circ H \circ (Q_1' \circ P_1' + Q_2' \circ P_2') \circ Q_1' \\ & = & P_2 \circ Q_1 \circ P_1 \circ H \circ Q_1' = 0_{\mathrm{Hom}(U_1,V_2)}. \end{array}$$

Next, to show that (3) implies (1) or (2), let $i = 1$ or $2$:

$$\begin{array}{rcl} Q_i \circ P_i \circ H & = & Q_i \circ P_i \circ H \circ (Q_1' \circ P_1' + Q_2' \circ P_2') \\ & = & Q_i \circ P_i \circ H \circ Q_i' \circ P_i' \\ & = & (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H \circ Q_i' \circ P_i' = H \circ Q_i' \circ P_i'. \end{array}$$

The construction in (4) is the same as in Lemma 1.86 (without requiring invertibility). The implication (4) $\implies$ (1) is straightforward. To show that (1) and (2) imply (4), let $H_1 = P_1 \circ H \circ Q_1'$ and $H_2 = P_2 \circ H \circ Q_2'$. Then,

$$\begin{array}{rcl} Q_1 \circ H_1 \circ P_1' + Q_2 \circ H_2 \circ P_2' & = & Q_1 \circ P_1 \circ H \circ Q_1' \circ P_1' + Q_2 \circ P_2 \circ H \circ Q_2' \circ P_2' \\ & = & Q_1 \circ P_1 \circ Q_1 \circ P_1 \circ H + Q_2 \circ P_2 \circ Q_2 \circ P_2 \circ H \\ & = & (Q_1 \circ P_1 + Q_2 \circ P_2) \circ H = H. \end{array}$$

∎

DEFINITION 1.88. For $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, and $H : U \to V$, $H$ <u>respects the direct sums</u> means: $H$ satisfies any of the equivalent conditions from Lemma 1.87. For such a map and $i = 1, 2$, the composites $P_i \circ H \circ Q_i' : U_i \to V_i$ are said to be induced by $H$.

LEMMA 1.89. *If $H : U \to V$ is an invertible map which respects the direct sums as in Definition 1.88, then $H^{-1}$ also respects the direct sums, and for each $i = 1, 2$, the induced map $P_i \circ H \circ Q_i' : U_i \to V_i$ is invertible, with inverse $P_i' \circ H^{-1} \circ Q_i$.*

PROOF.

$$\begin{array}{rcl} (P_i \circ H \circ Q_i') \circ (P_i' \circ H^{-1} \circ Q_i) & = & P_i \circ Q_i \circ P_i \circ H \circ H^{-1} \circ Q_i \\ & = & P_i \circ Q_i = Id_{V_i} \\ (P_I' \circ H^{-1} \circ Q_i) \circ (P_i \circ H \circ Q_i') & = & P_I' \circ H^{-1} \circ H \circ Q_i' \circ P_i' \circ Q_i' \\ & = & P_I' \circ Q_i' \end{array}$$

If $i = I$, this shows $P_i \circ H \circ Q'_i$ is invertible. If $i \neq I$, then $P'_I \circ H^{-1} \circ Q_i = 0_{\mathrm{Hom}(V_i, U_I)}$, so $H^{-1}$ respects the direct sums. ∎

LEMMA 1.90. *Given $U = U_1 \oplus U_2$, $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$, if $H : U \to V$ respects the direct sums and $H' : V \to W$ respects the direct sums, then $H' \circ H : U \to W$ respects the direct sums. A map induced by the composite is equal to the composite of the corresponding induced maps.* ∎

LEMMA 1.91. *Suppose $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, and $H : U \to V$ respects the direct sums, inducing maps $P_i \circ H \circ Q'_i$. Then, for any $A : W \to X$, the map $[H \otimes A] : U \otimes W \to V \otimes X$ respects the direct sums*

$$U_1 \otimes W \oplus U_2 \otimes W \to V_1 \otimes X \oplus V_2 \otimes X$$

*from Example 1.81. The induced map $[P_i \otimes Id_X] \circ [H \otimes A] \circ [Q'_i \otimes Id_W]$ is equal to $[(P_i \circ H \circ Q'_i) \otimes A]$.*

PROOF. All the claims follow immediately from Lemma 1.36. ∎

LEMMA 1.92. *Suppose $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, and $H : U \to V$ respects the direct sums, inducing maps $P_i \circ H \circ Q'_i$. Then, for any $A : W \to X$, the map $\mathrm{Hom}(A, H) : \mathrm{Hom}(X, U) \to \mathrm{Hom}(W, V)$ respects the direct sums*

$$\mathrm{Hom}(X, U_1) \oplus \mathrm{Hom}(X, U_2) \to \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(W, V_2)$$

*from Example 1.82. The induced map $\mathrm{Hom}(Id_W, P_i) \circ \mathrm{Hom}(A, H) \circ \mathrm{Hom}(Id_X, Q'_i)$ is equal to $\mathrm{Hom}(A, P_i \circ H \circ Q'_i)$. Analogously, the map $\mathrm{Hom}(H, A) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, X)$ respects the direct sums*

$$\mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W) \to \mathrm{Hom}(U_1, X) \oplus \mathrm{Hom}(U_2, X)$$

*from Example 1.83, and the induced map $\mathrm{Hom}(Q'_i, Id_W) \circ \mathrm{Hom}(H, A) \circ \mathrm{Hom}(P_i, Id_X)$ is equal to $\mathrm{Hom}(P_i \circ H \circ Q'_i, A)$.*

PROOF. All the claims follow immediately from Lemma 1.6. ∎

NOTATION 1.93. Given a direct sum $V = V_1 \oplus V_2$ with projection and inclusion pairs $(P_1, P_2)$, $(Q_1, Q_2)$, the pairs in the other order, $(V_2, V_1)$, $(P_2, P_1)$, $(Q_2, Q_1)$, also satisfy the definition of direct sum. The notation $V = V_2 \oplus V_1$ refers to these re-ordered pairs.

EXAMPLE 1.94. A map $H \in \mathrm{End}(V)$ which satisfies $Q_2 \circ P_2 \circ H = H \circ Q_1 \circ P_1$ respects the direct sums $H : V_1 \oplus V_2 \to V_2 \oplus V_1$ (and so $H$ also satisfies the other identities from Lemma 1.87).

LEMMA 1.95. *Given a vector space $V$ that admits two direct sums, $V = V_1 \oplus V_2$, $V = V''_1 \oplus V''_2$ with projections and inclusions $P_i$, $Q_i$, $P''_i$, $Q''_i$, respectively, the following are equivalent.*

(1) *The identity map $Id_V : V_1 \oplus V_2 \to V''_1 \oplus V''_2$ respects the direct sums.*
(2) $Q_1 \circ P_1 = Q''_1 \circ P''_1$.
(3) $Q_2 \circ P_2 = Q''_2 \circ P''_2$.
(4) $P''_I \circ Q_i = 0_{\mathrm{Hom}(V_i, V''_I)}$ *for $i \neq I$.*
(5) $P_I \circ Q''_i = 0_{\mathrm{Hom}(V''_i, V_I)}$ *for $i \neq I$.*

PROOF. The first statement is, by Definition 1.88 and Lemma 1.87, equivalent to any of the next three statements. The equivalence with the last statement follows from Lemma 1.89. ∎

DEFINITION 1.96. Given $V$, two direct sums $V = V_1 \oplus V_2$ and $V = V_1'' \oplus V_2''$ are equivalent direct sums means: they satisfy any of the properties from Lemma 1.95.

For a fixed $V$, this notion is clearly an equivalence relation on direct sum decompositions of $V$.

EXAMPLE 1.97. If $V = V_1 \oplus V_2$ and $H_1 : U_1 \to V_1$ and $H_2 : U_2 \to V_2$ are invertible, then $V$ is a direct sum $U_1 \oplus U_2$, with projections $H_i^{-1} \circ P_i$ and inclusions $Q_i \circ H_i$. The direct sums $V = V_1 \oplus V_2$ and $V = U_1 \oplus U_2$ are equivalent.

LEMMA 1.98. *Given $U$ and $V$, a direct sum $U = U_1 \oplus U_2$, and a map $H : U \to V$, suppose $V = V_1 \oplus V_2$ and $V = V_1'' \oplus V_2''$ are equivalent direct sums. Then $H$ respects the direct sums $U_1 \oplus U_2 \to V_1 \oplus V_2$ if and only if $H$ respects the direct sums $U_1 \oplus U_2 \to V_1'' \oplus V_2''$. Similarly, a map $A : V \to U$ respects the direct sums $V_1 \oplus V_2 \to U_1 \oplus U_2$ if and only if $A$ respects the direct sums $V_1'' \oplus V_2'' \to U_1 \oplus U_2$.* ∎

LEMMA 1.99. *Given $V = V_1 \oplus V_2$ and $V = V_1'' \oplus V_2''$ with respective direct sum data $P_i$, $Q_i$, $P_i''$, $Q_i''$, if $P_1 \circ Q_2'' = 0_{\mathrm{Hom}(V_2'', V_1)}$, and $P_1'' \circ Q_2 = 0_{\mathrm{Hom}(V_2, V_1'')}$, then $P_1'' \circ Q_1 : V_1 \to V_1''$ and $P_2'' \circ Q_2 : V_2 \to V_2''$ are both invertible.*

PROOF. The inverse of $P_i'' \circ Q_i$ is $P_i \circ Q_i'' : V_i'' \to V_i$. ∎

As a special case of both Lemma 1.99 and Lemma 1.89, if $V = V_1 \oplus V_2$ and $V = V_1'' \oplus V_2''$ are equivalent direct sums, then there are canonically induced invertible maps $P_i'' \circ Q_i : V_i \to V_i''$, $i = 1, 2$.

LEMMA 1.100. *Suppose $\phi \in V^*$. If $\phi \neq 0_{V^*}$ then there exists a direct sum $V = \mathbb{K} \oplus \ker(\phi)$.*

PROOF. Let $Q_2$ be the inclusion of the kernel subspace $\ker(\phi) = \{w \in V : \phi(w) = 0\}$ in $V$. Since $\phi \neq 0_{V^*}$, there exists some $v \in V$ so that $\phi(v) \neq 0$. Let $\alpha$, $\beta \in \mathbb{K}$ be any constants so that $\alpha \cdot \beta \cdot \phi(v) = 1$. Define $Q_1^\beta : \mathbb{K} \to V$ so that for $\gamma \in \mathbb{K}$, $Q_1^\beta(\gamma) = \beta \cdot \gamma \cdot v$. Define $P_1^\alpha = \alpha \cdot \phi : V \to \mathbb{K}$. Then,

$$P_1^\alpha \circ Q_1^\beta : \gamma \mapsto \alpha \cdot \phi(\beta \cdot \gamma \cdot v) = \alpha \cdot \beta \cdot \phi(v) \cdot \gamma = \gamma.$$

For any $w \in \ker(\phi)$, $(P_1^\alpha \circ Q_2)(w) = \alpha \cdot \phi(w) = 0$. Define $P_2 = Id_V - Q_1^\beta \circ P_1^\alpha$, which is a map from $V$ to $\ker(\phi)$: if $u \in V$, then

$$\begin{aligned} (\phi \circ P_2)(u) &= (\phi \circ Id_V - \phi \circ Q_1^\beta \circ P_1^\alpha)(u) \\ &= \phi(u) - \phi(Q_1^\beta(\alpha \cdot \phi(u))) \\ &= \phi(u) - \phi(\beta \cdot \alpha \cdot \phi(u) \cdot v) \\ &= \phi(u) - \alpha \cdot \beta \cdot \phi(v) \cdot \phi(u) = 0. \end{aligned}$$

Also, for $w \in \ker(\phi)$,

$$(P_2 \circ Q_2)(w) = ((Id_V - Q_1^\beta \circ P_1^\alpha) \circ Q_2)(w) = (Q_2 - Q_1^\beta \circ 0_{(\ker(\phi))^*})(w) = w,$$

so $P_2 \circ Q_2 = Id_{\ker(\phi)}$, and the claim follows from Lemma 1.78. ∎

Given $V$ and $\phi$, the direct sum from the previous Lemma is generally not unique, nor are two such direct sums, depending on $v$, $\alpha$, $\beta$, even equivalent in general. However, in some later examples, there will be a canonical element $v \notin \ker(\phi)$, and in such a case, different choices of $\alpha$, $\beta$ give equivalent direct sums.

LEMMA 1.101. *Given $V$, $\phi \in V^*$, and $v \in V$ so that $\phi(v) \neq 0$, let $\alpha$, $\beta$, $\alpha'$, $\beta' \in \mathbb{K}$ be any constants so that $\alpha \cdot \beta \cdot \phi(v) = \alpha' \cdot \beta' \cdot \phi(v) = 1$. Then the direct sum $V = \mathbb{K} \oplus \ker(\phi)$ constructed in the Proof of Lemma 1.100 is equivalent to the analogous direct sum with maps $Q_1^{\beta'} : \gamma \mapsto \beta' \cdot \gamma \cdot v$, $P_1^{\alpha'} = \alpha' \cdot \phi$, $Q_2$, and $P_2' = Id_V - Q_1^{\beta'} \circ P_1^{\alpha'}$.* ∎

The following result will be used as a step in Theorem 4.40.

THEOREM 1.102. *Suppose $U = U_1 \oplus U_2$ is a direct sum with projections and inclusions $P_i$, $Q_i$, and that there are vector spaces $V$, $V_1$, $V_2$, and maps $P_1' : V \to V_1$, $Q_2' : V_2 \to V$, $H : U \to V$, $H_1 : U_1 \to V_1$, $H_2 : U_2 \to V_2$, such that $P_1' \circ H = H_1 \circ P_1$, $Q_2' \circ H_2 = H \circ Q_2$, and $P_1' \circ Q_2' = 0_{\mathrm{Hom}(V_2, V_1)}$. Suppose further that $H$ and $H_1$ are invertible, and that $Q_2'$ is a linear monomorphism. Then, there exist maps $Q_1' : V_1 \to V$ and $P_2' : V \to V_2$ such that $V = V_1 \oplus V_2$. Also, $H$ respects the direct sums, and $H_2$ is invertible.*

PROOF. Let $Q_1' = H \circ Q_1 \circ H_1^{-1}$. Then

$$P_1' \circ Q_1' = P_1' \circ H \circ Q_1 \circ H_1^{-1} = H_1 \circ P_1 \circ Q_1 \circ H_1^{-1} = Id_{V_1}.$$

Let $P_2' = H_2 \circ P_2 \circ H^{-1}$. Then

$$\begin{aligned} Q_1' \circ P_1' + Q_2' \circ P_2' &= H \circ Q_1 \circ H_1^{-1} \circ P_1' + Q_2' \circ H_2 \circ P_2 \circ H^{-1} \\ &= H \circ Q_1 \circ P_1 \circ H^{-1} + H \circ Q_2 \circ P_2 \circ H^{-1} \\ &= Id_V. \end{aligned}$$

$$Q_2' \circ P_2' \circ Q_2' = (Id_V - Q_1' \circ P_1') \circ Q_2' = Q_2',$$

so $P_2' \circ Q_2' = Id_{V_2}$, by the monomorphism property (Definition 0.47), and this shows $V = V_1 \oplus V_2$ by Lemma 1.78. $H$ respects the direct sums:

$$\begin{aligned} P_1' \circ H \circ Q_2 &= H_1 \circ P_1 \circ Q_2 = 0_{\mathrm{Hom}(U_2, V_1)} \\ P_2' \circ H \circ Q_1 &= H_2 \circ P_2 \circ H^{-1} \circ H \circ Q_1 = 0_{\mathrm{Hom}(U_1, V_2)}. \end{aligned}$$

By Lemma 1.89, $P_2' \circ H \circ Q_2 = H_2 \circ P_2 \circ H^{-1} \circ H \circ Q_2 = H_2$ has inverse $P_2 \circ H^{-1} \circ Q_2'$. ∎

EXERCISE 1.103. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\nu$ be a direct sum, as in Definition 1.77, with projections $(P_1, P_2, \ldots, P_\nu)$ and inclusions $(Q_1, Q_2, \ldots, Q_\nu)$. For another vector space $W$ and additive functions $A : V \rightsquigarrow W$, $B : V \rightsquigarrow W$, the following are equivalent.

(1) For all $i = 1, \ldots, \nu$, $A \circ Q_i = B \circ Q_i : V_i \to W$.
(2) $A = B$.

HINT. The additive property is used in steps (1.9), (1.10).

$$\begin{aligned} A &= A \circ (Q_1 \circ P_1 + \cdots + Q_\nu \circ P_\nu) \\ &= (A \circ Q_1) \circ P_1 + \cdots + (A \circ Q_\nu) \circ P_\nu \qquad (1.9) \\ &= (B \circ Q_1) \circ P_1 + \cdots + (B \circ Q_\nu) \circ P_\nu \\ &= B \circ (Q_1 \circ P_1 + \cdots + Q_\nu \circ P_\nu) \qquad (1.10) \\ &= B. \end{aligned}$$

∎

EXERCISE 1.104. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\nu$ be a direct sum. Suppose there is some $i$ and some additive function $P' : V \rightsquigarrow V_i$ so that $P' \circ Q_i = Id_{V_i}$, and $P' \circ Q_I = 0_{\mathrm{Hom}(V_I, V_i)}$ for $I \neq i$. Then, $P' = P_i$.

HINT. This is a special case of Exercise 1.103. $P'$ is not assumed to be linear, but the above calculation uses the additive property to conclude that $P'$ is linear because it equals a given linear map. ∎

In the above sense, given all the data $P_1, \ldots, Q_\nu$ in a direct sum, each individual map $P_i$ is unique. Exercise 1.106 states an analogous uniqueness result for $Q_i$.

EXERCISE 1.105. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\nu$ be a direct sum. For any set $W$ and any functions $A : W \rightsquigarrow V$, $B : W \rightsquigarrow V$, the following are equivalent.

(1) For all $i = 1, \ldots, \nu$, $P_i \circ A = P_i \circ B : W \to V_i$.
(2) $A = B$.

HINT. The calculation is similar to that in Exercise 1.103, but does not assume any additive property. ∎

EXERCISE 1.106. For $V = V_1 \oplus V_2 \oplus \cdots \oplus V_\nu$, suppose there is some $i$ and some function $Q' : V_i \rightsquigarrow V$ so that $P_i \circ Q' = Id_{V_i}$, and $P_I \circ Q' = 0_{\mathrm{Hom}(V_i, V_I)}$ for $I \neq i$. Then, $Q' = Q_i$.

HINT. This is a special case of Exercise 1.105. ∎

EXERCISE 1.107. Let $V = V_1 \oplus V_2$ be a direct sum with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, and let $A : V_1 \to V_2$. Then the following maps $(P_1', P_2')$, $(Q_1', Q_2')$ also define a direct sum.

$$\begin{aligned} Q_1' &= Q_1 + Q_2 \circ A : V_1 \to V \\ Q_2' &= Q_2 : V_2 \to V \\ P_1' &= P_1 : V \to V_1 \\ P_2' &= P_2 - A \circ P_1 : V \to V_2. \end{aligned}$$

This is equivalent to the original direct sum if and only if $A = 0_{\mathrm{Hom}(V_1, V_2)}$.

HINT. The first equation from Definition 1.77 is:

$$\begin{aligned} Q_1' \circ P_1' + Q_2' \circ P_2' &= (Q_1 + Q_2 \circ A) \circ P_1 + Q_2 \circ (P_2 - A \circ P_1) \\ &= Q_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ P_2 - Q_2 \circ A \circ P_1 \\ &= Id_V. \end{aligned}$$

The remaining equations from Definition 1.77 (or Lemma 1.78) are also easy to check. If the direct sums are equivalent, then $0_{\mathrm{Hom}(V_1, V_2)} = P_2 \circ Q_1' = P_2 \circ (Q_1 + Q_2 \circ A) = A$, and conversely. ∎

The direct sum $P_i'$, $Q_i'$ is the <u>graph</u> of $A$. In a certain sense, Exercise 1.107 has a converse: if a space $V$ decomposes in two ways as a direct sum, with the same inclusion $Q_2$, then the two direct sums are related using the graph construction, up to equivalence.

EXERCISE 1.108. Given $V$, $V_1$, $V_2$, suppose the pairs $(P_1, P_2)$ and $(Q_1, Q_2)$ define a direct sum $V = V_1 \oplus V_2$, and the pairs $(P_1', P_2')$, $(Q_1', Q_2')$ also satisfy Definition 1.77. If $Q_2 = Q_2'$, then there exists a map $A : V_1 \to V_2$, and there exist $(P_1'', P_2'')$, $(Q_1'', Q_2'')$ which define a third direct sum, and which satisfy:

$$\begin{aligned} Q_1'' &= Q_1 + Q_2 \circ A : V_1 \to V \\ Q_2'' &= Q_2 : V_2 \to V \\ P_1'' &= P_1 : V \to V_1 \\ P_2'' &= P_2 - A \circ P_1 : V \to V_2. \end{aligned}$$

This $(P_1'', P_2'')$, $(Q_1'', Q_2'')$ direct sum is equivalent to the $(P_1', P_2')$, $(Q_1', Q_2')$ direct sum.

HINT. Note that by Exercise 1.106, the hypothesis $Q_2 = Q_2'$ is equivalent to assuming that $Q_2'$ satisfies $P_2 \circ Q_2' = Id_{V_2}$ and $P_1 \circ Q_2' = 0_{\mathrm{Hom}(V_2, V_1)}$. Choosing $Q_2'' = Q_2$ gives the identity $P_1' \circ Q_2'' = 0_{\mathrm{Hom}(V_2, V_1)}$.

The above four equations are the properties defining a graph. The claimed existence follows from checking that the following choices have the claimed properties.

$$\begin{aligned} A &= -P_2' \circ Q_1 : V_1 \to V_2 \\ Q_1'' &= Q_1' \circ P_1' \circ Q_1 : V_1 \to V \\ P_2'' &= P_2' : V \to V_2. \end{aligned}$$

The equivalence of direct sums as in Definition 1.96 is verified by checking $P_2' \circ Q_1'' = 0_{\mathrm{Hom}(V_1, V_2)}$. ∎

## 1.4. Idempotents and involutions

DEFINITION 1.109. An element $P \in \mathrm{End}(V)$ is an idempotent means: $P \circ P = P$.

LEMMA 1.110. *Given $V$ and $P_1$, $P_2 \in \mathrm{End}(V)$, any three out of the following four properties (1) – (4) imply the remaining fourth.*

1. *$P_1$ is an idempotent.*
2. *$P_2$ is an idempotent.*
3. *$P_1 \circ P_2 + P_2 \circ P_1 = 0_{\mathrm{End}(V)}$.*
4. *$P_1 + P_2$ is an idempotent.*

*Property (4) is equivalent to:*

5. *There exists $P_3 \in \mathrm{End}(V)$ such that $P_3$ is an idempotent and $P_1 + P_2 + P_3 = Id_V$.*

*If, further, either $\frac{1}{2} \in \mathbb{K}$ or $P_1 + P_2 = Id_V$, then $P_1$, $P_2$, $P_3$ satisfying properties (1) – (5) also satisfy:*

6. *For distinct $i_1, i_2 \in \{1, 2, 3\}$, $P_{i_1} \circ P_{i_2} = 0_{\mathrm{End}(V)}$.*

*Conversely, if $P_1, P_2, P_3 \in \mathrm{End}(V)$ satisfy (6) and $P_1 + P_2 + P_3 = Id_V$, then $P_1, P_2, P_3$ are idempotents satisfying (1) – (5).* ∎

EXERCISE 1.111. For $P \in \mathrm{End}(V)$, the following are equivalent.

1. $P$ is an idempotent.
2. For any $A \in \mathrm{End}(V)$, $P + P \circ A - P \circ A \circ P$ is an idempotent.
3. For any $A \in \mathrm{End}(V)$, $P + A \circ P - P \circ A \circ P$ is an idempotent. ∎

EXERCISE 1.112. If $P \in \mathrm{End}(V)$ is an idempotent, then the following are equivalent.

(1) $P' \circ P = P \circ P'$ for all idempotents $P' \in \mathrm{End}(V)$.
(2) $P = 0_{\mathrm{End}(V)}$ or $P = Id_V$.

HINT. This follows from Exercise 1.111 and Claim 0.57. ∎

EXAMPLE 1.113. An idempotent $P$ defines a direct sum structure as follows. Let $V_1 = P(V)$, the subspace of $V$ which is the image of $P$. Define $Q_1 : V_1 \to V$ to be the subspace inclusion map, and define $P_1 : V \to V_1$ by restricting the target of $P : V \to V$ to get $P_1 : V \to V_1$ with $P_1(v) = P(v)$ for all $v \in V$. Then $P = Q_1 \circ P_1 : V \to V$ by construction. The map $Id_V - P$ is also an idempotent (this is a special case of Lemma 1.110), so proceeding analogously, define $V_2 = (Id_V - P)(V)$, the image of the linear map $Id_V - P : V \to V$. Again, let $Q_2 : V_2 \to V$ be the subspace inclusion, and define $P_2 = Id_V - P$, with its target space restricted to $V_2$, so that $Q_2 \circ P_2 = Id_V - P$ by construction, and $Q_1 \circ P_1 + Q_2 \circ P_2 = P + (Id_V - P) = Id_V$. To show $V = V_1 \oplus V_2$, it remains only to check that these maps satisfy the two remaining equations from Lemma 1.78. For $v_1 \in V_1$, $Q_1(v_1) = v_1 = P(w_1)$ for some $w_1 \in V$, so $(P_1 \circ Q_1)(v_1) = P_1(P(w_1)) = P(P(w_1)) = P(w_1) = v_1$. Similarly, for $Q_2(v_2) = v_2 = (Id_V - P)(w_2)$, $(P_2 \circ Q_2)(v_2) = (Id_V - P)((Id_V - P)(w_2)) = (Id_V - P)(w_2) = v_2$. This construction of the direct sum $V = V_1 \oplus V_2$ is canonical up to re-ordering.

The statement of Lemma 1.85 in the special case of Example 1.113 is that the image of $Id_V - P$ is the kernel of $P$, and the image of $P$ is the kernel of $Id_V - P$.

EXAMPLE 1.114. Given any direct sum $V = U_1 \oplus U_2$ as in Definition 1.77 with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, the composite $Q_1 \circ P_1 : V \to V$ is an idempotent, and so is $Id_V - Q_1 \circ P_1 = Q_2 \circ P_2$. This is a converse to the construction of Example 1.113; any direct sum canonically defines an unordered pair of two idempotents. For $P = Q_1 \circ P_1$, the direct sum $V = V_1 \oplus V_2$ constructed in Example 1.113 is equivalent, as in Definition 1.96, to the original direct sum.

LEMMA 1.115. *Given idempotents $P : V \to V$, $P' : U \to U$ defining direct sums $V_1 \oplus V_2$ and $U_1 \oplus U_2$ as in Example 1.113, and a map $H : U \to V$, the following are equivalent.*

(1) *$H$ respects the direct sums (as in Definition 1.88).*
(2) *$H \circ P' = P \circ H$.* ∎

EXERCISE 1.116. Given maps $P_1 : V \to V_1$, $Q_1 : V_1 \to V$, $P_2 : V \to V_2$, $Q_2 : V_2 \to V_1$, if $Q_1 \circ P_1 + Q_2 \circ P_2 = Id_V$ and either $P_1 \circ Q_2 = 0_{\mathrm{Hom}(V_2, V_1)}$ or $P_2 \circ Q_1 = 0_{\mathrm{Hom}(V_1, V_2)}$, then $P_1 \circ Q_1 : V_1 \to V_1$ and $P_2 \circ Q_2 : V_2 \to V_2$ are both idempotents.

HINT. These are some of the composites from Definition 1.77 and this claim is related to Lemma 1.78 and Lemma 1.99. ∎

DEFINITION 1.117. An element $K \in \mathrm{End}(V)$ is an <u>involution</u> means: $K \circ K = Id_V$.

LEMMA 1.118. *If* $\frac{1}{2} \in \mathbb{K}$ *and* $K \in \mathrm{End}(V)$*, then the following are equivalent.*

(1) $K \in \mathrm{End}(V)$ *is an involution.*
(2) $P = \frac{1}{2} \cdot (Id_V + K)$ *is an idempotent.*
(3) $Id_V - P = \frac{1}{2} \cdot (Id_V - K)$ *is an idempotent.* ∎

LEMMA 1.119. *For an involution* $K \in \mathrm{End}(V)$*, let* $V_1 = \{v \in V : K(v) = v\}$*, and* $V_2 = \{v \in V : K(v) = -v\}$*. If* $\frac{1}{2} \in \mathbb{K}$*, then* $V = V_1 \oplus V_2$*, with* $Q_i$ *the subspace inclusion maps, and projections:*

$$P_1 = \frac{1}{2} \cdot (Id_V + K), \tag{1.11}$$

$$P_2 = \frac{1}{2} \cdot (Id_V - K). \tag{1.12}$$

PROOF. This can be proved directly, but also follows from the construction of Example 1.113. It is easy to check that $V_1$ is a subspace of $V$, equal to the image of the idempotent $P$ from Lemma 1.118 and that $V_2$ is equal to the image of $Id_V - P$. The composites $Q_1 \circ P_1$, $Q_2 \circ P_2 \in \mathrm{End}(V)$ are also given by the formulas $\frac{1}{2} \cdot (Id_V \pm K)$. ∎

NOTATION 1.120. We refer to the construction of $V = V_1 \oplus V_2$ as in Lemma 1.119 as the <u>direct sum produced by the involution</u> $K$. The subspaces $V_1$, $V_2$ and maps $P_1$, $P_2$ in (1.11), (1.12) are canonical, but Lemma 1.119 made a choice of order in the direct sum $V = V_1 \oplus V_2$. With this ordering convention, the involution $-K$ produces the direct sum $V = V_2 \oplus V_1$ as in Notation 1.93.

NOTATION 1.121. For the projection maps defined by formulas (1.11), (1.12) from a direct sum produced by an involution, the double arrowhead will appear in diagrams, $P_1 : V \twoheadrightarrow V_1$, $P_2 : V \twoheadrightarrow V_2$, and the same style arrow for composites of such projections. For the subspace inclusion maps as in Lemma 1.119, the hook arrow will appear: $Q_1 : V_1 \hookrightarrow V$ for the fixed point subspace of $K$, and $Q_2 : V_2 \hookrightarrow V$ for the fixed point subspace of $-K$, and similarly for composites of such inclusions.

EXAMPLE 1.122. Given any direct sum $V = U_1 \oplus U_2$ as in Definition 1.77 with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, the map

$$Q_1 \circ P_1 - Q_2 \circ P_2 = Id_V - 2 \cdot Q_2 \circ P_2 : V \to V$$

is an involution, and it respects the direct sums $U_1 \oplus U_2 \to U_1 \oplus U_2$. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by this involution, $V = V_1 \oplus V_2$ as in Lemma 1.119, is equivalent, as in Definition 1.96, to the original direct sum. As in Lemma 1.89 and Lemma 1.99, there are invertible maps $U_i \to V_i$. If the direct sum maps $Q_i$, $P_i$ were defined by some involution $K$ as in Lemma 1.119, then $Q_1 \circ P_1 - Q_2 \circ P_2 = K$.

LEMMA 1.123. *For an idempotent* $P : V \to V$*, let* $V = V_1 \oplus V_2$ *be the direct sum from Example 1.113. The maps* $K = 2 \cdot P - Id_V : V \to V$ *and* $Id_V - 2 \cdot P = -K$ *are involutions, and both* $K$ *and* $-K$ *respect the direct sums* $V_1 \oplus V_2 \to V_1 \oplus V_2$*. If* $\frac{1}{2} \in \mathbb{K}$*, then the direct sum from Lemma 1.119 produced by* $K$ *is the same as* $V = V_1 \oplus V_2$*.*

PROOF. The claim that $K$ and $-K$ respect the direct sums is a special case of Lemma 1.115. ∎

EXAMPLE 1.124. For any space $V$, the switching map $s : V \otimes V \to V \otimes V$ from Example 1.29 is an involution. For $\frac{1}{2} \in \mathbb{K}$, $s$ produces a direct sum on $V \otimes V$, denoted:

$$V \otimes V = S^2V \oplus \Lambda^2 V,$$

with projections $P_1 = \frac{1}{2} \cdot (Id_{V\otimes V} + s) : V \otimes V \twoheadrightarrow S^2V$ and $P_2 = \frac{1}{2} \cdot (Id_{V\otimes V} - s) : V \otimes V \twoheadrightarrow \Lambda^2 V$ and corresponding subspace inclusions $Q_1$, $Q_2$.

THEOREM 1.125. *Given $\frac{1}{2} \in \mathbb{K}$, an involution $K \in \mathrm{End}(V)$, let $V = V_1 \oplus V_2$ be the direct sum produced by $K$. For any $\phi \in V^*$, if $\phi \neq 0_{V^*}$ and $\phi \circ K = \phi$, then there is a direct sum $V = \mathbb{K} \oplus \ker(\phi \circ Q_1) \oplus V_2$.*

PROOF. Let $(P_1, P_2)$ and $(Q_1, Q_2)$ be the pairs as in Lemma 1.119; the inclusion $Q_1$ appears in the claim. From $\phi \neq 0_{V^*}$, there is some $w \in V$ so that $\phi(w) \neq 0$. Let $v = \frac{1}{2} \cdot (w + K(w)) \in V_1$. Then

$$(\phi \circ Q_1)(v) = \phi(v) = \phi(\frac{1}{2} \cdot (w + K(w))) = \frac{1}{2} \cdot \phi(w) + \frac{1}{2} \cdot \phi(K(w)) = \phi(w) \neq 0.$$

So $\phi \circ Q_1 \neq 0_{V_1^*}$ and Lemma 1.100 applies to get a direct sum $V_1 = \mathbb{K} \oplus \ker(\phi \circ Q_1)$, depending on parameters $\alpha$, $\beta \in \mathbb{K}$ such that $\alpha \cdot \beta \cdot \phi(w) = 1$. The inclusions are $Q_3^\beta : \mathbb{K} \to V_1 : \gamma \mapsto \beta \cdot \gamma \cdot v$, and the subspace inclusion $Q_4 : \ker(\phi \circ Q_1) \to V_1$. The projections are $P_3^\alpha = \alpha \cdot \phi \circ Q_1 : V_1 \to \mathbb{K}$, $P_4 = Id_{V_1} - Q_3^\beta \circ P_3^\alpha : V_1 \to \ker(\phi \circ Q_1)$. Theorem 1.79 applies to get the claimed direct sum. In particular, the first inclusion is $Q_1 \circ Q_3^\beta : \mathbb{K} \to V$, the second is a subspace inclusion $Q_1 \circ Q_4 : \ker(\phi \circ Q_1) \to V$, and the third is the subspace inclusion not depending on $\phi$, $Q_2 : V_2 \hookrightarrow V$. The three projections are $\alpha \cdot \phi \circ Q_1 \circ P_1 : V \to \mathbb{K}$, $(Id_{V_1} - Q_3^\beta \circ (\alpha \cdot \phi \circ Q_1)) \circ P_1 : V \to \ker(\phi \circ Q_1)$, and $P_2 : V \twoheadrightarrow V_2$. ∎

For $K$ and $\phi$ as in Theorem 1.125, any element $u \in V$ can be written in the following way as a sum of terms that do not depend on $\alpha$ or $\beta$:

$$\begin{aligned}
(1.13)\quad u &= Id_V(u) \\
&= (Q_1 \circ Q_3^\beta \circ (\alpha \cdot \phi \circ Q_1 \circ P_1) \\
&\quad + Q_1 \circ Q_4 \circ (Id_{V_1} - Q_3^\beta \circ (\alpha \cdot \phi \circ Q_1)) \circ P_1 + Q_2 \circ P_2)(u) \\
&= \frac{\phi(u)}{2\phi(w)}(w + K(w)) + \left(\frac{1}{2}(u + K(u)) - \frac{\phi(u)}{2\phi(w)}(w + K(w))\right) + \frac{1}{2}(u - K(u)).
\end{aligned}$$

The second and third terms are in the kernel of $\phi$; the third term is the projection of $u$ onto the $-1$ eigenspace of $K$, not depending on $\phi$ or $w$. The first two terms are both in the $+1$ eigenspace of $K$, and they both depend on $\phi$ and on $v = \frac{1}{2} \cdot (w + K(w))$.

LEMMA 1.126. *Given $\frac{1}{2} \in \mathbb{K}$ and two involutions $K : V \to V$ and $K' : U \to U$, which produce direct sums $V_1 \oplus V_2$, $U_1 \oplus U_2$ as in Lemma 1.119, a map $H : U \to V$ respects the direct sums $U_1 \oplus U_2 \to V_1 \oplus V_2$ if and only if $K \circ H = H \circ K'$.* ∎

LEMMA 1.127. *Given $\frac{1}{2} \in \mathbb{K}$ and two involutions $K : V \to V$ and $K' : U \to U$, which produce direct sums $V_1 \oplus V_2$, $U_1 \oplus U_2$ as in Lemma 1.119, a map $H : U \to V$ respects the direct sums $U_1 \oplus U_2 \to V_2 \oplus V_1$ if and only if $K \circ H = -H \circ K'$.*

PROOF. Note the order of the spaces $V_2 \oplus V_1$ is different from that appearing in Lemma 1.126, so the notation refers to the identities $Q_1 \circ P_1 \circ H = H \circ Q_2' \circ P_2'$

and $Q_2 \circ P_2 \circ H = H \circ Q_1' \circ P_1'$. The claims can be checked directly, but also follow from applying Lemma 1.126 to the involutions $K$ and $-K'$. $\blacksquare$

Lemma 1.128. *Given $V$ and a pair of involutions on $V$, $K_1$ and $K_2$, if $\frac{1}{2} \in \mathbb{K}$, then the following are equivalent.*

(1) *The involutions commute, i.e., $K_1 \circ K_2 = K_2 \circ K_1$.*
(2) *The composite $K_1 \circ K_2$ is an involution.*
(3) *$K_2$ respects the direct sum $V = V_1 \oplus V_2$ produced by $K_1$.*
(4) *$K_1$ respects the direct sum $V = V_3 \oplus V_4$ produced by $K_2$.*

Proof. The equivalence (1) $\iff$ (2) is elementary and does not require $\frac{1}{2} \in \mathbb{K}$. The direct sums in (3), (4) are as in Lemma 1.119. The equivalences (1) $\iff$ (3) and (1) $\iff$ (4) are special cases of Lemma 1.126. $\blacksquare$

In statement (3) of Lemma 1.128, $K_2$ induces an involution on both $V_1$ and $V_2$ as in Definition 1.88 and Lemma 1.89, and similarly for $K_1$ in statement (4).

Given $\frac{1}{2} \in \mathbb{K}$, and $V$ with commuting involutions $K_1$, $K_2$ as in Lemma 1.128 and Lemma 1.129, and corresponding direct sums $V = V_1 \oplus V_2$, $V = V_3 \oplus V_4$, respectively, as in Lemma 1.128, let $V = V_5 \oplus V_6$ be the direct sum produced by the involution $K_1 \circ K_2$.

Lemma 1.129. *Given $V$, subspaces $V_1, \ldots, V_5$ as above, and commuting involutions $K_1$, $K_2 \in \mathrm{End}(V)$, for $v \in V$, any pair of two of the following three statements implies the remaining one:*

(1) *$v = K_1(v) \in V_1$.*
(2) *$v = K_2(v) \in V_3$.*
(3) *$v = (K_1 \circ K_2)(v) \in V_5$.* $\blacksquare$

It follows from Lemma 1.129 that these subspaces of $V$ are equal:

$$V_1 \cap V_3 = V_1 \cap V_5 = V_3 \cap V_5 = V_1 \cap V_3 \cap V_5. \tag{1.14}$$

Let $(P_5, P_6)$, $(Q_5, Q_6)$ denote the projections and inclusions for the above direct sum $V = V_5 \oplus V_6$ produced by $K_1 \circ K_2$. Since $K_1 \circ Q_5 = K_2 \circ Q_5$, the maps induced by $K_1$ and $K_2$ are equal:

$$P_5 \circ K_1 \circ Q_5 = P_5 \circ K_2 \circ Q_5 : V_5 \to V_5; \tag{1.15}$$

this map is a canonical involution on $V_5$, producing a direct sum $V_5 = V_5' \oplus V_5''$ with projection $P_5' : V_5 \twoheadrightarrow V_5'$ from (1.11). Similarly, there is an involution induced by $K_1$ or $K_1 \circ K_2$ on $V_3$, producing $V_3 = V_3' \oplus V_3''$, and there is another involution induced by $K_2$ or $K_1 \circ K_2$ on $V_1$, producing $V_1 = V_1' \oplus V_1''$. On the set $V_6$, the induced involutions are opposite:

$$P_6 \circ K_1 \circ Q_6 = -P_6 \circ K_2 \circ Q_6 : V_6 \to V_6, \tag{1.16}$$

if one produces a direct sum $V_6 = V_6' \oplus V_6''$, the other produces $V_6 = V_6'' \oplus V_6'$. Similarly, there are opposite induced involutions on $V_2$ and $V_4$.

Theorem 1.130. *Given $\frac{1}{2} \in \mathbb{K}$, and commuting involutions on $V$ with the above notation,*

$$V_5' = V_3' = V_1' = V_1 \cap V_3 \cap V_5.$$

*The composite projections are all equal:*

$$P_5' \circ P_5 = P_3' \circ P_3 = P_1' \circ P_1 : V \twoheadrightarrow V_1 \cap V_3 \cap V_5.$$

*Also,* $V_5'' = V_2 \cap V_4$*,* $V_3'' = V_2 \cap V_6$*, and* $V_1'' = V_4 \cap V_6$*.*

PROOF. $V_5'$ is the set of fixed points $v \in V_5$ of the involution $P_5 \circ K_1 \circ Q_5$. Denote the maps from Lemma 1.119 $P_5'$, $Q_5'$, so

$$Q_5' \circ P_5' = \frac{1}{2} \cdot (Id_{V_5} + P_5 \circ K_1 \circ Q_5).$$

To establish the first claim, it is enough to show $V_5' = V_1 \cap V_3$; the claims $V_3' = V_1 \cap V_5$ and $V_1' = V_3 \cap V_5$ are similar, and then (1.14) applies. To show $V_5' \subseteq V_1$, use the fact that $K_1$ commutes with $Q_5 \circ P_5 = \frac{1}{2} \cdot (Id_V + K_1 \circ K_2)$; if $v \in V_5' \subseteq V_5$, then

$$v = Q_5(v) = (P_5 \circ K_1 \circ Q_5)(v) = Q_5((P_5 \circ K_1 \circ Q_5)(v)) = K_1(Q_5(v)),$$

so $v \in V_1$. Showing $V_5' \subseteq V_3$ is similar, so $V_5' \subseteq V_1 \cap V_3$.

Another argument would be to consider the subspace $V_5'$ as the image of $Q_5 \circ Q_5' \circ P_5' \circ P_5$ in $V$. Then

$$\begin{aligned} Q_5 \circ Q_5' \circ P_5' \circ P_5 &= Q_5 \circ \frac{1}{2} \cdot (Id_{V_5} + P_5 \circ K_1 \circ Q_5) \circ P_5 \\ &= \frac{1}{2} \cdot Q_5 \circ P_5 + \frac{1}{2} \cdot K_1 \circ Q_5 \circ P_5 \\ &= Q_1 \circ P_1 \circ Q_5 \circ P_5, \end{aligned} \tag{1.17}$$

which shows $V_5'$ is contained in $V_1$, the image of $Q_1$ in $V$.

Conversely, if $v \in V_1 \cap V_3$, then $v = K_1(v) = Q_5(v) \in V_5$ (Lemma 1.129) and $(P_5 \circ K_1 \circ Q_5)(v) = (P_5 \circ Q_5)(v) = v \in V_5'$.

The equality of the composites of projections follows from using the commutativity of the involutions to get $Q_1 \circ P_1 \circ Q_5 \circ P_5 = Q_5 \circ P_5 \circ Q_1 \circ P_1$, and then (1.17) implies $Q_5 \circ Q_5' \circ P_5' \circ P_5 = Q_1 \circ Q_1' \circ P_1' \circ P_1$.

The last claim of the Theorem follows from similar calculations. However, the three subspaces are in general not equal to each other. ∎

The projection $P_5 : V \twoheadrightarrow V_5$ satisfies $P_5 \circ K_1 = (P_5 \circ K_1 \circ Q_5) \circ P_5$, so Lemma 1.126 applies: $P_5$ respects the direct sums $V_1 \oplus V_2 \to V_5' \oplus V_5''$ and the map $V_1 \to V_5'$ induced by $P_5$ is $P_5' \circ P_5 \circ Q_1$. By Theorem 1.130,

$$P_5' \circ P_5 \circ Q_1 = P_1' \circ P_1 \circ Q_1 = P_1' : V_1 \twoheadrightarrow V_1' = V_5'. \tag{1.18}$$

This gives an alternate construction of $P_1'$ as a map induced by $P_5$, or similarly, any $P_i'$ is equal to a map induced by $P_I$ for any distinct $i = 1, 3, 5$, $I = 1, 3, 5$.

THEOREM 1.131. *Given* $\frac{1}{2} \in \mathbb{K}$*, suppose* $K_V^1$*,* $K_V^2$ *are commuting involutions on* $V$ *as in Theorem 1.130. Similarly, let* $K_U^1$*,* $K_U^2$ *be commuting involutions on* $U$*, with corresponding notation for the direct sums:* $U = U_1 \oplus U_2$*,* $U = U_3 \oplus U_4$*, etc. If a map* $H : U \to V$ *satisfies* $H \circ K_U^1 = K_V^1 \circ H$ *and* $H \circ K_U^2 = K_V^2 \circ H$*, then* $H$ *respects the corresponding direct sums* $U_1 \oplus U_2 \to V_1 \oplus V_2$ *and* $U_3 \oplus U_4 \to V_3 \oplus V_4$*. Further, the induced map* $P_V^1 \circ H \circ Q_U^1 : U_1 \to V_1$ *respects the direct sums* $U_1' \oplus U_1'' \to V_1' \oplus V_1''$ *and similarly for the maps* $U_3 \to V_3$*,* $U_5 \to V_5$ *induced by* $H$*. The induced map* $U_1' \to V_1'$ *is equal to the map* $U_3' \to V_3'$ *induced by* $P_V^3 \circ H \circ Q_U^3 : U_3 \to V_3$*.*

PROOF. The fact that $H$ respects each pair of direct sums is Lemma 1.126. The subspace $U_1$ has a canonical involution $P_U^1 \circ K_U^2 \circ Q_U^1$, and since $K_U^1$, $K_U^2$

commute, $K_U^2$ also commutes with $Q_U^1 \circ P_U^1 = \frac{1}{2} \cdot (Id_U + K_U^1)$. The map induced by $H$, $P_V^1 \circ H \circ Q_U^1 : U_1 \to V_1$, satisfies:

$$\begin{aligned}(P_V^1 \circ H \circ Q_U^1) \circ (P_U^1 \circ K_U^2 \circ Q_U^1) &= P_V^1 \circ H \circ K_U^2 \circ Q_U^1 \circ P_U^1 \circ Q_U^1 \\ &= P_V^1 \circ K_V^2 \circ H \circ Q_U^1 \\ &= (P_V^1 \circ K_V^2 \circ Q_V^1) \circ (P_V^1 \circ H \circ Q_U^1).\end{aligned}$$

It follows from Lemma 1.126 again that $P_V^1 \circ H \circ Q_U^1$ respects the direct sums as claimed. The induced map is $P_V^{1\prime} \circ (P_V^1 \circ H \circ Q_U^1) \circ Q_U^{1\prime} : U_1' \to V_1'$.

The last claim of the Theorem is that this induced map is equal to the map $P_V^{3\prime} \circ (P_V^3 \circ H \circ Q_U^3) \circ Q_U^{3\prime} : U_3' \to V_3'$. The claim follows from the idea that the induced maps are restrictions of $H$ to the same subspace $U_1' = U_3' = U_1 \cap U_3$, by Theorem 1.130. More specifically, the subspace inclusions are equal: $Q_U^3 \circ Q_U^{3\prime} = Q_U^1 \circ Q_U^{1\prime} : U_1 \cap U_3 \hookrightarrow U$, and the composites of projections are equal: $P_V^{1\prime} \circ P_V^1 = P_V^{3\prime} \circ P_V^3$. ∎

Example 1.132. Given $\frac{1}{2} \in \mathbb{K}$, suppose $K_1$ and $K_2$ are commuting involutions on $V$ as in Theorem 1.130, and suppose $H$ is another involution on $V$ so that $K_1 \circ H = H \circ K_2$. The three involutions $K_1$, $K_2$, $K_1 \circ K_2$ produce direct sums $V = V_1 \oplus V_2$, $V_3 \oplus V_4$, and $V_5 \oplus V_6$. Similarly, because $K_1 \circ K_2$ commutes with $H$, the three involutions $K_1 \circ K_2$, $H$, and $K_1 \circ K_2 \circ H$ produce corresponding direct sums $V = V_5 \oplus V_6$, $V_7 \oplus V_8$, and $V_9 \oplus V_{10}$. As in (1.15), there are induced involutions on $V_5$, $P_5 \circ K_1 \circ Q_5 = P_5 \circ K_2 \circ Q_5$ and $P_5 \circ H \circ Q_5 = P_5 \circ K_1 \circ K_2 \circ H \circ Q_5$. These two involutions commute: for $v = Q_5(v) = (K_1 \circ K_2)(v) \in V_5$, $v$ satisfies $K_1(v) = K_2(v)$ and

$$\begin{aligned}(P_5 \circ K_1 \circ Q_5 \circ P_5 \circ H \circ Q_5)(v) &= (P_5 \circ K_1 \circ H)(v) = (P_5 \circ H \circ K_2)(v), \\ (P_5 \circ H \circ Q_5 \circ P_5 \circ K_1 \circ Q_5)(v) &= (P_5 \circ H \circ K_1)(v).\end{aligned}$$

By Lemma 1.128, their product

$$(P_5 \circ K_1 \circ Q_5) \circ (P_5 \circ H \circ Q_5) = P_5 \circ K_1 \circ H \circ Q_5 = P_5 \circ K_2 \circ H \circ Q_5$$

is also an involution on $V_5$ (although in general, $K_1 \circ H$ and $K_2 \circ H$ need not be involutions). So, Theorem 1.130 applies to these three commuting involutions on $V_5$, with $P_5 \circ K_1 \circ Q_5$ producing a direct sum $V_5 = V_5' \oplus V_5''$, where $V_5' = V_1 \cap V_3 \cap V_5$ is the fixed point subspace of $P_5 \circ K_1 \circ Q_5$. Similarly, $V_5 \cap V_7 \cap V_9$ is the fixed point subspace of $P_5 \circ H \circ Q_5$, and denoting by $V_{11}$ the fixed point subspace of $P_5 \circ K_1 \circ H \circ Q_5$, the three fixed point subspaces have the following intersection:

$$\begin{aligned}(V_1 \cap V_3 \cap V_5) \cap (V_5 \cap V_7 \cap V_9) \cap V_{11} &= (V_1 \cap V_3) \cap (V_5 \cap V_7) \\ &= (V_1 \cap V_3 \cap V_5) \cap V_7 = V_1 \cap V_3 \cap V_7 \\ &= \{v \in V : v = K_1(v) = K_2(v) = H(v)\}.\end{aligned}$$

The projections from Theorem 1.130 appear in the following commutative diagram.

$$\begin{tikzcd}
 & & & & V \arrow[dllll, two heads] \arrow[dll, two heads] \arrow[d, two heads] \arrow[drr, two heads] \arrow[drrrr, two heads] & & & & \\
V_1 \arrow[drr, two heads] & & V_3 \arrow[d, two heads] & & V_5 \arrow[dll, two heads] \arrow[d, two heads] \arrow[drr, two heads] & & V_7 \arrow[d, two heads] & & V_9 \arrow[dll, two heads] \\
 & & V_1 \cap V_3 \cap V_5 \arrow[drr, two heads] & & V_{11} \arrow[d, two heads] & & V_5 \cap V_7 \cap V_9 \arrow[dll, two heads] & & \\
 & & & & V_1 \cap V_3 \cap V_7 & & & &
\end{tikzcd}$$

EXAMPLE 1.133. The construction in Example 1.132 also works under the hypothesis that $H$ commutes with both $K_1$ and $K_2$ (instead of $K_1 \circ H = H \circ K_2$).

EXERCISE 1.134. For involutions on $V$ as in Example 1.132 satisfying $K_1 \circ K_2 = K_2 \circ K_1$ and $K_1 \circ H = H \circ K_2$, the set

$$\{Id_V, K_1, K_2, K_1 \circ K_2, H, K_1 \circ H, K_2 \circ H, K_1 \circ K_2 \circ H\}$$

is the image of a representation $D_4 \rightsquigarrow \mathrm{End}(V)$, where $D_4$ is the eight-element dihedral group. ■

REMARK 1.135. An example of a vector space over $\mathbb{K} = \mathbb{Q}$ admitting three involutions satisfying the relations of Example 1.132 is $\mathbb{Q}(\sqrt{-1}, \sqrt[4]{2})$, as considered by [**Cox**] (Example 7.3.4), along with diagrams analogous to the above diagram.

THEOREM 1.136. *Given $V$, the following statements (1) to (7) are equivalent, and any implies (8). Further, if $\frac{1}{2} \in \mathbb{K}$, then all eight statements are equivalent.*

(1) *$V$ admits a direct sum of the form $V = U \oplus U$.*
(2) *$V = U_1 \oplus U_2$ and there exist invertible maps $A_1 : U_3 \to U_1$ and $A_2 : U_3 \to U_2$.*
(3) *$V = U' \oplus U''$ and there exists an invertible map $A : U'' \to U'$.*
(4) *$V = U' \oplus U''$ and there exists an involution $K \in \mathrm{End}(V)$ that respects the direct sums $U' \oplus U'' \to U'' \oplus U'$.*
(5) *$V$ admits an idempotent $P \in \mathrm{End}(V)$ and an involution $K \in \mathrm{End}(V)$ such that $P \circ K = K \circ (Id_V - P)$.*
(6) *$V = U' \oplus U''$ and there exists an invertible $H \in \mathrm{End}(V)$ that respects the direct sums $U' \oplus U'' \to U'' \oplus U'$.*
(7) *$V$ admits an idempotent $P \in \mathrm{End}(V)$ and an invertible map $H \in \mathrm{End}(V)$ such that $P \circ H = H \circ (Id_V - P)$.*
(8) *$V$ admits anticommuting involutions $K_1$, $K_2$ (i.e., $K_1 \circ K_2 = -K_2 \circ K_1$).*

PROOF. The implication (1) $\implies$ (2) is canonical: let $U_1 = U_2 = U_3 = U$, and $A_1 = A_2 = Id_U$.

The implication (2) $\implies$ (1) is canonical. Given $V = U_1 \oplus U_2$ with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, Example 1.97 applies. Let $U = U_3$, to get $V = U \oplus U$ with projections $(A_1^{-1} \circ P_1, A_2^{-1} \circ P_2)$ and inclusions $(Q_1 \circ A_1, Q_2 \circ A_2)$. The direct sums are equivalent.

The implication (1) $\implies$ (3) is canonical: let $U' = U'' = U$ and $A = Id_U$.

The implication (2) $\implies$ (3) is canonical: let $U' = U_1$, $U'' = U_2$ and $A = A_1 \circ A_2^{-1}$.

For (3) $\implies$ (1), there are two choices. Given $V = U' \oplus U''$ with projections $(P', P'')$ and inclusions $(Q', Q'')$, one choice is to let $U = U'$. Then $V = U \oplus U$ with projections $(P_1, A \circ P_2)$ and inclusions $(Q_1, Q_2 \circ A^{-1})$. The other choice is to let $U = U''$, with projections $(A^{-1} \circ P_1, P_2)$ and inclusions $(Q_1 \circ A, Q_2)$. As in Example 1.97, either of the two constructions gives a direct sum equivalent to $V = U' \oplus U''$, so they are equivalent to each other.

For (3) $\implies$ (2), there are two choices. One choice is to let $U_3 = U'$, $A_1 = Id_{U'}$, $A_2 = A^{-1}$. Applying the canonical (2) $\implies$ (1) construction then gives projections $(P_1, A \circ P_2)$ as in the first choice of the previous implication. The second choice is to let $U_3 = U''$, $A_1 = A$, $A_2 = Id_{U''}$, which similarly corresponds to the second choice in the previous implication.

The implication (3) $\implies$ (4) is canonical. Given $A : U'' \to U'$, let

$$K = Q'' \circ A^{-1} \circ P' + Q' \circ A \circ P''. \tag{1.19}$$

It is straightforward to check that $K$ is an involution, and $Q' \circ P' \circ K = K \circ Q'' \circ P''$, so $K$ respects the direct sums as in Example 1.94.

The implication (4) $\implies$ (3) is canonical. Given $K$, let $A = P' \circ K \circ Q'' : U'' \to U'$, which by Lemma 1.89 has inverse $A^{-1} = P'' \circ K \circ Q'$.

The implication (4) $\implies$ (6) is canonical: let $H = K$.

For (6) $\implies$ (3), there are two choices. One choice is to let $A = P' \circ H \circ Q'' : U'' \to U'$, so $A^{-1} = P'' \circ H^{-1} \circ Q'$. The canonical involution (1.19) from the implication (3) $\implies$ (4) is then $K = Q'' \circ P'' \circ H^{-1} \circ Q' \circ P' + Q' \circ P' \circ H \circ Q'' \circ P''$. The second choice is to let $A = P' \circ H^{-1} \circ Q''$. This similarly leads to an involution $Q'' \circ P'' \circ H \circ Q' \circ P' + Q' \circ P' \circ H^{-1} \circ Q'' \circ P''$, which, unless $H$ is an involution, may be different from the involution from the first choice.

For (4) $\implies$ (5), and for (6) $\implies$ (7), there are two choices: $P = Q' \circ P'$, or $P = Q'' \circ P''$. This choice between two idempotents was already mentioned in Example 1.114.

Conversely, for (5) $\implies$ (4) (and similarly for (7) $\implies$ (6)), there are two choices. For $U' = \ker(P)$ and $U'' = P(V)$, as in Example 1.113, there are two ways to form a direct sum: $V = U' \oplus U''$ or $V = U'' \oplus U'$. The map $K$ (similarly $H$) respects the direct sums as in Lemma 1.115.

For (4) $\implies$ (8), which does not require $\frac{1}{2} \in \mathbb{K}$, there are two choices (assuming the ordering of the pair $K_1$, $K_2$ does not matter). Given $K$, let $K_1 = K$. One choice is to let $K_2 = Q' \circ P' - Q'' \circ P''$, as in Example 1.122. It follows from $K \circ Q' \circ P' = Q'' \circ P'' \circ K$ that $K_1 \circ K_2 = -K_2 \circ K_1$. The second choice is to let $K_2 = -Q' \circ P' + Q'' \circ P''$.

Similarly for (5) $\implies$ (8), there are two choices. Given $K$, let $K_1 = K$. One choice is to let $K_2 = 2 \cdot P - Id_V$, as in Lemma 1.123. The second choice is to let $K_2 = Id_V - 2 \cdot P$.

For (8) $\implies$ (4) using $\frac{1}{2} \in \mathbb{K}$, the involution $K_1$ produces a direct sum $V = V_1 \oplus V_2$ as in Lemma 1.119, with projections $P_1 = \frac{1}{2}(Id_V + K_1)$, $P_2 = \frac{1}{2}(Id_V - K_1)$ (the order of the direct sum could be chosen the other way, $V_2 \oplus V_1$). By Lemma 1.127, $K = K_2$ satisfies (4). In this case, the invertible map from (3) is, by Lemma 1.89, the composite

$$A = P_1 \circ K_2 \circ Q_2 : V_2 \to V_1, \tag{1.20}$$

with inverse $P_2 \circ K_2 \circ Q_1 : V_1 \to V_2$. Another choice for (8) $\implies$ (4) is to use $K_2$ to produce a different direct sum, and then let $K = K_1$. ∎

THEOREM 1.137. *Given $\frac{1}{2} \in \mathbb{K}$ and two involutions $K, K' \in \mathrm{End}(V)$, which produce direct sums $V = V_1 \oplus V_2$, $V = V_1' \oplus V_2'$ as in Lemma 1.119, if $K$ and $K'$ anticommute, then for $i = 1, 2$, $I = 1, 2$, and $\beta \in \mathbb{K}$, $\beta \neq 0$, the map*

$$\beta \cdot P_I' \circ Q_i : V_i \to V_I'$$

*is invertible.*

PROOF. Consider $P_I' \circ Q_i : V_i \to V_I'$ and $P_i \circ Q_I' : V_I' \to V_i$. Then

$$P_I' \circ Q_i \circ P_i \circ Q_I' = P_I' \circ \frac{1}{2} \cdot (Id_V \pm K) \circ Q_I'.$$

Since $K$ respects the direct sums $V_1' \oplus V_2' \to V_2' \oplus V_1'$ by Lemma 1.127, $P_I' \circ K \circ Q_I' = 0_{\mathrm{End}(V_I')}$ by Lemma 1.87. In the other order,

$$P_i \circ Q_I' \circ P_I' \circ Q_i = P_i \circ \frac{1}{2} \cdot (Id_V \pm K') \circ Q_i,$$

and similarly, $P_i \circ K' \circ Q_i = 0_{\mathrm{End}(V_i)}$.

Since $P_I' \circ Q_I' = Id_{V_I'}$ and $P_i \circ Q_i = Id_{V_i}$, the conclusion is that for any scalar $\beta \in \mathbb{K}$, $\beta \neq 0$, the map $\beta \cdot P_I' \circ Q_i$ has inverse $\frac{2}{\beta} \cdot P_i \circ Q_I'$. ∎

LEMMA 1.138. *Given involutions $K_1$, $K_2$, $K_3 \in \mathrm{End}(V)$, any pair of two of the following three statements implies the remaining one.*

(1) *$K_3$ commutes with $K_1 \circ K_2$.*
(2) *$K_3$ anticommutes with $K_1$.*
(3) *$K_3$ anticommutes with $K_2$.*

∎

EXERCISE 1.139. If $K_1$, $K_2$, $K_3$ are involutions such that $K_1$ and $K_2$ commute and $K_3$ satisfies the three conditions from Lemma 1.138, then the set

$$\{\pm Id_V, \pm K_1, \pm K_2, \pm K_3, \pm K_2 \circ K_3, \pm K_1 \circ K_3, \pm K_1 \circ K_2, \pm K_1 \circ K_2 \circ K_3\}$$

is the image of a representation $D_4 \times \mathbb{Z}_2 \rightsquigarrow \mathrm{End}(V)$, where $D_4$ is the eight-element dihedral group and $\mathbb{Z}_2$ is the two-element group. ∎

For $\frac{1}{2} \in \mathbb{K}$ and commuting involutions $K_1$, $K_2$, recall the direct sums $V = V_1 \oplus V_2$, $V = V_3 \oplus V_4$, $V = V_5 \oplus V_6$ from Lemma 1.129 produced by $K_1$, $K_2$, $K_1 \circ K_2$. Further, suppose $K_3$ is another involution satisfying the three conditions from Lemma 1.138, and let $V = V_7 \oplus V_8$ and $V = V_9 \oplus V_{10}$ be the direct sums produced by the involutions $K_3$ and $K_1 \circ K_2 \circ K_3$. Theorem 1.130 applies to $V_5$ twice: first, to the pair $K_1$, $K_2$ to get the canonical involution $P_5 \circ K_1 \circ Q_5$ from (1.15) producing $V_5 = V_5' \oplus V_5''$ with $V_5' = V_1 \cap V_3 \cap V_5$, and second, to the other pair $K_1 \circ K_2$, $K_3$ to get another involution $P_5 \circ K_3 \circ Q_5 = P_5 \circ K_1 \circ K_2 \circ K_3 \circ Q_5$ as in (1.15), producing another direct sum $V_5 = V_5''' \oplus V_5''''$, with $V_5''' = V_5 \cap V_7 \cap V_9$.

COROLLARY 1.140. *Given $\frac{1}{2} \in \mathbb{K}$, $0 \neq \beta \in \mathbb{K}$, commuting involutions $K_1$, $K_2$, and an involution $K_3$ as in Lemma 1.138, the map*

$$\beta \cdot P_5''' \circ Q_5' : V_5' \to V_5'''$$

*is invertible.*

PROOF. $Q'_5$ is as in the Proof of Theorem 1.130. The projection $P'''_5 : V_5 \twoheadrightarrow V'''_5$ is from the direct sum $V_5 = V'''_5 \oplus V''''_5$ produced by $P_5 \circ K_3 \circ Q_5$. The involutions $P_5 \circ K_1 \circ Q_5$, $P_5 \circ K_3 \circ Q_5 \in \mathrm{End}(V_5)$ anticommute, and Theorem 1.137 applies. ∎

Using a step analogous to (1.17), the output of the above invertible map, for input $v \in V'_5$, can be written as:

$$\begin{array}{rcl} \beta \cdot P'''_5 \circ Q'_5 : v & \mapsto & \beta \cdot (P'''_5 \circ Q'_5)(v) \\ & = & Q_5(Q'''_5(\beta \cdot (P'''_5 \circ (P_5 \circ Q_5) \circ Q'_5)(v))) \\ & = & \beta \cdot ((Q_5 \circ Q'''_5 \circ P'''_5 \circ P_5) \circ Q_5 \circ Q'_5)(v) \\ & = & \beta \cdot ((Q_7 \circ P_7 \circ Q_5 \circ P_5) \circ Q_5 \circ Q'_5)(v) \\ & = & \beta \cdot (Q_7 \circ P_7 \circ Q_5 \circ Q'_5)(v) \\ & = & \dfrac{\beta}{2} \cdot (v + K_3(v)). \end{array} \tag{1.21}$$

Corollary 1.140 could be re-stated as constructing an invertible map between these subspaces of $V_5 = \{v \in V : v = (K_1 \circ K_2)(v)\}$:

$$\{v = K_1(v) = K_2(v)\} \to \{v = K_3(v) = (K_1 \circ K_2 \circ K_3)(v)\}.$$

Two more subspaces of $V_5$, from Theorem 1.130, are:

$$\begin{array}{rcl} V''_5 & = & \{v \in V : v = -K_1(v) = -K_2(v)\} = V_2 \cap V_4, \\ V''''_5 & = & \{v \in V : v = -K_3(v) = -(K_1 \circ K_2 \circ K_3)(v)\} = V_8 \cap V_{10}, \end{array}$$

and Theorem 1.137 also gives a construction of invertible maps: $V'_5 \to V''''_5$, $V''_5 \to V'''_5$, and $V''_5 \to V''''_5$.

EXAMPLE 1.141. Given any spaces $V$ and $W$, and an involution $K$ on $V$, the map $[Id_W \otimes K]$ is an involution on $W \otimes V$. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by $[Id_W \otimes K]$ has projections $\frac{1}{2} \cdot (Id_{W\otimes V} \pm [Id_W \otimes K])$. For the direct sum $V = V_1 \oplus V_2$ as in Lemma 1.119 with inclusions $Q_1$, $Q_2$, there is also a direct sum $W \otimes V = W \otimes V_1 \oplus W \otimes V_2$ as in Example 1.81, with projections $[Id_W \otimes \frac{1}{2} \cdot (Id_V \pm K)]$ and inclusions $[Id_W \otimes Q_i]$. The two constructions lead to the same formula for the projection maps, so the projections are canonical and $K$ produces a direct sum $W \otimes V = W \otimes V_1 \oplus W \otimes V_2$. The space $W \otimes V_1$ is a subspace of $W \otimes V$, equal to the fixed point set of $[Id_W \otimes K]$, with inclusion map $[Id_W \otimes Q_1]$, and similarly $W \otimes V_2$ is the fixed point subspace of $-[Id_W \otimes K]$. The space $V \otimes W$ admits an analogous involution and direct sum.

EXAMPLE 1.142. Given any spaces $U$, $W$ with involutions $K_U$ on $U$ and $K_W$ on $W$, the involutions $[Id_U \otimes K_W]$ and $[K_U \otimes Id_W]$ on $U \otimes W$ commute, so Lemma 1.129 applies, and if $\frac{1}{2} \in \mathbb{K}$, then Lemma 1.128 and Theorem 1.130 apply. For the direct sums $U = U_1 \oplus U_2$ and $W = W_1 \oplus W_2$ produced as in Lemma 1.119, $[K_U \otimes Id_W]$ respects the direct sum $U \otimes W_1 \oplus U \otimes W_2$ from Example 1.141; the induced involution on $U \otimes W_1$ is exactly $[K_U \otimes Id_{W_1}]$, so $U \otimes W_1$ admits a direct sum $U_1 \otimes W_1 \oplus U_2 \otimes W_1$. Similarly, $[Id_U \otimes K_W]$ induces an involution on $U_1 \otimes W$ and a direct sum $U_1 \otimes W_1 \oplus U_1 \oplus W_2$. The subspace $U_1 \otimes W_1$ appears in two different ways, but there is no conflict in naming it: by Theorem 1.130, $U_1 \otimes W_1 = (U \otimes W_1) \cap (U_1 \otimes W)$.

EXAMPLE 1.143. Given any spaces $V$ and $W$, and an involution $K$ on $V$, the map $\mathrm{Hom}(Id_W, K)$ is an involution on $\mathrm{Hom}(W, V)$. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by $\mathrm{Hom}(Id_W, K)$ has projections

$$\frac{1}{2} \cdot (Id_{\mathrm{Hom}(W,V)} \pm \mathrm{Hom}(Id_W, K)) : A \mapsto \frac{1}{2} \cdot (A \pm K \circ A).$$

For the direct sum $V = V_1 \oplus V_2$ as in Lemma 1.119, there is also a direct sum $\mathrm{Hom}(W, V) = \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(W, V_2)$ as in Example 1.82, with projections

$$\mathrm{Hom}(Id_W, P_i) : \mathrm{Hom}(W, V) \to \mathrm{Hom}(W, V_i) : A \mapsto P_i \circ A = \frac{1}{2} \cdot (Id_V \pm K) \circ A.$$

The two constructions lead to the same formula for the projection maps. The only difference is in the target space: the fixed point set of $\mathrm{Hom}(Id_W, K)$ is the set of maps $A : W \to V$ such that $A = K \circ A$, while the image of the projection $\mathrm{Hom}(Id_W, P_1)$ is a set of maps with domain $W$ and target $V_1 = \{v \in V : v = K(v)\}$, which is a subspace of $V$. It will not cause any problems to consider $\mathrm{Hom}(W, V_i)$ as a subspace of $\mathrm{Hom}(W, V)$; more precisely, in the case where $V = V_1 \oplus V_2$ is a direct sum produced by an involution, the map $\mathrm{Hom}(Id_W, Q_i)$ from Example 1.82 can be regarded as a subspace inclusion as in Lemma 1.119, so $A$ and $Q_i \circ A$ are identified. Then the above two direct sum constructions have the same projections and inclusions, so the projections are canonical and $K$ produces a direct sum $\mathrm{Hom}(W, V) = \mathrm{Hom}(W, V_1) \oplus \mathrm{Hom}(W, V_2)$.

EXAMPLE 1.144. Given any spaces $V$ and $W$, and an involution $K$ on $V$, the map $\mathrm{Hom}(K, Id_W)$ is an involution on $\mathrm{Hom}(V, W)$. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by $\mathrm{Hom}(K, Id_W)$ has projections

$$\frac{1}{2} \cdot (Id_{\mathrm{Hom}(V,W)} \pm \mathrm{Hom}(K, Id_W)) : A \mapsto \frac{1}{2} \cdot (A \pm A \circ K). \tag{1.22}$$

For the direct sum $V = V_1 \oplus V_2$ as in Lemma 1.119, there is also a direct sum $\mathrm{Hom}(V, W) = \mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W)$ as in Example 1.83, with projections

$$\mathrm{Hom}(Q_i, Id_W) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V_i, W) : A \mapsto A \circ Q_i.$$

Unlike Example 1.143, the two constructions lead to different formulas for the projection maps. The fixed point set of $\mathrm{Hom}(K, Id_W)$ is the set of maps $A : V \to W$ such that $A = A \circ K$, while the image of the projection $\mathrm{Hom}(Q_1, Id_W)$ is a set of maps with domain $V_1$ and target $W$ — which does not look like a subspace of $\mathrm{Hom}(V, W)$. The conclusion is that the two direct sum constructions are different. However, they are equivalent, as in Definition 1.96. Checking statements (2) and (3) of Lemma 1.95,

$$\mathrm{Hom}(P_i, Id_W) \circ \mathrm{Hom}(Q_i, Id_W) : A \mapsto A \circ Q_i \circ P_i = A \circ \frac{1}{2} \cdot (Id_V \pm K),$$

which is the same as (1.22).

EXAMPLE 1.145. Given any spaces $U$, $V$, $W$, with involutions $K_U$ on $U$, $K_V$ on $V$, and $\mathrm{Hom}(K_V, Id_W)$ on $\mathrm{Hom}(V, W)$ as in Example 1.144, suppose $\frac{1}{2} \in \mathbb{K}$ and $H : U \to \mathrm{Hom}(V, W)$ satisfies $\mathrm{Hom}(K_V, Id_W) \circ H = H \circ K_U$. Let $U = U_1 \oplus U_2$ be the direct sum produced by $K_U$, and consider the direct sum on $\mathrm{Hom}(V, W)$ produced by $\mathrm{Hom}(K_V, Id_W)$ as in (1.22) from Example 1.144. Then, by Lemma 1.126, $H$ respects the direct sums:

$$H : U_1 \oplus U_2 \to \{A : V \to W : A \circ K_V = A\} \oplus \{A : V \to W : A \circ K_V = -A\}.$$

Let $V = V_1 \oplus V_2$ be the direct sum produced by $K_V$; then by Lemma 1.98, $H$ also respects the other, equivalent direct sum on $\mathrm{Hom}(V, W)$ from Example 1.144:

$$H : U_1 \oplus U_2 \to \mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W).$$

EXAMPLE 1.146. Given $\frac{1}{2} \in \mathbb{K}$, any spaces $U$ and $W$, and involutions $K_U \in \mathrm{End}(U)$ and $K_W \in \mathrm{End}(W)$, let $U = U_1 \oplus U_2$ be the direct sum produced by $K_U$, with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, and let $W = W_1 \oplus W_2$ be the direct sum produced by $K_W$, with data $(P_1', P_2')$, $(Q_1', Q_2')$. Then there are commuting involutions $\mathrm{Hom}(Id_U, K_W)$, $\mathrm{Hom}(K_U, Id_W)$, on $\mathrm{Hom}(U, W)$, and their composite $\mathrm{Hom}(K_U, K_W)$ is another involution. Theorem 1.130 applies, and we re-use its $V_1, \ldots, V_6$ notation for the produced direct sums. As in Example 1.143, there is a direct sum

$$\begin{aligned}
\mathrm{Hom}(U, W) &= \{A = K_W \circ A\} \oplus \{A = -K_W \circ A\} \\
&= \mathrm{Hom}(U, W_1) \oplus \mathrm{Hom}(U, W_2) \\
&= V_1 \oplus V_2.
\end{aligned}$$

As in Example 1.144, there are two different but equivalent direct sums, the first is

$$\mathrm{Hom}(U, W) = \{A = A \circ K_U\} \oplus \{A = -A \circ K_U\} = V_3 \oplus V_4,$$

with projections and inclusions denoted $(P_3, P_4)$, $(Q_3, Q_4)$, and the second is

$$\mathrm{Hom}(U, W) = \mathrm{Hom}(U_1, W) \oplus \mathrm{Hom}(U_2, W),$$

with projections and inclusions

$$(\mathrm{Hom}(Q_1, Id_W), \mathrm{Hom}(Q_2, Id_W)), \ (\mathrm{Hom}(P_1, Id_W), \mathrm{Hom}(P_2, Id_W)).$$

From Lemma 1.99, there are canonical, invertible maps

$$\begin{aligned}
P_3 \circ \mathrm{Hom}(P_1, Id_W) &: \mathrm{Hom}(U_1, W) \to V_3, \\
\mathrm{Hom}(Q_1, Id_W) \circ Q_3 &: V_3 \to \mathrm{Hom}(U_1, W).
\end{aligned}$$

There is also the direct sum produced by the composite involution,

$$\begin{aligned}
\mathrm{Hom}(U, W) &= \{A = K_W \circ A \circ K_U\} \oplus \{A = -K_W \circ A \circ K_U\} \\
&= V_5 \oplus V_6.
\end{aligned}$$

It follows from Theorem 1.130 that $V_1$, $V_3$, and $V_5$ admit canonical involutions and direct sums. For example, $P_3 \circ \mathrm{Hom}(Id_U, K_W) \circ Q_3$ is the involution on $V_3$, and it produces the direct sum $V_3 = V_3' \oplus V_3''$, where $V_3' = V_1 \cap V_3 \cap V_5$. The above invertible map $\mathrm{Hom}(Q_1, Id_W) \circ Q_3 : V_3 \to \mathrm{Hom}(U_1, W)$ satisfies

$$\begin{aligned}
& \mathrm{Hom}(Id_{U_1}, K_W) \circ (\mathrm{Hom}(Q_1, Id_W) \circ Q_3) \\
= \ & \mathrm{Hom}(Q_1, K_W) \circ Q_3 \\
= \ & (\mathrm{Hom}(Q_1, Id_W) \circ Q_3) \circ (P_3 \circ \mathrm{Hom}(Id_U, K_W) \circ Q_3),
\end{aligned}$$

so by Lemma 1.126, it respects the direct sums

$$V_3' \oplus V_3'' \to \mathrm{Hom}(U_1, W_1) \oplus \mathrm{Hom}(U_1, W_2).$$

By Lemma 1.89, there is a canonical, invertible map from

$$V_3' = V_1 \cap V_3 \cap V_5 = \{A : U \to W : A = K_W \circ A = A \circ K_U = K_W \circ A \circ K_U\}$$

to $\mathrm{Hom}(U_1,W_1)$, specifically, the map

$$\begin{aligned} A &\mapsto P_1'\circ(Q_3(Q_3'(A)))\circ Q_1 \\ &= P_1'\circ A\circ Q_1=\frac{1}{2}\cdot(Id_W+K_W)\circ A\circ Q_1=A\circ Q_1. \end{aligned}$$

The inverse is defined for $B\in\mathrm{Hom}(U_1,W_1)$ by

$$B \mapsto P_3'(P_3(Q_1'\circ B\circ P_1)),$$

which, since $Q_1'\circ B\circ P_1$ is an element of the subspace $V_1\cap V_3$, simplifies to

$$Q_1'\circ B\circ P_1=B\circ P_1=B\circ\frac{1}{2}\cdot(Id_U+K_U).$$

EXAMPLE 1.147. For $\frac{1}{2}\in\mathbb{K}$, and involutions $K_U$ on $U$ and $K_W$ on $W$ as in the previous Example, suppose $K_3$ is an involution on $\mathrm{Hom}(U,W)$ that commutes with $\mathrm{Hom}(K_U,K_W)$ and anticommutes with either $\mathrm{Hom}(K_U,Id_W)$ or $\mathrm{Hom}(Id_U,K_W)$. Then Lemma 1.138 and Corollary 1.140 apply. Continuing with the $V_1,\dots,V_6$ notation from Theorem 1.130 and Example 1.146, and also the $V_5=V_5'''\oplus V_5''''$ and $V_7,\dots,V_{10}$ notation from Corollary 1.140, the result of the Corollary is that for $0\neq\beta\in\mathbb{K}$, there is an invertible map:

$$\beta\cdot P_5'''\circ Q_5':V_5' \to V_5'''$$

which maps

$$\{A\in\mathrm{Hom}(U,W):A=A\circ K_U=K_W\circ A\}$$

to

$$\{A\in\mathrm{Hom}(U,W):A=K_W\circ A\circ K_U=K_3(A)\}.$$

There is also the canonical, invertible map from Example 1.146,

$$P_3'\circ(P_3\circ\mathrm{Hom}(P_1,Id_W))\circ\mathrm{Hom}(Id_{U_1},Q_1'),$$

which maps $\mathrm{Hom}(U_1,W_1)$ to $V_3'=V_5'$. The composite of these maps is an invertible map $\mathrm{Hom}(U_1,W_1)\to V_5'''$:

$$(\beta\cdot P_5'''\circ Q_5')\circ(P_3'\circ(P_3\circ\mathrm{Hom}(P_1,Id_W))\circ\mathrm{Hom}(Id_{U_1},Q_1')).$$

For $B\in\mathrm{Hom}(U_1,W_1)$, its output in $V_5'''\subseteq\mathrm{Hom}(U,W)$ under the above map simplifies as follows, using the equality of subspace inclusions $Q_5\circ Q_5'=Q_3\circ Q_3'$ and steps similar to (1.21):

$$\begin{aligned} B &\mapsto ((\beta\cdot P_5'''\circ Q_5')\circ(P_3'\circ P_3\circ\mathrm{Hom}(P_1,Q_1')))(B) \\ &= Q_5(Q_5'''(\beta\cdot(P_5'''\circ(P_5\circ Q_5)\circ Q_5'\circ P_3'\circ P_3\circ\mathrm{Hom}(P_1,Q_1'))(B))) \\ &= \beta\cdot((Q_5\circ Q_5'''\circ P_5'''\circ P_5)\circ(Q_3\circ Q_3'\circ P_3'\circ P_3)\circ\mathrm{Hom}(P_1,Q_1'))(B) \\ &= \beta\cdot((Q_7\circ P_7\circ Q_5\circ P_5)\circ(Q_5\circ P_5\circ Q_3\circ P_3)\circ\mathrm{Hom}(P_1,Q_1'))(B), \end{aligned}$$

which, since $Q_1'\circ B\circ P_1$ is an element of the subspace $V_3\cap V_5$, simplifies to

$$\beta\cdot(Q_7\circ P_7)(Q_1'\circ B\circ P_1)=\frac{\beta}{2}\cdot(Q_1'\circ B\circ P_1+K_3(Q_1'\circ B\circ P_1)).$$

The inverse map

$$\{A=K_W\circ A\circ K_U=K_3(A)\}\to\mathrm{Hom}(U_1,W_1)$$

is the composite:

$$\begin{aligned}
& (\mathrm{Hom}(Id_{U_1}, P_1') \circ (\mathrm{Hom}(Q_1, Id_W) \circ Q_3) \circ Q_3') \circ (\frac{2}{\beta} \cdot P_5' \circ Q_5''') \\
= \quad & \frac{2}{\beta} \cdot \mathrm{Hom}(Q_1, P_1') \circ (Q_5 \circ Q_5') \circ P_5' \circ (P_5 \circ Q_5) \circ Q_5''' \\
= \quad & \frac{2}{\beta} \cdot \mathrm{Hom}(Q_1, P_1') \circ (Q_3 \circ P_3 \circ Q_5 \circ P_5) \circ Q_5 \circ Q_5''' \\
= \quad & \frac{2}{\beta} \cdot \mathrm{Hom}(Q_1, P_1') \circ Q_3 \circ P_3 \circ Q_5 \circ Q_5''',
\end{aligned}$$

which acts as $A \mapsto \frac{2}{\beta} \cdot P_1' \circ (\frac{1}{2} \cdot (A + A \circ K_U)) \circ Q_1 = \frac{2}{\beta} \cdot A \circ Q_1$.

CHAPTER 2

# A Survey of Trace Elements

## 2.1. Endomorphisms: the scalar valued trace

In the following diagram, the canonical maps $k_{VV}$, $e_{VV}$, and $f_{VV}$ are abbreviated $k$, $e$, $f$, and the double duality $d_{V^*\otimes V}$ is abbreviated $d$.

Lemma 2.1. *For any vector space $V$, the following diagram is commutative.*

$$\begin{tikzcd}
 & & (V^*\otimes V)^* & & \\
\mathrm{End}(V) \arrow[urr, "e"] & & & & \mathrm{End}(V)^* \arrow[ull, "k^*"'] \\
 & V^*\otimes V \arrow[ul, "k"] \arrow[uur, "f"] \arrow[rr, "d"] & & (V^*\otimes V)^{**} \arrow[uul, "f^*"'] \arrow[ur, "e^*"'] &
\end{tikzcd}$$

Proof. The left triangle is commutative by the definition of $f$ from Notation 1.69. The right triangle is commutative by Lemma 1.6, and the middle by Lemma 1.71. ∎

The spaces $\mathrm{End}(V)$ and $(V^*\otimes V)^*$ each have the interesting property of containing a distinguished element, which is nonzero when $V$ has nonzero dimension. The identity $Id_V : v \mapsto v$ is the distinguished element of $\mathrm{End}(V)$.

Definition 2.2. The distinguished element of $(V^*\otimes V)^*$ is the evaluation map, $Ev_V : \phi\otimes v \mapsto \phi(v)$.

The two distinguished elements are related by $e : Id_V \mapsto Ev_V$:

$$(e(Id_V))(\phi\otimes v) = \phi(Id_V(v)) = Ev_V(\phi\otimes v). \tag{2.1}$$

Definition 2.3. For finite-dimensional $V$, define the trace by

$$Tr_V = (k^*)^{-1}(Ev_V) \in \mathrm{End}(V)^*.$$

This distinguished element is the output of either of the two previously mentioned distinguished elements under any path of maps in the above diagram leading to $\mathrm{End}(V)^*$, by Lemma 2.1, and the fact that all the arrows are invertible when $V$ is finite-dimensional. At least one arrow in any path taking $Id_V$ or $Ev_V$ to $Tr_V$ is the inverse of one of the arrows indicated in the diagram.

Remark 2.4. Using Definition 2.3 as the definition of trace, so that $Tr_V(A) = Ev_V(k^{-1}(A))$, is exactly the approach of [**MB**], [**B**] §II.4.3, and [**K**] §II.3. In [$\mathbf{G_2}$] §I.8, this formula is stated as a consequence of a different definition of trace.

LEMMA 2.5. *For finite-dimensional $V$, and $H \in \mathrm{End}(V)$,*

$$Tr_{V^*}(H^*) = Tr_V(H).$$

PROOF. In this case, $H^*$ is $t_{VV}(H)$. In the following diagram, $t_{VV}$, $k_{V^*V^*}$, $e_{V^*V^*}$, $f_{V^*V^*}$, and $d_{V^{**}\otimes V^*}$ are abbreviated $t$, $k'$, $e'$, $f'$, and $d'$. There is also a map $p : V^* \otimes V \to V^{**} \otimes V^*$ from Notation 1.72, and $p^*$ maps the distinguished element $Ev_{V^*} \in (V^{**} \otimes V^*)^*$ to $Ev_V$:

$$(p^*(Ev_{V^*}))(\phi \otimes v) = Ev_{V^*}((d_V(v)) \otimes \phi) = (d_V(v))(\phi) = \phi(v) = Ev_V(\phi \otimes v).$$

$$\begin{tikzcd}
 & & (V^* \otimes V)^* & & \\
\mathrm{End}(V) \arrow[urr, "e"] \arrow[ddd, "t"] & & & & \mathrm{End}(V)^* \arrow[ull, "k^*"'] \\
 & V^* \otimes V \arrow[ul, "k"] \arrow[uur, "f"] \arrow[rr, "d"] \arrow[ddd, "p"] & & (V^* \otimes V)^{**} \arrow[uul, "f^*"'] \arrow[ur, "e^*"] \arrow[ddd, "p^{**}"] & \\
 & & (V^{**} \otimes V^*)^* \arrow[uuu, "p^*"] & & \\
\mathrm{End}(V^*) \arrow[urr, "e'"] & & & & \mathrm{End}(V^*)^* \arrow[ull, "k'^*"'] \arrow[uuu, "t^*"] \\
 & V^{**} \otimes V^* \arrow[ul, "k'"] \arrow[uur, "f'"] \arrow[rr, "d'"] & & (V^{**} \otimes V^*)^{**} \arrow[uul, "f'^*"'] \arrow[ur, "e'^*"] &
\end{tikzcd}$$

Some of the squares in the diagram are commutative, for example, $k' \circ p = t \circ k$ from Lemma 1.75, and then $p^* \circ (k'^*) = k^* \circ t^*$ by Lemma 1.6, and this is enough to give the result:

$$t^*(Tr_{V^*}) = (t^* \circ (k'^*)^{-1})(Ev_{V^*}) = ((k^*)^{-1} \circ p^*)(Ev_{V^*}) = (k^*)^{-1}(Ev_V) = Tr_V.$$

∎

The equality $q \circ t = e$ from Lemma 1.58, which could fit in the back left square of the above diagram, shows that $q$ maps the distinguished element $Id_{V^*}$ to $Ev_V$. So, there is another formula for the trace,

$$Tr_V = ((k^*)^{-1} \circ q)(Id_{V^*}). \tag{2.2}$$

LEMMA 2.6. *For maps $A : V \to U$ and $B : U \to V$ between vector spaces of finite, but possibly different, dimensions, $Tr_V(B \circ A) = Tr_U(A \circ B)$.*

PROOF. Abbreviated names for maps are used again in the following diagram, with primes in the lower pentagon.

$$\begin{tikzcd}
 & & (V^*\otimes V)^* \arrow[ddd, shift left, "{[A^*\otimes B]^*}"] & & \\
\mathrm{End}(V) \arrow[urr, "e"] \arrow[ddd, shift left, "{\mathrm{Hom}(B,A)}"] & & & & \mathrm{End}(V)^* \arrow[ull, "k^*"'] \arrow[ddd, shift left, "{\mathrm{Hom}(A,B)^*}"] \\
 & V^*\otimes V \arrow[ul, "k"] \arrow[uur, "f"] \arrow[rr, "d"] \arrow[ddd, shift left, "{[B^*\otimes A]}"] & & (V^*\otimes V)^{**} \arrow[uul, "f^*"] \arrow[ur, "e^*"] \arrow[ddd, shift left, "{[B^*\otimes A]^{**}}"] & \\
 & & (U^*\otimes U)^* \arrow[uuu, shift left, "{[B^*\otimes A]^*}"] & & \\
\mathrm{End}(U) \arrow[urr, "e'"] \arrow[uuu, shift left, "{\mathrm{Hom}(A,B)}"] & & & & \mathrm{End}(U)^* \arrow[ull, "k'^*"'] \arrow[uuu, shift left, "{\mathrm{Hom}(B,A)^*}"] \\
 & U^*\otimes U \arrow[ul, "k'"] \arrow[uur, "f'"] \arrow[rr, "d'"] \arrow[uuu, shift left, "{[A^*\otimes B]}"] & & (U^*\otimes U)^{**} \arrow[uul, "f'^*"] \arrow[ur, "e'^*"] \arrow[uuu, shift left, "{[A^*\otimes B]^{**}}"] &
\end{tikzcd}$$

Some of the squares in the diagram are commutative. In the back left square with upward arrows, it follows from Lemma 1.57 (and can also be checked directly) that:

$$e \circ \mathrm{Hom}(A, B) = [B^* \otimes A]^* \circ e'.$$

For a front left square, by Lemma 1.62,

$$\mathrm{Hom}(B, A) \circ k = k' \circ [B^* \otimes A],$$

then by Lemma 1.6, $k^* \circ \mathrm{Hom}(B, A)^* = [B^* \otimes A]^* \circ k'^*$, corresponding to a back right square. The claimed equality follows from the following steps, including Lemma 2.1:

$$\begin{aligned}
Tr_U(A \circ B) &= Tr_U(\mathrm{Hom}(B, A)(Id_V)) = (\mathrm{Hom}(B, A)^*(Tr_U))(Id_V) \\
&= ((\mathrm{Hom}(B, A)^* \circ (k'^*)^{-1} \circ e')(Id_U))(Id_V) \\
&= ((k^*)^{-1} \circ [B^* \otimes A]^* \circ e')(Id_U))(Id_V) \\
&= ((k^*)^{-1} \circ e \circ \mathrm{Hom}(A, B))(Id_U))(Id_V) \\
&= ((e^* \circ d \circ k^{-1} \circ \mathrm{Hom}(A, B))(Id_U))(Id_V) \\
&= (d(k^{-1}(\mathrm{Hom}(A, B)(Id_U))))(e(Id_V)) \\
&= (e(Id_V))(k^{-1}(\mathrm{Hom}(A, B)(Id_U))) \\
&= (((k^*)^{-1} \circ e)(Id_V))(\mathrm{Hom}(A, B)(Id_U)) = Tr_V(B \circ A).
\end{aligned}$$

∎

EXAMPLE 2.7. In the case $V = \mathbb{K}$, $k(Id_{\mathbb{K}} \otimes 1) = Id_{\mathbb{K}} \in \mathrm{End}(\mathbb{K}) = \mathbb{K}^*$. The trace is $Tr_{\mathbb{K}} = e^*(d(Id_{\mathbb{K}} \otimes 1)) \in \mathbb{K}^{**}$, and for $\phi \in \mathbb{K}^*$, $Tr_{\mathbb{K}}(\phi) = (e(\phi))(Id_{\mathbb{K}} \otimes 1) = Id_{\mathbb{K}}(\phi(1)) = \phi(1)$. So, $Tr_{\mathbb{K}} = d_{\mathbb{K}}(1)$, and in particular, $Tr_{\mathbb{K}}(Id_{\mathbb{K}}) = 1$.

EXAMPLE 2.8. If $V$ is finite-dimensional and admits a direct sum of the form $V = \mathbb{K} \oplus U$, with projection $P_1 : V \to \mathbb{K}$ and $Q_1 : \mathbb{K} \to V$, then by Lemma 2.6 and Example 2.7, $Tr_V(Q_1 \circ P_1) = Tr_{\mathbb{K}}(P_1 \circ Q_1) = Tr_{\mathbb{K}}(Id_{\mathbb{K}}) = 1$. Similarly, if $V$ is a direct sum of finitely many copies of $\mathbb{K}$, $V = \mathbb{K} \oplus \mathbb{K} \oplus \cdots \oplus \mathbb{K}$, then $Tr_V(Id_V) = Tr_V(\Sigma Q_i \circ P_i) = \Sigma Tr_{\mathbb{K}}(P_i \circ Q_i) = \Sigma 1$.

EXAMPLE 2.9. Assume $Tr_V(Id_V) \neq 0$. Let $\mathrm{End}_0(V)$ denote the kernel of $Tr_V$, i.e., the subspace of trace 0 endomorphisms. Recall from Lemma 1.100 that there exists a direct sum $\mathrm{End}(V) = \mathbb{K} \oplus \mathrm{End}_0(V)$, and in particular, there exist constants $\alpha$, $\beta \in \mathbb{K}$ so that $\alpha \cdot \beta \cdot Tr_V(Id_V) = 1$, and a direct sum is defined by

$$\begin{aligned} P_1^\alpha &= \alpha \cdot Tr_V, \\ Q_1^\beta : \mathbb{K} &\to \mathrm{End}(V) : \gamma \mapsto \beta \cdot \gamma \cdot Id_V, \\ P_2 &= Id_{\mathrm{End}(V)} - Q_1^\beta \circ P_1^\alpha, \end{aligned} \tag{2.3}$$

and the subspace inclusion map $Q_2 : \mathrm{End}_0(V) \to \mathrm{End}(V)$. Such a direct sum admits a free parameter and is generally not unique, but since $Id_V$ is a canonical element of $\mathrm{End}(V)$ and is not in $\ker(Tr_V)$ by assumption, Lemma 1.101 applies, and any choice of constants $\alpha$, $\beta$ leads to an equivalent direct sum. So, any endomorphism $H$ can be written as a sum of a scalar multiple of $Id_V$, and a trace zero endomorphism:

$$H = \frac{Tr_V(H)}{Tr_V(Id_V)} \cdot Id_V + \left( H - \frac{Tr_V(H)}{Tr_V(Id_V)} \cdot Id_V \right),$$

and this decomposition of $H$ is canonical.

THEOREM 2.10. *For $V$ finite-dimensional, and $A \in \mathrm{End}(V)$,*

$$Tr_V(A) = (Ev_V \circ [Id_{V^*} \otimes A] \circ k^{-1})(Id_V).$$

PROOF. By Lemma 1.62,

$$[Id_{V^*} \otimes A] \circ k^{-1} = [Id_V^* \otimes A] \circ k^{-1} = k^{-1} \circ \mathrm{Hom}(Id_V, A),$$

so

$$(Ev_V \circ [Id_{V^*} \otimes A] \circ k^{-1})(Id_V) = Ev_V(k^{-1}(A)) = Tr_V(A).$$

∎

REMARK 2.11. The idea of Theorem 2.10 (as in [K] §II.3) is that the trace of $A$ is the output of the distinguished element $k^{-1}(Id_V)$ under the composite of maps in this diagram:

$$V^* \otimes V \xrightarrow{[Id_{V^*} \otimes A]} V^* \otimes V \xrightarrow{Ev_V} \mathbb{K}.$$

The statement of Theorem 2.10 could also be written as

$$Tr_V(A) = ((d_{\mathrm{End}(V)}(Id_V)) \circ \mathrm{Hom}(k^{-1}, Ev_V) \circ j)(Id_{V^*} \otimes A).$$

In terms of the scalar multiplication map $l$ from Example 1.28 and the $\beta = 1$ case of (2.3) from Example 2.9,

$$Q_1^1 : \mathbb{K} \to \mathrm{End}(V^*) : 1 \mapsto Id_{V^*}, \tag{2.4}$$

the composite $[Q_1^1 \otimes Id_{\mathrm{End}(V)}] \circ l^{-1} : \mathrm{End}(V) \to \mathrm{End}(V^*) \otimes \mathrm{End}(V)$ takes $A$ to $Id_{V^*} \otimes A$, so

$$Tr_V = (d_{\mathrm{End}(V)}(Id_V)) \circ \mathrm{Hom}(k^{-1}, Ev_V) \circ j \circ [Q_1^1 \otimes Id_{\mathrm{End}(V)}] \circ l^{-1}. \tag{2.5}$$

The map $Q_1^1$ from (2.4) is used in (2.5), and again in later Sections, without any assumption on the identity map's trace, so $Q_1^1$ is not necessarily part of the data for some direct sum as in Example 2.9.

PROPOSITION 2.12. *For $V = V_1 \oplus V_2$, $A \in \operatorname{End}(V_1)$, and $B \in \operatorname{End}(V_2)$, let $A \oplus B$ be the element of $\operatorname{End}(V)$ defined by $A \oplus B = Q_1 \circ A \circ P_1 + Q_2 \circ B \circ P_2$. If $V$ is finite-dimensional, then*

$$Tr_V(A \oplus B) = Tr_{V_1}(A) + Tr_{V_2}(B).$$

PROOF. Recall $V_1$ and $V_2$ are finite-dimensional by Exercise 0.50. The construction of $A \oplus B$ is as in Lemma 1.86.

$$\begin{aligned} Tr_V(A \oplus B) &= Tr_V(Q_1 \circ A \circ P_1) + Tr_V(Q_2 \circ B \circ P_2) \\ &= Tr_{V_1}(P_1 \circ Q_1 \circ A) + Tr_{V_2}(P_2 \circ Q_2 \circ B) \\ &= Tr_{V_1}(A) + Tr_{V_2}(B), \end{aligned}$$

using Lemma 2.6. ∎

PROPOSITION 2.13. *If $V$ is finite-dimensional, $V = V_1 \oplus V_2$, and $K \in \operatorname{End}(V)$, then*

$$Tr_V(K) = Tr_{V_1}(P_1 \circ K \circ Q_1) + Tr_{V_2}(P_2 \circ K \circ Q_2).$$

PROOF. Using Lemma 2.6,

$$Tr_V(K) = Tr_V((Q_1 \circ P_1 + Q_2 \circ P_2) \circ K) = Tr_{V_1}(P_1 \circ K \circ Q_1) + Tr_{V_2}(P_2 \circ K \circ Q_2).$$

∎

The formula $Tr_V(Id_V) = Tr_{V_1}(Id_{V_1}) + Tr_{V_2}(Id_{V_2})$ can be considered as a special case of either Proposition 2.12 or Proposition 2.13.

EXERCISE 2.14. Given $V$ finite-dimensional and $A, P \in \operatorname{End}(V)$, suppose $P$ is an idempotent with image subspace $V_1 = P(V)$, and let $P_1$ and $Q_1$ be the projections and inclusions from Example 1.113. Then

$$Tr_V(P \circ A) = Tr_{V_1}(P_1 \circ A \circ Q_1).$$

In particular, for $A = Id_V$, $Tr_V(P) = Tr_{V_1}(Id_{V_1})$. ∎

EXERCISE 2.15. Suppose $\mathbb{K}$ has the property that $\sum_{\iota=1}^{\nu} 1 \neq 0$ for all $\nu \geq 1$. For $V$ finite-dimensional and $A \in \operatorname{End}(V)$, any pair of two of the following three statements implies the remaining third statement.

(1) There exist $\phi \in V^*$, $v \in V$ so that $A = k_{VV}(\phi \otimes v)$.
(2) $Tr_V(A) = 1$.
(3) $A$ is a non-zero idempotent.

HINT. The implications $(1), (2) \implies (3)$ and $(1), (3) \implies (2)$ do not require the assumption on $\mathbb{K}$ (known as characteristic zero). The implication $(1), (2) \implies (3)$ is considered in ([AFMC] **P1993-3**). For $(2), (3) \implies (1)$, by Lemma 1.64, any $A$ is of the form $A = k_{VV}\left(\sum_{\iota=1}^{\mu} \phi_\iota \otimes v_\iota\right)$, and the exercise is to show that there is such an expression with $\mu = 1$. ∎

Proposition 2.16. ([G$_1$] §IV.7) *For $V$ finite-dimensional and $A \in \mathrm{End}(V)$, the following are equivalent.*

(1) $Tr_V(A \circ B) = 0$ *for all* $B$.
(2) $A = 0_{\mathrm{End}(V)}$.

Proof.

$$\begin{aligned} \mathrm{Hom}(Id_V, A)^*(Tr_V) &= (\mathrm{Hom}(Id_V, A)^* \circ (k^*)^{-1} \circ e)(Id_V) \\ &= ((k^*)^{-1} \circ e \circ \mathrm{Hom}(A, Id_V))(Id_V) \\ &= ((k^*)^{-1} \circ e)(A), \end{aligned}$$

by the commutativity of the diagram from Lemma 2.6, with $U = V$. If

$$Tr_V(A \circ B) = (\mathrm{Hom}(Id_V, A)^*(Tr_V))(B) = (((k^*)^{-1} \circ e)(A))(B)$$

is always zero, then $((k^*)^{-1} \circ e)(A) = 0_{\mathrm{End}(V)^*}$, and since $(k^*)^{-1} \circ e$ has zero kernel, $A$ must be $0_{\mathrm{End}(V)}$. The converse is trivial. ∎

Proposition 2.17. ([B] §II.10.11) *If $V$ is finite-dimensional, then for any $\Phi \in \mathrm{End}(V)^*$, there exists $F \in \mathrm{End}(V)$ such that $\Phi(A) = Tr_V(F \circ A)$ for all $A \in \mathrm{End}(V)$.*

Proof. Let $F = e^{-1}(k^*(\Phi))$. The result follows from the commutativity of the appropriate paths in the diagram for the Proof of Lemma 2.6 in the case $U = V$.

$$\begin{aligned} \Phi(A) &= (\mathrm{Hom}(Id_V, A)^*(\Phi))(Id_V) \\ &= ((e^* \circ d \circ k^{-1} \circ \mathrm{Hom}(A, Id_V) \circ e^{-1} \circ k^*)(\Phi))(Id_V) \\ &= ((k^{-1})^*(e(Id_V)))(\mathrm{Hom}(A, Id_V)(e^{-1}(k^*(\Phi)))) \\ &= Tr_V(F \circ A). \end{aligned}$$

∎

Proposition 2.18. ([B] §II.10.11) *For $V$ finite-dimensional and $\Phi \in \mathrm{End}(V)^*$, the following are equivalent.*

(1) $\Phi$ *satisfies* $\Phi(A \circ B) = \Phi(B \circ A)$ *for all* $A$, $B \in \mathrm{End}(V)$.
(2) *There exists* $\lambda \in \mathbb{K}$ *such that* $\Phi = \lambda \cdot Tr_V$.

Proof. By Proposition 2.17, $\Phi(A \circ B) = Tr_V(F \circ A \circ B) = \Phi(B \circ A) = Tr_V(F \circ B \circ A)$. By Lemma 2.6, $Tr_V(F \circ B \circ A) = Tr_V(A \circ B \circ F)$ for all $B$, so $\mathrm{Hom}(A, F)^*(Tr_V) = \mathrm{Hom}(F, A)^*(Tr_V)$. Then

$$\begin{aligned} \mathrm{Hom}(A, F)^*((k^*)^{-1}(e(Id_V))) &= \mathrm{Hom}(F, A)^*((k^*)^{-1}(e(Id_V))) \\ (k^*)^{-1}(e(\mathrm{Hom}(F, A)(Id_V))) &= (k^*)^{-1}(e(\mathrm{Hom}(A, F)(Id_V))) \\ ((k^*)^{-1} \circ e)(A \circ F) &= ((k^*)^{-1} \circ e)(F \circ A), \end{aligned}$$

so $A \circ F = F \circ A$ for all $A$, and so $F = \lambda \cdot Id_V$ by Claim 0.57. The converse follows from Lemma 2.6. ∎

PROPOSITION 2.19. ([G1] §IV.7) *Suppose* $Tr_V(Id_V) \neq 0$*, and that the map* $\Omega \in \mathrm{End}(\mathrm{End}(V))$ *satisfies* $\Omega(A \circ B) = (\Omega(A)) \circ (\Omega(B))$ *for all* $A$*,* $B$*, and* $\Omega(Id_V) = Id_V$*. Then* $Tr_V(\Omega(H)) = Tr_V(H)$ *for all* $H \in \mathrm{End}(V)$*.*

PROOF.

$$\begin{aligned} Tr_V(\Omega(A \circ B)) &= Tr_V((\Omega(A)) \circ (\Omega(B)))) \\ &= Tr_V((\Omega(B)) \circ (\Omega(A))) \\ &= Tr_V(\Omega(B \circ A)), \end{aligned}$$

so Proposition 2.18 applies to $\Omega^*(Tr_V)$ and $Tr_V(\Omega(H)) = \lambda \cdot Tr_V(H)$. The second property of $\Omega$ implies $Tr_V(\Omega(Id_V)) = Tr_V(Id_V) = \lambda \cdot Tr_V(Id_V)$, so either $Tr_V(Id_V) = 0$, or $\lambda = 1$ and $Tr_V(\Omega(H)) = Tr_V(H)$ for all $H$. ∎

REMARK 2.20. The following Lemma is a generalization of Example 2.7, motivated by a property of a "line bundle."

LEMMA 2.21. *Suppose that* $L$ *is a finite-dimensional vector space, and that the evaluation map* $Ev_L : L^* \otimes L \to \mathbb{K}$ *is invertible. Then* $Tr_L : \mathrm{End}(L) \to \mathbb{K}$ *is invertible and* $Tr_L(Id_L) = 1$*.*

PROOF. $k = k_{LL}$ and $e = e_{LL}$ are invertible. $Ev_L \neq 0_{(L^* \otimes L)^*}$, so $Id_L = e^{-1}(Ev_L) \neq 0_{\mathrm{End}(L)}$. $Tr_L = Ev_L \circ k^{-1}$ is invertible, so $Tr_L(Id_L) \neq 0$, and Example 2.9 applies. In particular, there is some $\beta \in \mathbb{K}$ so that $\beta \cdot Tr_L(Id_L) = 1$, and a map $Q_1^\beta : \mathbb{K} \to \mathrm{End}(L) : \gamma \mapsto \beta \cdot \gamma \cdot Id_L$ as in Equation (2.3) so that $Tr_L \circ Q_1^\beta = Id_{\mathbb{K}}$. It follows that

$$(Ev_L \circ k^{-1}) \circ Q_1^\beta = Id_{\mathbb{K}} \implies Q_1^\beta \circ Ev_L = k.$$

There is also some $v \in L$, and there is some $\phi \in L^*$, so that $Ev_L(\phi \otimes v) \neq 0$, and so $\phi(v) \neq 0$, and $v \neq 0_L$. Then,

$$k(\phi \otimes v) = (Q_1^\beta \circ Ev_L)(\phi \otimes v) = \beta \cdot \phi(v) \cdot Id_L,$$

so

$$(k(\phi \otimes v))(v) = \phi(v) \cdot v = (\beta \cdot \phi(v) \cdot Id_L)(v) = \beta \cdot \phi(v) \cdot v,$$

which implies $\beta = 1$. ∎

REMARK 2.22. The following Proposition is proved in a different way by [AS2].

PROPOSITION 2.23. *Given* $V$ *finite-dimensional and any positive integer* $\nu$*, if* $P_1 : V \to V$ *and* $P_2 : V \to V$ *are any idempotents, then*

$$Tr_V((P_1 - P_2)^{2\nu+1}) = Tr_V(P_1 - P_2).$$

PROOF. The odd power refers to a composite $(P_1 - P_2) \circ \cdots \circ (P_1 - P_2)$. It can be shown by induction on $\nu$ that there exist constants $\alpha_{\nu,\iota} \in \mathbb{K}$, $\iota = 1, \ldots, \nu$, so that the composite expands:

$$(P_1 - P_2)^{2\nu+1} = P_1 - P_2 + \sum_{\iota=1}^{\nu} \alpha_{\nu,\iota} \cdot ((P_1 \circ P_2)^\iota \circ P_1 - (P_2 \circ P_1)^\iota \circ P_2).$$

The claim then follows from Lemma 2.6. ∎

## 2.2. The generalized trace

An analogue of the trace $Tr_V : \mathrm{End}(V) \to \mathbb{K}$ is the generalized trace, a map

$$Tr_{V;U,W} : \mathrm{Hom}(V \otimes U, V \otimes W) \to \mathrm{Hom}(U, W),$$

constructed in Definition 2.24.

### 2.2.1. Defining the generalized trace.

The following particular cases of the canonical $j$ maps will be used repeatedly:

$$\begin{aligned} j_1 : \mathrm{End}(V)^* \otimes \mathrm{End}(\mathrm{Hom}(U,W)) &\to \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U,W), \mathbb{K} \otimes \mathrm{Hom}(U,W)) \\ j_2 : \mathrm{End}(V) \otimes \mathrm{Hom}(U,W) &\to \mathrm{Hom}(V \otimes U, V \otimes W). \end{aligned}$$

If $V$ is finite-dimensional, then both $j_1$ and $j_2$ are invertible, by Claim 1.34. Denote by $l_1$ the scalar multiplication map $\mathbb{K} \otimes \mathrm{Hom}(U,W) \to \mathrm{Hom}(U,W)$. The domain of $j_1$ contains the distinguished element $Tr_V \otimes Id_{\mathrm{Hom}(U,W)}$.

Definition 2.24. For finite-dimensional $V$, define

$$\begin{aligned} Tr_{V;U,W} &= (\mathrm{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\ &= l_1 \circ [Tr_V \otimes Id_{\mathrm{Hom}(U,W)}] \circ j_2^{-1}. \end{aligned}$$

Note that the finite-dimensionality of $V$ is used in the Definition, since $j_2$ must be invertible, but $U$ and $W$ may be arbitrary vector spaces.

Example 2.25. A map of the form $j_2(A \otimes B) : V \otimes U \to V \otimes W$, for finite-dimensional $V$, and $A : V \to V$, $B : U \to W$, has trace

$$Tr_{V;U,W}(j_2(A \otimes B)) = l_1((j_1(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}))(A \otimes B)) = Tr_V(A) \cdot B.$$

In the $V = \mathbb{K}$ case, the trace is an invertible map. Denote scalar multiplication maps $l_W : \mathbb{K} \otimes W \to W$ and $l_U : \mathbb{K} \otimes U \to U$.

Theorem 2.26. *For any vector spaces $U$, $W$,*

$$Tr_{\mathbb{K};U,W} = \mathrm{Hom}(l_U^{-1}, l_W) : \mathrm{Hom}(\mathbb{K} \otimes U, \mathbb{K} \otimes W) \to \mathrm{Hom}(U, W).$$

Proof. For any $\phi \in \mathbb{K}^*$, $F \in \mathrm{Hom}(U,W)$, the equation $(\phi(1) \cdot F) \circ l_U = l_W \circ [\phi \otimes F]$ from Lemma 1.38 can be rewritten using $j_2 : \mathbb{K}^* \otimes \mathrm{Hom}(U,W) \to \mathrm{Hom}(\mathbb{K} \otimes U, \mathbb{K} \otimes W)$,

$$\mathrm{Hom}(l_U^{-1}, l_W) \circ j_2 = l_1 \circ (j_1((d_{\mathbb{K}}(1)) \otimes Id_{\mathrm{Hom}(U,W)})) : \phi \otimes F \mapsto \phi(1) \cdot F.$$

The equality follows from Example 2.7, where $Tr_{\mathbb{K}} = d_{\mathbb{K}}(1)$:

$$\mathrm{Hom}(l_U^{-1}, l_W) = l_1 \circ (j_1((d_{\mathbb{K}}(1)) \otimes Id_{\mathrm{Hom}(U,W)})) \circ j_2^{-1} = Tr_{\mathbb{K};U,W}.$$

∎

In the $U = W = \mathbb{K}$ case, the generalized trace is related to the scalar trace as in the following Theorem.

Theorem 2.27. *For finite-dimensional $V$, the scalar multiplication map $l_V : V \otimes \mathbb{K} \to V$, and any $H \in \mathrm{End}(V)$,*

$$(Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V))(1) = Tr_V(H).$$

*Equivalently,*

$$Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V) = Tr_V(H) \cdot Id_{\mathbb{K}}. \tag{2.6}$$

Proof. In the following diagram,

$$
\begin{tikzcd}
\mathrm{End}(V)^*\otimes\mathrm{End}(\mathbb{K}^*) \arrow[r, "a_1"] \arrow[d, "j_1"] & \mathrm{End}(V)^*\otimes\mathrm{Hom}(\mathbb{K},\mathbb{K}^*) \arrow[d, "j_3"] & \mathrm{End}(V)^*\otimes\mathbb{K}^* \arrow[l, "a_5"'] \arrow[ddd, "k_{\mathrm{End}(V),\mathbb{K}^*}"'] \\
\mathrm{Hom}(\mathrm{End}(V)\otimes\mathbb{K}^*,\mathbb{K}\otimes\mathbb{K}^*) \arrow[r, "a_2"] \arrow[d, "\mathrm{Hom}(Id_{\mathrm{End}(V)\otimes\mathbb{K}^*},l_1)"] & \mathrm{Hom}(\mathrm{End}(V)\otimes\mathbb{K},\mathbb{K}\otimes\mathbb{K}^*) \arrow[d, "\mathrm{Hom}(Id_{\mathrm{End}(V)\otimes\mathbb{K}},l_1)"] & \\
\mathrm{Hom}(\mathrm{End}(V)\otimes\mathbb{K}^*,\mathbb{K}^*) \arrow[r, "a_3"] & \mathrm{Hom}(\mathrm{End}(V)\otimes\mathbb{K},\mathbb{K}^*) & \\
\mathrm{Hom}(\mathrm{End}(V\otimes\mathbb{K}),\mathbb{K}^*) \arrow[u, "\mathrm{Hom}(j_2,Id_{\mathbb{K}^*})"] & \mathrm{Hom}(\mathrm{Hom}(V\otimes\mathbb{K},V),\mathbb{K}^*) \arrow[l, "a_4"'] \arrow[r, "a_6"] & \mathrm{Hom}(\mathrm{End}(V),\mathbb{K}^*) \arrow[ul, "\mathrm{Hom}(l_2,Id_{\mathbb{K}^*})"']
\end{tikzcd}
$$

the horizontal arrows are

$$
\begin{aligned}
a_1 &= [Id_{\mathrm{End}(V)^*}\otimes\mathrm{Hom}(m,Id_{\mathbb{K}^*})] \\
a_2 &= \mathrm{Hom}([Id_{\mathrm{End}(V)}\otimes m],Id_{\mathbb{K}\otimes\mathbb{K}^*}) \\
a_3 &= \mathrm{Hom}([Id_{\mathrm{End}(V)}\otimes m],Id_{\mathbb{K}^*}) \\
a_4 &= \mathrm{Hom}(\mathrm{Hom}(Id_{V\otimes\mathbb{K}},l_V),Id_{\mathbb{K}^*}) \\
a_5 &= [Id_{\mathrm{End}(V)^*}\otimes\mathrm{Hom}(Id_{\mathbb{K}},m)] \\
a_6 &= \mathrm{Hom}(\mathrm{Hom}(l_V,Id_V),Id_{\mathbb{K}^*}),
\end{aligned}
$$

for $m:\mathbb{K}\to\mathbb{K}^*$ as in Definition 1.20, so that $m(\alpha):\lambda\mapsto\lambda\cdot\alpha$. So there is not much going on besides scalar multiplication, including the map $l_2:\mathrm{End}(V)\otimes\mathbb{K}\to\mathrm{End}(V)$.

Starting with the element $Tr_V\otimes Id_{\mathbb{K}^*}$ in the space in the upper left corner, its output under the composite map going downward and then right to the lower right corner is, using Definition 2.24, $Tr_{V;\mathbb{K},\mathbb{K}}\circ\mathrm{Hom}(l_V,l_V^{-1})$, as in the LHS of the claim (2.6). The output in the path going right and then downward is

$$
k_{\mathrm{End}(V),\mathbb{K}^*}(Tr_V\otimes(m^{-1}\circ Id_{\mathbb{K}^*}\circ m)) = k_{\mathrm{End}(V),\mathbb{K}^*}(Tr_V\otimes Id_{\mathbb{K}}): H\mapsto Tr_V(H)\cdot Id_{\mathbb{K}},
$$

corresponding to the RHS of the claim. The claimed equality follows from just the commutativity of the diagram, without using any special properties of the trace $Tr_V$.

The upper left block is commutative by Lemma 1.37 and the middle left block is commutative by Lemma 1.6. For the commutativity of the lower block, it is enough, by Lemma 1.6, to check this equality, for $A\in\mathrm{End}(V)$, $\alpha,\beta\in\mathbb{K}$, $v\in V$:

$$
\begin{aligned}
& & &\mathrm{Hom}(Id_{V\otimes\mathbb{K}},l_V)\circ j_2\circ[Id_{\mathrm{End}(V)}\otimes m]: \\
&A\otimes\alpha &\mapsto\ & l_V\circ[A\otimes(m(\alpha))]: v\otimes\beta\mapsto l_V((A(v))\otimes(\beta\cdot\alpha)) = \beta\cdot\alpha\cdot A(v), \\
& & &\mathrm{Hom}(l_V,Id_V)\circ l_2: \\
&A\otimes\alpha &\mapsto\ & (\alpha\cdot A)\circ l_V: v\otimes\beta\mapsto(\alpha\cdot A)(\beta\cdot v).
\end{aligned}
$$

Checking the commutativity of the block on the right, for $\Phi \in \mathrm{End}(V)^*$, $\phi \in \mathbb{K}^*$,

$$\begin{array}{rcl} & & \mathrm{Hom}(Id_{\mathrm{End}(V)\otimes\mathbb{K}}, l_1) \circ j_3 \circ [Id_{\mathrm{End}(V)^*} \otimes \mathrm{Hom}(Id_{\mathbb{K}}, m)] : \\ \Phi \otimes \phi & \mapsto & l_1 \circ [\Phi \otimes (m \circ \phi)] : \\ A \otimes \alpha & \mapsto & (\Phi(A)) \cdot (m(\phi(\alpha))) : \beta \mapsto (\Phi(A)) \cdot \beta \cdot \phi(\alpha), \\ & & \mathrm{Hom}(l_2, Id_{\mathbb{K}^*}) \circ k_{\mathrm{End}(V),\mathbb{K}^*} : \\ \Phi \otimes \phi & \mapsto & (k_{\mathrm{End}(V),\mathbb{K}^*}(\Phi \otimes \phi)) \circ l_2 : \\ A \otimes \alpha & \mapsto & (k_{\mathrm{End}(V),\mathbb{K}^*}(\Phi \otimes \phi))(\alpha \cdot A) = (\Phi(\alpha \cdot A)) \cdot \phi : \beta \mapsto (\Phi(\alpha \cdot A)) \cdot \phi(\beta). \end{array}$$

$\blacksquare$

**2.2.2. Properties of the generalized trace.**

The next Theorems in this Section are straightforward linear algebra identities for the generalized trace.

REMARK 2.28. Versions of some of the results in this Section are stated in a more general context of category theory, and given different proofs, in [**Maltsiniotis**] §3.5 or [**JSV**] §2. The result of Theorem 2.26 is related to a property called "vanishing" by [**JSV**], and Theorem 2.29 and Theorem 2.30 are "naturality" properties ([**JSV**]).

An analogue of Lemma 2.6 applies to maps $A : V \to V'$ and $B : V' \otimes U \to V \otimes W$, using the canonical maps

$$\begin{array}{rcl} j_U & : & \mathrm{Hom}(V, V') \otimes \mathrm{End}(U) \to \mathrm{Hom}(V \otimes U, V' \otimes U), \\ j_W & : & \mathrm{Hom}(V, V') \otimes \mathrm{End}(W) \to \mathrm{Hom}(V \otimes W, V' \otimes W). \end{array}$$

THEOREM 2.29. *For finite-dimensional $V$ and $V'$,*

$$Tr_{V;U,W}(B \circ (j_U(A \otimes Id_U))) = Tr_{V';U,W}((j_W(A \otimes Id_W)) \circ B).$$

PROOF. In the following diagram,

$$\begin{tikzcd} M_{11} \arrow[r, "a_1"] \arrow[d, "j_1"] & M_{12} \arrow[d, "j_1''"] & M_{13} \arrow[l, "a_2"'] \arrow[d, "j_1'"] \\ M_{21} \arrow[r, "a_3"] \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1}, l_1)"] & M_{22} \arrow[d, shift left, "\mathrm{Hom}((j_2'')^{-1}, l_1)"] & M_{23} \arrow[l, "a_4"'] \arrow[d, shift left, "\mathrm{Hom}((j_2')^{-1}, l_1)"] \\ M_{31} \arrow[r, "a_5"] \arrow[u, shift left, "\mathrm{Hom}(j_2, l_1^{-1})"] & M_{32} \arrow[u, shift left, "\mathrm{Hom}(j_2'', l_1^{-1})"] & M_{33} \arrow[l, "a_6"'] \arrow[u, shift left, "\mathrm{Hom}(j_2', l_1^{-1})"] \end{tikzcd}$$

the objects are

$$\begin{array}{rcl} M_{11} & = & \mathrm{End}(V)^* \otimes \mathrm{End}(\mathrm{Hom}(U, W)) \\ M_{21} & = & \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U, W), \mathbb{K} \otimes \mathrm{Hom}(U, W)) \\ M_{31} & = & \mathrm{Hom}(\mathrm{Hom}(V \otimes U, V \otimes W), \mathrm{Hom}(U, W)) \\ M_{12} & = & \mathrm{Hom}(V', V)^* \otimes \mathrm{End}(\mathrm{Hom}(U, W)) \\ M_{22} & = & \mathrm{Hom}(\mathrm{Hom}(V', V) \otimes \mathrm{Hom}(U, W), \mathbb{K} \otimes \mathrm{Hom}(U, W)) \\ M_{32} & = & \mathrm{Hom}(\mathrm{Hom}(V' \otimes U, V \otimes W), \mathrm{Hom}(U, W)) \\ M_{13} & = & \mathrm{End}(V')^* \otimes \mathrm{End}(\mathrm{Hom}(U, W)) \\ M_{23} & = & \mathrm{Hom}(\mathrm{End}(V') \otimes \mathrm{Hom}(U, W), \mathbb{K} \otimes \mathrm{Hom}(U, W)) \\ M_{33} & = & \mathrm{Hom}(\mathrm{Hom}(V' \otimes U, V' \otimes W), \mathrm{Hom}(U, W)), \end{array}$$

where the left and right columns are the maps from the definition of trace and the horizontal arrows in the diagram are

$$\begin{aligned}
a_1 &= [\mathrm{Hom}(A, Id_V)^* \otimes Id_{\mathrm{End}(\mathrm{Hom}(U,W))}] \\
a_2 &= [\mathrm{Hom}(Id_{V'}, A)^* \otimes Id_{\mathrm{End}(\mathrm{Hom}(U,W))}] \\
a_3 &= \mathrm{Hom}([\mathrm{Hom}(A, Id_V) \otimes Id_{\mathrm{Hom}(U,W)}], Id_{\mathbb{K}\otimes\mathrm{Hom}(U,W)}) \\
a_4 &= \mathrm{Hom}([\mathrm{Hom}(Id_{V'}, A) \otimes Id_{\mathrm{Hom}(U,W)}], Id_{\mathbb{K}\otimes\mathrm{Hom}(U,W)}) \\
a_5 &= \mathrm{Hom}(\mathrm{Hom}(j_U(A \otimes Id_U), Id_{V\otimes W}), Id_{\mathrm{Hom}(U,W)}) \\
a_6 &= \mathrm{Hom}(\mathrm{Hom}(Id_{V'\otimes U}, j_W(A \otimes Id_W)), Id_{\mathrm{Hom}(U,W)}).
\end{aligned}$$

The two quantities in the statement of the Theorem are

$$\begin{aligned}
Tr_{V;U,W}(B \circ (j_U(A \otimes Id_U))) &= (a_5(Tr_{V;U,W}))(B), \\
Tr_{V';U,W}((j_W(A \otimes Id_W)) \circ B) &= (a_6(Tr_{V';U,W}))(B).
\end{aligned}$$

Each of the squares in the diagram is commutative, by Lemma 1.6 and Lemma 1.37. By Lemma 2.6, $\mathrm{Hom}(A, Id_V)^*(Tr_V) = \mathrm{Hom}(Id_{V'}, A)^*(Tr_{V'})$, so

$$\begin{aligned}
a_1(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) &= (\mathrm{Hom}(A, Id_V)^*(Tr_V)) \otimes Id_{\mathrm{Hom}(U,W)} \\
&= (\mathrm{Hom}(Id_{V'}, A)^*(Tr_{V'})) \otimes Id_{\mathrm{Hom}(U,W)} \\
&= a_2(Tr_{V'} \otimes Id_{\mathrm{Hom}(U,W)}).
\end{aligned}$$

The Theorem follows from the commutativity of the diagram:

$$\begin{aligned}
a_5(Tr_{V;U,W}) &= (a_5 \circ \mathrm{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\
&= (\mathrm{Hom}((j_2'')^{-1}, l_1) \circ j_1'' \circ a_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\
&= (\mathrm{Hom}((j_2'')^{-1}, l_1) \circ j_1'' \circ a_2)(Tr_{V'} \otimes Id_{\mathrm{Hom}(U,W)}) \\
&= (a_6 \circ \mathrm{Hom}((j_2')^{-1}, l_1) \circ j_1')(Tr_{V'} \otimes Id_{\mathrm{Hom}(U,W)}) \\
&= a_6(Tr_{V';U,W}).
\end{aligned}$$

∎

The general strategy for the preceding proof will be repeated in some subsequent proofs. To derive an equality involving the generalized trace, a diagram is set up with the maps from Definition 2.24 on the left and right. The lowest row will be the desired theorem, and the top row is the "key step," which is either obvious, or which uses the previously derived properties of the scalar valued trace. There will be little choice in selecting canonical maps as horizontal arrows, and the commutativity of the diagram will give the theorem as a consequence of the key step. We remark that the canonical maps $m$, $n$, and $q$ will not be needed until Section 2.3.

Theorem 2.30. *If $V$ is finite-dimensional, then for any maps $A : V \otimes U \to V \otimes W$, $B : W \to W'$, and $C : U' \to U$, the composite $[Id_V \otimes B] \circ A \circ [Id_V \otimes C] : V \otimes U' \to V \otimes W'$ has trace*

$$Tr_{V;U',W'}([Id_V \otimes B] \circ A \circ [Id_V \otimes C]) = B \circ (Tr_{V;U,W}(A)) \circ C.$$

Proof. In the following diagram,

$$\begin{tikzcd}
M_{11} \arrow[r, "a_1"] \arrow[d, "j_1"] & M_{12} \arrow[d, "j_1''"] & M_{13} \arrow[l, "a_2"'] \arrow[d, "j_1'"] \\
M_{21} \arrow[r, "a_3"] \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1}, l_1)"] & M_{22} \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1}, l_1')"] & M_{23} \arrow[l, "a_4"'] \arrow[d, shift left, "\mathrm{Hom}((j_2')^{-1}, l_1')"] \\
M_{31} \arrow[r, "a_5"] \arrow[u, shift left, "\mathrm{Hom}(j_2, l_1^{-1})"] & M_{32} \arrow[u, shift left, "\mathrm{Hom}(j_2, (l_1')^{-1})"] & M_{33} \arrow[l, "a_6"'] \arrow[u, shift left, "\mathrm{Hom}(j_2', (l_1')^{-1})"]
\end{tikzcd}$$

the objects are

$$\begin{array}{rcl}
M_{11} & = & \mathrm{End}(V)^* \otimes \mathrm{End}(\mathrm{Hom}(U, W)) \\
M_{21} & = & \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U, W), \mathbb{K} \otimes \mathrm{Hom}(U, W)) \\
M_{31} & = & \mathrm{Hom}(\mathrm{Hom}(V \otimes U, V \otimes W), \mathrm{Hom}(U, W)) \\
M_{12} & = & \mathrm{End}(V)^* \otimes \mathrm{Hom}(\mathrm{Hom}(U, W), \mathrm{Hom}(U', W')) \\
M_{22} & = & \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U, W), \mathbb{K} \otimes \mathrm{Hom}(U', W')) \\
M_{32} & = & \mathrm{Hom}(\mathrm{Hom}(V \otimes U, V \otimes W), \mathrm{Hom}(U', W')) \\
M_{13} & = & \mathrm{End}(V)^* \otimes \mathrm{End}(\mathrm{Hom}(U', W')) \\
M_{23} & = & \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U', W'), \mathbb{K} \otimes \mathrm{Hom}(U', W')) \\
M_{33} & = & \mathrm{Hom}(\mathrm{Hom}(V \otimes U', V \otimes W'), \mathrm{Hom}(U', W')),
\end{array}$$

where the left and right columns are the maps from the definition of trace and the horizontal arrows in the diagram are

$$\begin{array}{rcl}
a_1 & = & [Id_{\mathrm{End}(V)^*} \otimes \mathrm{Hom}(Id_{\mathrm{Hom}(U,W)}, \mathrm{Hom}(C, B))] \\
a_2 & = & [Id_{\mathrm{End}(V)^*} \otimes \mathrm{Hom}(\mathrm{Hom}(C, B), Id_{\mathrm{Hom}(U',W')})] \\
a_3 & = & \mathrm{Hom}(Id_{\mathrm{End}(V) \otimes \mathrm{Hom}(U,W)}, [Id_{\mathbb{K}} \otimes \mathrm{Hom}(C, B)]) \\
a_4 & = & \mathrm{Hom}([Id_{\mathrm{End}(V)} \otimes \mathrm{Hom}(C, B)], Id_{\mathbb{K} \otimes \mathrm{Hom}(U',W')}) \\
a_5 & = & \mathrm{Hom}(Id_{\mathrm{Hom}(V \otimes U, V \otimes W)}, \mathrm{Hom}(C, B)) \\
a_6 & = & \mathrm{Hom}(\mathrm{Hom}([Id_V \otimes C], [Id_V \otimes B]), Id_{\mathrm{Hom}(U',W')}).
\end{array}$$

The two quantities in the statement of the Theorem are

$$\begin{array}{rcl}
Tr_{V;U',W'}([Id_V \otimes B] \circ A \circ [Id_V \otimes C]) & = & (a_6(Tr_{V;U',W'}))(A) \\
B \circ (Tr_{V;U,W}(A)) \circ C & = & (a_5(Tr_{V;U,W}))(A).
\end{array}$$

Squares except the lower left commute by Lemma 1.6 and Lemma 1.37; for the remaining square, let $\lambda \in \mathbb{K}$, $E \in \mathrm{Hom}(U, W)$:

$$\begin{array}{rcl}
(l_1' \circ [Id_{\mathbb{K}} \otimes \mathrm{Hom}(C, B)])(\lambda \otimes E) & = & l_1'(\lambda \otimes (B \circ E \circ C)) = \lambda \cdot (B \circ E \circ C) \\
(\mathrm{Hom}(C, B) \circ l_1)(\lambda \otimes E) & = & B \circ (\lambda \cdot E) \circ C = \lambda \cdot (B \circ E \circ C).
\end{array}$$

The "key step" uses a property of the identity map, and not any properties of the trace:

$$a_1(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) = Tr_V \otimes \mathrm{Hom}(C,B) = a_2(Tr_V \otimes Id_{\mathrm{Hom}(U',W')})$$

The Theorem follows from the commutativity of the diagram:

$$\begin{aligned} a_5(Tr_{V;U,W}) &= (a_5 \circ \mathrm{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\ &= (\mathrm{Hom}(j_2^{-1}, l_1') \circ j_1'' \circ a_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\ &= (\mathrm{Hom}(j_2^{-1}, l_1') \circ j_1'' \circ a_2)(Tr_V \otimes Id_{\mathrm{Hom}(U',W')}) \\ &= (a_6 \circ \mathrm{Hom}((j_2')^{-1}, l_1') \circ j_1')(Tr_V \otimes Id_{\mathrm{Hom}(U',W')}) \\ &= a_6(Tr_{V;U',W'}). \end{aligned}$$

∎

COROLLARY 2.31. *If $V$ and $V'$ are finite-dimensional then for any maps $A : V \to V'$, $B : W \to W'$, $C : U' \to U$, the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Hom}(V \otimes U, V \otimes W) \arrow[rr, "Tr_{V;U,W}"] & & \mathrm{Hom}(U,W) \arrow[dd, "\mathrm{Hom}(C,B)"] \\ \mathrm{Hom}(V' \otimes U, V \otimes W) \arrow[u, "{\mathrm{Hom}([A \otimes Id_U], Id_{V\otimes W})}"] \arrow[d, "{\mathrm{Hom}([Id_{V'} \otimes C], [A \otimes B])}"'] & & \\ \mathrm{Hom}(V' \otimes U', V' \otimes W') \arrow[rr, "Tr_{V';U',W'}"] & & \mathrm{Hom}(U', W') \end{tikzcd}$$

PROOF. This follows from Theorem 2.29 and Theorem 2.30. ∎

LEMMA 2.32. *The following diagram is commutative.*

$$\begin{tikzcd} V_1^* \otimes V_2^* \otimes W_1 \otimes W_2 \arrow[r, "s"] \arrow[d, "{[j \otimes Id_{W_1 \otimes W_2}]}"] & V_1^* \otimes W_1 \otimes V_2^* \otimes W_2 \arrow[d, "{[k_{V_1W_1} \otimes k_{V_2W_2}]}"] \\ \mathrm{Hom}(V_1 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) \otimes W_1 \otimes W_2 \arrow[d, "{[\mathrm{Hom}(Id_{V_1 \otimes V_2}, l) \otimes Id_{W_1 \otimes W_2}]}"] & \mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \arrow[d, "j"] \\ (V_1 \otimes V_2)^* \otimes W_1 \otimes W_2 \arrow[r, "k_{V_1 \otimes V_2, W_1 \otimes W_2}"] & \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2) \end{tikzcd}$$

PROOF. The map $s$ switches the middle two factors of the tensor product (as in Example 1.29), and $l$ is multiplication of elements of $\mathbb{K}$.

$$\begin{aligned} \phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2 &\mapsto (j \circ [k_{V_1W_1} \otimes k_{V_2W_2}] \circ s)(\phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2) \\ &= [(k_{V_1W_1}(\phi_1 \otimes w_1)) \otimes (k_{V_2W_2}(\phi_2 \otimes w_2))] : \\ v_1 \otimes v_2 &\mapsto ((\phi(v_1)) \cdot w_1) \otimes ((\phi_2(v_2)) \cdot w_2), \\ \phi_1 \otimes \phi_2 \otimes w_1 \otimes w_2 &\mapsto k_{V_1 \otimes V_2, W_1 \otimes W_2}((l \circ [\phi_1 \otimes \phi_2]) \otimes w_1 \otimes w_2) : \\ v_1 \otimes v_2 &\mapsto \phi_1(v_1) \cdot \phi_2(v_2) \cdot w_1 \otimes w_2. \end{aligned}$$

∎

REMARK 2.33. The above result appears in [**K**] §II.2, and is related to a matrix algebra equation in [**Magnus**] §3.6.

THEOREM 2.34. *For finite-dimensional $V$ and $U$, and $A : V \otimes U \to V \otimes U$,*

$$Tr_U(Tr_{V;U,U}(A)) = Tr_{V\otimes U}(A).$$

PROOF. As in Lemma 2.6, the maps $k_{VV}$ and $k_{UU}$ are abbreviated $k$ and $k'$, and the corresponding map for $V\otimes U$ is denoted $k'' : (V\otimes U)^*\otimes V\otimes U \to \mathrm{End}(V\otimes U)$. By Lemma 2.32, these $k$ maps are related by the following commutative diagram.

$$\begin{tikzcd}
V^*\otimes V\otimes U^*\otimes U \arrow[r, "{[k\otimes k']}"] \arrow[d, "s"] & \mathrm{End}(V)\otimes\mathrm{End}(U) \arrow[ddd, "j_2"] \\
V^*\otimes U^*\otimes V\otimes U \arrow[d, "{[j\otimes Id_{V\otimes U}]}"] & \\
\mathrm{Hom}(V\otimes U, \mathbb{K}\otimes\mathbb{K})\otimes V\otimes U \arrow[d, "{[\mathrm{Hom}(Id_{V\otimes U}, l)\otimes Id_{V\otimes U}]}"] & \\
(V\otimes U)^*\otimes V\otimes U \arrow[r, "k''"] & \mathrm{End}(V\otimes U)
\end{tikzcd}$$

The composite of maps in the left column is abbreviated $a_1$. In particular, since $U$ and $V$ are assumed finite-dimensional, all the arrows in the square are invertible. In the following diagram,

$$\begin{tikzcd}
\mathrm{End}(V)^*\otimes\mathrm{End}(\mathrm{End}(U)) \arrow[r, shift left, "a_2"] \arrow[d, "j_1"] & (V^*\otimes V)^*\otimes\mathrm{End}(U^*\otimes U) \arrow[l, shift left, "a_2^{-1}"] \arrow[d, "j_1'"] \\
\mathrm{Hom}(\mathrm{End}(V)\otimes\mathrm{End}(U), \mathbb{K}\otimes\mathrm{End}(U)) \arrow[r, "a_3"] \arrow[d, shift left, "{\mathrm{Hom}(j_2^{-1}, l_1)}"] & \mathrm{Hom}(V^*\otimes V\otimes U^*\otimes U, \mathbb{K}\otimes U^*\otimes U) \arrow[d, shift left, "{\mathrm{Hom}(a_1^{-1}, k'\circ l_1')}"] \\
\mathrm{Hom}(\mathrm{End}(V\otimes U), \mathrm{End}(U)) \arrow[r, "a_4"] \arrow[u, shift left, "{\mathrm{Hom}(j_2, l_1^{-1})}"] \arrow[d, "{\mathrm{Hom}(Id_{\mathrm{End}(V\otimes U)}, Tr_U)}"] & \mathrm{Hom}((V\otimes U)^*\otimes V\otimes U, \mathrm{End}(U)) \arrow[u, shift left, "{\mathrm{Hom}(a_1, (l_1')^{-1}\circ(k')^{-1})}"] \arrow[d, "{\mathrm{Hom}(Id_{(V\otimes U)^*\otimes V\otimes U}, Tr_U)}"'] \\
\mathrm{End}(V\otimes U)^* \arrow[r, "k''^*"] & ((V\otimes U)^*\otimes V\otimes U)^*
\end{tikzcd}$$

the horizontal arrows are

$$\begin{aligned}
a_2 &= [k^*\otimes\mathrm{Hom}(k', (k')^{-1})] \\
a_2^{-1} &= [(k^*)^{-1}\otimes\mathrm{Hom}((k')^{-1}, k')] \\
a_3 &= \mathrm{Hom}([k\otimes k'], [Id_{\mathbb{K}}\otimes(k')^{-1}]) \\
a_4 &= \mathrm{Hom}(k'', Id_{\mathrm{End}(U)}),
\end{aligned}$$

and the statement of the Theorem is that

$$\mathrm{Hom}(Id_{\mathrm{End}(V\otimes U)}, Tr_U)(Tr_{V;U,U}) = Tr_{V\otimes U}.$$

The upper square is commutative by Lemma 1.37, and Lemma 1.6 applies easily to the commutativity of the lower square, and to that of the middle square using $k''\circ a_1 = j_2\circ[k\otimes k']$, from the first diagram, and $l_1\circ[Id_{\mathbb{K}}\otimes k'] = k'\circ l_1'$, by Lemma 1.38.

The commutativity of this square,

$$\begin{tikzcd}
V^*\otimes V\otimes U^*\otimes U \arrow[r, "l_1'\circ(j_1'(Ev_V\otimes Id_{U^*\otimes U}))"] \arrow[d, "a_1"] & U^*\otimes U \arrow[d, "Ev_U"] \\
(V\otimes U)^*\otimes V\otimes U \arrow[r, "Ev_{V\otimes U}"] & \mathbb{K}
\end{tikzcd}$$

$$\begin{aligned}
(Ev_{V\otimes U}\circ a_1)(\phi\otimes v\otimes\xi\otimes u) &= Ev_{V\otimes U}((l\circ[\phi\otimes\xi])\otimes v\otimes u) \\
&= \phi(v)\cdot\xi(u) \\
(Ev_U\circ l_1'\circ(j_1'(Ev_V\otimes Id_{U^*\otimes U})))(\phi\otimes v\otimes\xi\otimes u) &= Ev_U(\phi(v)\cdot\xi\otimes u) \\
&= \phi(v)\cdot\xi(u),
\end{aligned}$$

implies that the distinguished elements $Ev_V\otimes Id_{U^*\otimes U}$ and $Ev_{V\otimes U}$ are related by the right column of maps in the second diagram:

$$\begin{aligned}
Ev_V\otimes Id_{U^*\otimes U} &\mapsto Tr_U\circ k'\circ l_1'\circ(j_1'(Ev_V\otimes Id_{U^*\otimes U}))\circ a_1^{-1} \\
&= Ev_U\circ l_1'\circ(j_1'(Ev_V\otimes Id_{U^*\otimes U}))\circ a_1^{-1} \\
&= Ev_{V\otimes U}.
\end{aligned}$$

The above equation used the definition of $Tr_U$. Along the top row, the key step uses the definition of $Tr_V$:

$$a_2^{-1}(Ev_V\otimes Id_{U^*\otimes U}) = ((k^*)^{-1}(Ev_V))\otimes(k'\circ Id_{U^*\otimes U}\circ(k')^{-1}) = Tr_V\otimes Id_{\mathrm{End}(U)}.$$

The Theorem follows:

$$\begin{aligned}
Tr_U\circ Tr_{V;U,U} &= (\mathrm{Hom}(Id_{\mathrm{End}(V\otimes U)},Tr_U)\circ\mathrm{Hom}(j_2^{-1},l_1)\circ j_1)(Tr_V\otimes Id_{\mathrm{End}(U)}) \\
&= (\mathrm{Hom}(Id_{\mathrm{End}(V\otimes U)},Tr_U)\circ\mathrm{Hom}(j_2^{-1},l_1)\circ j_1\circ a_2^{-1})(Ev_V\otimes Id_{U^*\otimes U}) \\
&= (k''^*)^{-1}(Ev_{V\otimes U}) \\
&= Tr_{V\otimes U}.
\end{aligned}$$

∎

Remark 2.35. The previous Theorem appears in slightly different form in [K] §II.3. The following Corollary is a well-known identity for the (scalar valued) trace ([B] §II.4.4, [Magnus] §1.10, [K] §II.6).

Corollary 2.36. *For finite-dimensional $V$ and $U$, and $A:V\to V$, $B:U\to U$,*

$$Tr_{V\otimes U}(j_2(A\otimes B)) = Tr_V(A)\cdot Tr_U(B).$$

Proof. As in Example 2.25,

$$\begin{aligned}
Tr_{V\otimes U}(j_2(A\otimes B)) &= Tr_U(Tr_{V;U,U}(j_2(A\otimes B))) \\
&= Tr_U(Tr_V(A)\cdot B) \\
&= Tr_V(A)\cdot Tr_U(B).
\end{aligned}$$

∎

The result of Corollary 2.36 could also be proved directly using methods similar to the previous proof, and could be stated as the equality

$$j_2^*(Tr_{V\otimes U}) = (\mathrm{Hom}(Id_{\mathrm{End}(V)\otimes\mathrm{End}(U)}, l)\circ j)(Tr_V\otimes Tr_U) \in (\mathrm{End}(V)\otimes\mathrm{End}(U))^*,$$

or

$$Tr_{V\otimes U}\circ j_2 = l\circ[Tr_V\otimes Tr_U].$$

COROLLARY 2.37. ([$\mathbf{G}_2$] §I.5) *For finite-dimensional $V$ and $U$, and $A : V\to V$, $B : U\to U$,*

$$Tr_{\mathrm{Hom}(V,U)}(\mathrm{Hom}(A,B)) = Tr_V(A)\cdot Tr_U(B).$$

PROOF. By Lemma 1.62, $\mathrm{Hom}(A,B) = k_{VU}\circ[A^*\otimes B]\circ k_{VU}^{-1}$, so Lemma 2.6, Corollary 2.36, and Lemma 2.5 apply:

$$\begin{aligned}
Tr_{\mathrm{Hom}(V,U)}(\mathrm{Hom}(A,B)) &= Tr_{\mathrm{Hom}(V,U)}(k_{VU}\circ[A^*\otimes B]\circ k_{VU}^{-1})\\
&= Tr_{V^*\otimes U}([A^*\otimes B])\\
&= Tr_{V^*}(A^*)\cdot Tr_U(B)\\
&= Tr_V(A)\cdot Tr_U(B).
\end{aligned}$$

$\blacksquare$

THEOREM 2.38. *For finite-dimensional $V$ and $V'$, and $A : V\otimes V'\otimes U\to V\otimes V'\otimes W$,*

$$Tr_{V\otimes V';U,W}(A) = Tr_{V';U,W}(Tr_{V;V'\otimes U,V'\otimes W}(A)).$$

PROOF. In the following diagram,

$$\begin{tikzcd}
M_{11} \arrow[r, "a_1"] \arrow[d, "j_1'"] & M_{12} \arrow[d, "j_1''"] \\
M_{21} \arrow[r, "a_2"] \arrow[d, shift left, "\mathrm{Hom}((j_2')^{-1},l_1)"] & M_{22} \arrow[d, shift left, "\mathrm{Hom}((j_2'')^{-1},l_1)"] \\
M_{31} \arrow[r, "a_3"] \arrow[u, shift left, "\mathrm{Hom}(j_2',l_1^{-1})"] & M_{32} \arrow[u, shift left, "\mathrm{Hom}(j_2'',l_1^{-1})"]
\end{tikzcd}$$

the objects are

$$\begin{aligned}
M_{11} &= \mathrm{End}(V')^*\otimes\mathrm{End}(\mathrm{Hom}(U,W))\\
M_{21} &= \mathrm{Hom}(\mathrm{End}(V')\otimes\mathrm{Hom}(U,W), \mathbb{K}\otimes\mathrm{Hom}(U,W))\\
M_{31} &= \mathrm{Hom}(\mathrm{Hom}(V'\otimes U, V'\otimes W), \mathrm{Hom}(U,W))\\
M_{12} &= \mathrm{End}(V\otimes V')^*\otimes\mathrm{End}(\mathrm{Hom}(U,W))\\
M_{22} &= \mathrm{Hom}(\mathrm{End}(V\otimes V')\otimes\mathrm{Hom}(U,W), \mathbb{K}\otimes\mathrm{Hom}(U,W))\\
M_{32} &= \mathrm{Hom}(\mathrm{Hom}(V\otimes V'\otimes U, V\otimes V'\otimes W), \mathrm{Hom}(U,W)),
\end{aligned}$$

the horizontal arrows are

$$\begin{aligned}
a_1 &= [(Tr_{V;V',V'})^*\otimes Id_{\mathrm{End}(\mathrm{Hom}(U,W))}]\\
a_2 &= \mathrm{Hom}([Tr_{V;V',V'}\otimes Id_{\mathrm{Hom}(U,W)}], Id_{\mathbb{K}\otimes\mathrm{Hom}(U,W)})\\
a_3 &= \mathrm{Hom}(Tr_{V;V'\otimes U,V'\otimes W}, Id_{\mathrm{Hom}(U,W)}),
\end{aligned}$$

and the statement of the Theorem is that

$$a_3(Tr_{V';U,W}) = Tr_{V\otimes V';U,W}.$$

The commutativity of this square,

$$\begin{tikzcd}
\mathrm{End}(V)\otimes\mathrm{End}(V')\otimes\mathrm{Hom}(U,W) \arrow[r, "{[j_2'''\otimes Id_{\mathrm{Hom}(U,W)}]}"] \arrow[d, "{[Id_{\mathrm{End}(V)}\otimes j_2']}"] & \mathrm{End}(V\otimes V')\otimes\mathrm{Hom}(U,W) \arrow[d, "j_2''"'] \\
\mathrm{End}(V)\otimes\mathrm{Hom}(V'\otimes U, V'\otimes W) \arrow[r, "j_2"] & \mathrm{Hom}(V\otimes V'\otimes U, V\otimes V'\otimes W)
\end{tikzcd}$$

is easy to check, and together with that of the diagram:

$$\begin{tikzcd}
\mathrm{End}(V)\otimes\mathrm{End}(V')\otimes\mathrm{Hom}(U,W) \arrow[r, "{[j_2'''\otimes Id_{\mathrm{Hom}(U,W)}]}"] \arrow[d, "{[Id_{\mathrm{End}(V)}\otimes j_2']}"] & \mathrm{End}(V\otimes V')\otimes\mathrm{Hom}(U,W) \arrow[d, "{[Tr_{V;V',V'}\otimes Id_{\mathrm{Hom}(U,W)}]}"'] \\
\mathrm{End}(V)\otimes\mathrm{Hom}(V'\otimes U, V'\otimes W) \arrow[d, "{j_1(Tr_V\otimes Id_{\mathrm{Hom}(V'\otimes U,V'\otimes W)})}"] & \mathrm{End}(V')\otimes\mathrm{Hom}(U,W) \arrow[d, "j_2'"] \\
\mathbb{K}\otimes\mathrm{Hom}(V'\otimes U, V'\otimes W) \arrow[r, "l_1'"] & \mathrm{Hom}(V'\otimes U, V'\otimes W)
\end{tikzcd}$$

$$\begin{aligned}
D\otimes E\otimes F &\mapsto (j_2'\circ[(Tr_{V;V',V'}\circ j_2''')\otimes Id_{\mathrm{Hom}(U,W)}])(D\otimes E\otimes F)\\
&= j_2'((Tr_{V;V',V'}(j_2'''(D\otimes E)))\otimes F)\\
&= (Tr_V(D))\cdot j_2'(E\otimes F)\\
D\otimes E\otimes F &\mapsto (l_1'\circ(j(Tr_V\otimes j_2')))(D\otimes E\otimes F)\\
&= l_1'((Tr_V(D))\otimes(j_2'(E\otimes F)))\\
&= (Tr_V(D))\cdot j_2'(E\otimes F),
\end{aligned}$$

implies

$$\begin{aligned}
Tr_{V;V'\otimes U,V'\otimes W}\circ j_2'' &= l_1'\circ(j_1(Tr_V\otimes Id_{\mathrm{Hom}(V'\otimes U,V'\otimes W)}))\circ j_2^{-1}\circ j_2''\\
&= l_1'\circ(j_1(Tr_V\otimes Id_{\mathrm{Hom}(V'\otimes U,V'\otimes W)}))\\
&\quad\circ[Id_{\mathrm{End}(V)}\otimes j_2']\circ[(j_2''')^{-1}\otimes Id_{\mathrm{Hom}(U,W)}]\\
&= j_2'\circ[Tr_{V;V',V'}\otimes Id_{\mathrm{Hom}(U,W)}],
\end{aligned}$$

which is what is needed to show that the lower square of the first diagram is commutative. Its upper square is commutative by Lemma 1.37, and the distinguished elements in the top row are related by Theorem 2.34:

$$a_1(Tr_{V'}\otimes Id_{\mathrm{Hom}(U,W)}) = (Tr_{V'}\circ Tr_{V;V',V'})\otimes Id_{\mathrm{Hom}(U,W)} = Tr_{V\otimes V'}\otimes Id_{\mathrm{Hom}(U,W)}.$$

The Theorem follows:

$$\begin{aligned}
a_3(Tr_{V;U,W}) &= (a_3\circ\mathrm{Hom}((j_2')^{-1},l_1)\circ j_1')(Tr_{V'}\otimes Id_{\mathrm{Hom}(U,W)})\\
&= (\mathrm{Hom}((j_2'')^{-1},l_1)\circ j_1''\circ a_1)(Tr_{V'}\otimes Id_{\mathrm{Hom}(U,W)})\\
&= (\mathrm{Hom}((j_2'')^{-1},l_1)\circ j_1'')(Tr_{V\otimes V'}\otimes Id_{\mathrm{Hom}(U,W)})\\
&= Tr_{V\otimes V';U,W}.
\end{aligned}$$

∎

REMARK 2.39. The result of the above Theorem is another "vanishing" property of the generalized trace ([**JSV**]).

The maps

$$\begin{aligned} j_3 &: \operatorname{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \operatorname{Hom}(V_2 \otimes U_2, V_2 \otimes W_2) \\ &\to \operatorname{Hom}(V_1 \otimes U_1 \otimes V_2 \otimes U_2, V_1 \otimes W_1 \otimes V_2 \otimes W_2), \\ j_4 &: \operatorname{Hom}(U_1, W_1) \otimes \operatorname{Hom}(U_2, W_2) \to \operatorname{Hom}(U_1 \otimes U_2, W_1 \otimes W_2) \end{aligned}$$

appear in the following Theorem comparing the trace of a tensor product to the tensor product of traces. There are also some switching maps, as in Theorem 2.34,

$$\begin{aligned} s_1 : V_1 \otimes W_1 \otimes V_2 \otimes W_2 \to V_1 \otimes V_2 \otimes W_1 \otimes W_2 \\ s_2 : V_1 \otimes V_2 \otimes U_1 \otimes U_2 \to V_1 \otimes U_1 \otimes V_2 \otimes U_2. \end{aligned}$$

THEOREM 2.40. *For finite-dimensional $V_1$ and $V_2$, and maps $A : V_1 \otimes U_1 \to V_1 \otimes W_1$ and $B : V_2 \otimes U_2 \to V_2 \otimes W_2$,*

$$Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}(s_1 \circ (j_3(A \otimes B)) \circ s_2) = j_4((Tr_{V_1; U_1, W_1}(A)) \otimes (Tr_{V_2; U_2, W_2}(B))).$$

PROOF. In the following diagram,

$$\begin{tikzcd}
M_{11} \arrow[r, "a_1"] \arrow[d, "j_1"] & M_{12} \arrow[d, "j_5"] & M_{13} \arrow[l, "a_2"'] \arrow[d, "j_6"] & M_{14} \arrow[l, "{[j \otimes j] \circ s_3}"'] \arrow[d, "{[j_1^1 \otimes j_1^2]}"] \\
M_{21} \arrow[r, "a_3"] \arrow[d, shift left, "a_5^{-1}"] & M_{22} \arrow[d, shift left, "a_6^{-1}"] & M_{23} \arrow[l, "a_4"'] \arrow[d, shift left, "a_7^{-1}"] & M_{24} \arrow[l, "{\operatorname{Hom}(s_4, s_5) \circ j_7}"'] \arrow[d, shift left, "a_8^{-1}"] \\
M_{31} \arrow[r, "a_9"] \arrow[u, shift left, "a_5"] & M_{32} \arrow[u, shift left, "a_6"] & M_{33} \arrow[l, "a_{10}"'] \arrow[u, shift left, "a_7"] & M_{34} \arrow[l, "j_8"'] \arrow[u, shift left, "a_8"]
\end{tikzcd}$$

the objects are

$$\begin{aligned}
M_{11} &= \mathrm{End}(V_1 \otimes V_2)^* \otimes \mathrm{End}(\mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{21} &= \mathrm{Hom}(\mathrm{End}(V_1 \otimes V_2) \otimes \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2), \\
&\quad \mathbb{K} \otimes \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{31} &= \mathrm{Hom}(\mathrm{Hom}(V_1 \otimes V_2 \otimes U_1 \otimes U_2, V_1 \otimes V_2 \otimes W_1 \otimes W_2), \\
&\quad \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{12} &= (\mathrm{End}(V_1) \otimes \mathrm{End}(V_2))^* \otimes \\
&\quad \mathrm{Hom}(\mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2), \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{22} &= \mathrm{Hom}(\mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \otimes \mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2), \\
&\quad \mathbb{K} \otimes \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{32} &= \mathrm{Hom}(\mathrm{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \\
&\quad \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)) \\
M_{13} &= \mathrm{Hom}(\mathrm{End}(V_1) \otimes \mathrm{End}(V_2), \mathbb{K} \otimes \mathbb{K}) \otimes \mathrm{End}(\mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2)) \\
M_{23} &= \mathrm{Hom}(\mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \otimes \mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2), \\
&\quad \mathbb{K} \otimes \mathbb{K} \otimes \mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2)) \\
M_{33} &= \mathrm{Hom}(\mathrm{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \\
&\quad \mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2)) \\
M_{14} &= \mathrm{End}(V_1)^* \otimes \mathrm{End}(\mathrm{Hom}(U_1, W_1)) \otimes \mathrm{End}(V_2)^* \otimes \mathrm{End}(\mathrm{Hom}(U_2, W_2)) \\
M_{24} &= \mathrm{Hom}(\mathrm{End}(V_1) \otimes \mathrm{Hom}(U_1, W_1), \mathbb{K} \otimes \mathrm{Hom}(U_1, W_1)) \otimes \\
&\quad \mathrm{Hom}(\mathrm{End}(V_2) \otimes \mathrm{Hom}(U_2, W_2), \mathbb{K} \otimes \mathrm{Hom}(U_2, W_2)) \\
M_{34} &= \mathrm{Hom}(\mathrm{Hom}(V_1 \otimes U_1, V_1 \otimes W_1), \mathrm{Hom}(U_1, W_1)) \otimes \\
&\quad \mathrm{Hom}(\mathrm{Hom}(V_2 \otimes U_2, V_2 \otimes W_2), \mathrm{Hom}(U_2, W_2));
\end{aligned}$$

the left, right columns define $Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}$ and $Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2}$. The arrows are

$$\begin{aligned}
a_1 &= [(j_2')^* \otimes \mathrm{Hom}(j_4, Id_{\mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)})] \\
a_2 &= [\mathrm{Hom}(Id_{\mathrm{End}(V_1) \otimes \mathrm{End}(V_2)}, l_{\mathbb{K}}) \otimes \mathrm{Hom}(Id_{\mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2)}, j_4)] \\
a_3 &= \mathrm{Hom}([j_2' \otimes j_4], Id_{\mathbb{K} \otimes \mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
a_4 &= \mathrm{Hom}(Id_{\mathrm{End}(V_1) \otimes \mathrm{End}(V_2) \otimes \mathrm{Hom}(U_1, W_1) \otimes \mathrm{Hom}(U_2, W_2)}, [l_{\mathbb{K}} \otimes j_4]) \\
a_5 &= \mathrm{Hom}(j_2, l_1^{-1}) \\
a_6 &= \mathrm{Hom}([j_2^1 \otimes j_2^2] \circ s_4, l_1^{-1}) \\
a_7 &= \mathrm{Hom}([j_2^1 \otimes j_2^2] \circ s_4, (l \circ l)^{-1}) \\
a_8 &= [\mathrm{Hom}(j_2^1, (l_1^1)^{-1}) \otimes \mathrm{Hom}(j_2^2, (l_1^2)^{-1})] \\
a_9 &= \mathrm{Hom}(\mathrm{Hom}(s_2, s_1) \circ j_3, Id_{\mathrm{Hom}(U_1 \otimes U_2, W_1 \otimes W_2)}) \\
a_{10} &= \mathrm{Hom}(Id_{\mathrm{Hom}(V_1 \otimes U_1, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2 \otimes U_2, V_2 \otimes W_2)}, j_4).
\end{aligned}$$

The Theorem claims the two maps

$$\begin{aligned}
Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2} \circ \mathrm{Hom}(s_2, s_1) \circ j_3 &= a_9(Tr_{V_1 \otimes V_2; U_1 \otimes U_2, W_1 \otimes W_2}), \\
j_4 \circ (j_8(Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2})) &= (a_{10} \circ j_8)(Tr_{V_1; U_1, W_1} \otimes Tr_{V_2; U_2, W_2})
\end{aligned}$$

are equal. The diagram is commutative— all six squares are easy to check, for example, the upper left and upper middle follow from Lemma 1.37, and each of the remaining four involves two arrows with switching maps. The equality along the top row,

$$\begin{array}{rcl} a_1 & : & Tr_{V_1\otimes V_2}\otimes Id_{\mathrm{Hom}(U_1\otimes U_2,W_1\otimes W_2)}\mapsto (Tr_{V_1\otimes V_2}\circ j_2')\otimes j_4, \\ a_2\circ[j\otimes j]\circ s_3 & : & Tr_{V_1}\otimes Id_{\mathrm{Hom}(U_1,W_1)}\otimes Tr_{V_2}\otimes Id_{\mathrm{Hom}(U_2,W_2)} \\ & \mapsto & (l_{\mathbb{K}}\circ[Tr_{V_1}\otimes Tr_{V_2}])\otimes j_4, \end{array}$$

follows from Corollary 2.36. This key step, together with the commutativity of the diagram, proves the Theorem:

$$\begin{array}{rcl} a_9 & : & (Tr_{V_1\otimes V_2;U_1\otimes U_2,W_1\otimes W_2}) \\ & \mapsto & (a_9\circ a_5^{-1}\circ j_1)(Tr_{V_1\otimes V_2}\otimes Id_{\mathrm{Hom}(U_1\otimes U_2,W_1\otimes W_2)}) \\ & = & (a_6^{-1}\circ j_5\circ a_1)(Tr_{V_1\otimes V_2}\otimes Id_{\mathrm{Hom}(U_1\otimes U_2,W_1\otimes W_2)}) \\ & = & (a_6^{-1}\circ j_5\circ a_2\circ[j\otimes j]\circ s_3)(Tr_{V_1}\otimes Id_{\mathrm{Hom}(U_1,W_1)}\otimes Tr_{V_2}\otimes Id_{\mathrm{Hom}(U_2,W_2)}) \\ & = & (a_{10}\circ j_8\circ a_8^{-1}\circ[j_1^1\otimes j_1^2])(Tr_{V_1}\otimes Id_{\mathrm{Hom}(U_1,W_1)}\otimes Tr_{V_2}\otimes Id_{\mathrm{Hom}(U_2,W_2)}) \\ & = & (a_{10}\circ j_8)(Tr_{V_1;U_1,W_1}\otimes Tr_{V_2;U_2,W_2}). \end{array}$$

∎

REMARK 2.41. The maps $j_4$, from the previous Theorem, and

$$j_0:\mathrm{Hom}(V\otimes U_1,V\otimes W_1)\otimes\mathrm{Hom}(U_2,W_2)\to\mathrm{Hom}(V\otimes U_1\otimes U_2,V\otimes W_1\otimes W_2)$$

appear in the following Corollary about the compatibility of the trace and the tensor product, related to a "superposing" identity of [**JSV**].

COROLLARY 2.42. *For finite-dimensional $V$, and maps $A:V\otimes U_1\to V\otimes W_1$, $B:U_2\to W_2$,*

$$Tr_{V;U_1\otimes U_2,W_1\otimes W_2}(j_0(A\otimes B)) = j_4((Tr_{V;U_1,W_1}(A))\otimes B).$$

PROOF. It can be checked that the following diagram is commutative.

$$\begin{tikzcd} V\otimes U_1\otimes\mathbb{K}\otimes U_2 \arrow[rr, "j_3(A\otimes(l_{W_2}^{-1}\circ B\circ l_{U_2}))"] & & V\otimes W_1\otimes\mathbb{K}\otimes W_2 \arrow[d, "s_1"] \\ V\otimes\mathbb{K}\otimes U_1\otimes U_2 \arrow[u, "s_2"] \arrow[d, "{[l_V\otimes Id_{U_1\otimes U_2}]}"] & & V\otimes\mathbb{K}\otimes W_1\otimes W_2 \arrow[d, "{[l_V\otimes Id_{W_1\otimes W_2}]}"] \\ V\otimes U_1\otimes U_2 \arrow[rr, "j_0(A\otimes B)"] & & V\otimes W_1\otimes W_2 \end{tikzcd}$$

Theorem 2.29, the diagram, the previous Theorem, and finally Theorem 2.26 apply:

$$\begin{array}{rcl} LHS & = & Tr_{V;U_1\otimes U_2,W_1\otimes W_2}((j_0(A\otimes B))\circ[l_V\otimes Id_{U_1\otimes U_2}]\circ[l_V^{-1}\otimes Id_{U_1\otimes U_2}]) \\ & = & Tr_{V\otimes\mathbb{K};U_1\otimes U_2,W_1\otimes W_2}([l_V^{-1}\otimes Id_{W_1\otimes W_2}]\circ(j_0(A\otimes B))\circ[l_V\otimes Id_{U_1\otimes U_2}]) \\ & = & Tr_{V\otimes\mathbb{K};U_1\otimes U_2,W_1\otimes W_2}(s_1\circ(j_3(A\otimes(l_{W_2}^{-1}\circ B\circ l_{U_2})))\circ s_2) \\ & = & j_4((Tr_{V;U_1,W_1}(A))\otimes(Tr_{\mathbb{K};U_2,W_2}(\mathrm{Hom}(l_{U_2},l_{W_2}^{-1})(B)))) \\ & = & j_4((Tr_{V;U_1,W_1}(A))\otimes B). \end{array}$$

∎

NOTATION 2.43. Denote the composite maps

$$\begin{array}{rcl}\tilde{j}_U = \mathrm{Hom}(Id_{V\otimes U}, l)\circ j & : & V^*\otimes U^* \to (V\otimes U)^* \\ \tilde{j}_W = \mathrm{Hom}(Id_{V\otimes W}, l)\circ j & : & V^*\otimes W^* \to (V\otimes W)^*,\end{array}$$

where $l$ is multiplication $\mathbb{K}\otimes\mathbb{K}\to\mathbb{K}$.

If $V$ is finite-dimensional then these $\tilde{j}$ maps are invertible. A composition like this already appeared in Lemma 2.32, and these maps appear in the next Theorem 2.44, relating the trace of the transpose to the transpose of the trace, using $t$ maps as in Notation 1.9. The $\tilde{j}$ maps appear again in the next Chapter, but the tilde notation will only be used when abbreviating is more useful than not.

THEOREM 2.44. *For finite-dimensional $V$ and a map $H : V\otimes U \to V\otimes W$,*

$$Tr_{V^*;W^*,U^*}(\tilde{j}_U^{-1}\circ(t_{V\otimes U,V\otimes W}(H))\circ\tilde{j}_W) = t_{UW}(Tr_{V;U,W}(H)).$$

PROOF. In the following diagram,

$$\begin{tikzcd}
M_{11} \arrow[r, "a_1"] \arrow[d, "j_1"] & M_{12} \arrow[d, "j_1''"] & M_{13} \arrow[l, "a_2"'] \arrow[d, "j_1'"] \\
M_{21} \arrow[r, "a_3"] \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1},l_1)"] & M_{22} \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1},l_1')"] & M_{23} \arrow[l, "a_4"'] \arrow[d, shift left, "\mathrm{Hom}((j_2')^{-1},l_1')"] \\
M_{31} \arrow[r, "a_5"] \arrow[u, shift left, "\mathrm{Hom}(j_2,l_1^{-1})"] & M_{32} \arrow[u, shift left, "\mathrm{Hom}(j_2,(l_1')^{-1})"] & M_{33} \arrow[l, "a_6"'] \arrow[u, shift left, "\mathrm{Hom}(j_2',(l_1')^{-1})"]
\end{tikzcd}$$

the objects are

$$\begin{array}{rcl}
M_{11} & = & \mathrm{End}(V)^*\otimes\mathrm{End}(\mathrm{Hom}(U,W)) \\
M_{21} & = & \mathrm{Hom}(\mathrm{End}(V)\otimes\mathrm{Hom}(U,W),\mathbb{K}\otimes\mathrm{Hom}(U,W)) \\
M_{31} & = & \mathrm{Hom}(\mathrm{Hom}(V\otimes U,V\otimes W),\mathrm{Hom}(U,W)) \\
M_{12} & = & \mathrm{End}(V)^*\otimes\mathrm{Hom}(\mathrm{Hom}(U,W),\mathrm{Hom}(W^*,U^*)) \\
M_{22} & = & \mathrm{Hom}(\mathrm{End}(V)\otimes\mathrm{Hom}(U,W),\mathbb{K}\otimes\mathrm{Hom}(W^*,U^*)) \\
M_{32} & = & \mathrm{Hom}(\mathrm{Hom}(V\otimes U,V\otimes W),\mathrm{Hom}(W^*,U^*)) \\
M_{13} & = & \mathrm{End}(V^*)^*\otimes\mathrm{End}(\mathrm{Hom}(W^*,U^*)) \\
M_{23} & = & \mathrm{Hom}(\mathrm{End}(V^*)\otimes\mathrm{Hom}(W^*,U^*),\mathbb{K}\otimes\mathrm{Hom}(W^*,U^*)) \\
M_{33} & = & \mathrm{Hom}(\mathrm{Hom}(V^*\otimes W^*,V^*\otimes U^*),\mathrm{Hom}(W^*,U^*)),
\end{array}$$

the arrows are

$$\begin{array}{rcl}
a_1 & = & [Id_{\mathrm{End}(V)^*}\otimes\mathrm{Hom}(Id_{\mathrm{Hom}(U,W)},t_{UW})] \\
a_2 & = & [t_{VV}^*\otimes\mathrm{Hom}(t_{UW},Id_{\mathrm{Hom}(W^*,U^*)})] \\
a_3 & = & \mathrm{Hom}(Id_{\mathrm{End}(V)\otimes\mathrm{Hom}(U,W)},[Id_{\mathbb{K}}\otimes t_{UW}]) \\
a_4 & = & \mathrm{Hom}([t_{VV}\otimes t_{UW}],Id_{\mathbb{K}\otimes\mathrm{Hom}(W^*,U^*)}) \\
a_5 & = & \mathrm{Hom}(Id_{\mathrm{Hom}(V\otimes U,V\otimes W)},t_{UW}) \\
a_6 & = & \mathrm{Hom}(\mathrm{Hom}(\tilde{j}_W,\tilde{j}_U^{-1})\circ t_{V\otimes U,V\otimes W},Id_{\mathrm{Hom}(W^*,U^*)}),
\end{array}$$

and the two quantities in the statement of the Theorem are

$$\begin{array}{rcl}
t_{UW}(Tr_{V;U,W}(H)) & = & (a_5(Tr_{V;U,W}))(H) \\
Tr_{V^*;W^*,U^*}(\mathrm{Hom}(\tilde{j}_W,\tilde{j}_U^{-1})(t_{V\otimes U,V\otimes W}(H))) & = & (a_6(Tr_{V^*;W^*,U^*}))(H).
\end{array}$$

The diagram is commutative, for example, the lower right square:

$$\begin{array}{rcl} E\otimes F & \mapsto & (\mathrm{Hom}(Id_{V^*\otimes W^*},\tilde{j}_U)\circ j_2'\circ[t_{VV}\otimes t_{UW}])(E\otimes F) \\ & = & \tilde{j}_U\circ(j_2'((t_{VV}(E))\otimes(t_{UW}(F)))): \\ \phi\otimes\xi & \mapsto & \tilde{j}_U((E^*(\phi))\otimes(F^*(\xi))): \\ v\otimes u & \mapsto & \phi(E(v))\cdot\xi(F(u)), \\ E\otimes F & \mapsto & (\mathrm{Hom}(\tilde{j}_W, Id_{V^*\otimes U^*})\circ t_{V\otimes U,V\otimes W}\circ j_2)(E\otimes F) \\ & = & (t_{V\otimes U,V\otimes W}(j_2(E\otimes F)))\circ\tilde{j}_W: \\ \phi\otimes\xi & \mapsto & (\tilde{j}_W(\phi\otimes\xi))\circ(j_2(E\otimes F)): \\ v\otimes u & \mapsto & \phi(E(v))\cdot\xi(F(u)). \end{array}$$

Lemma 2.5 implies the equality of the outputs of the distinguished elements in the top row:

$$\begin{array}{rcl} a_1(Tr_V\otimes Id_{\mathrm{Hom}(U,W)}) & = & Tr_V\otimes t_{UW} \\ a_2(Tr_{V^*}\otimes Id_{\mathrm{Hom}(W^*,U^*)}) & = & (t_{VV}^*(Tr_{V^*}))\otimes t_{UW} \\ & = & Tr_V\otimes t_{UW}, \end{array}$$

and the Theorem follows from the commutativity of the diagram:

$$\begin{array}{rcl} a_5(Tr_{V;U,W}) & = & (a_5\circ\mathrm{Hom}(j_2^{-1},l_1)\circ j_1)(Tr_V\otimes Id_{\mathrm{Hom}(U,W)}) \\ & = & (\mathrm{Hom}(j_2^{-1},l_1')\circ j_1''\circ a_1)(Tr_V\otimes Id_{\mathrm{Hom}(U,W)}) \\ & = & (\mathrm{Hom}(j_2^{-1},l_1')\circ j_1''\circ a_2)(Tr_{V^*}\otimes Id_{\mathrm{Hom}(W^*,U^*)}) \\ & = & (a_6\circ\mathrm{Hom}((j_2')^{-1},l_1')\circ j_1')(Tr_{V^*}\otimes Id_{\mathrm{Hom}(W^*,U^*)}) \\ & = & a_6(Tr_{V^*;W^*,U^*}). \end{array}$$

∎

EXERCISE 2.45. Given a direct sum $V=V_1\oplus V_2$, and $A:V_1\otimes U\to V_1\otimes W$, and $B:V_2\otimes U\to V_2\otimes W$, define $A\oplus B:V\otimes U\to V\otimes W$ using the projections and inclusions from Example 1.81:

$$A\oplus B=[Q_1\otimes Id_W]\circ A\circ[P_1\otimes Id_U]+[Q_2\otimes Id_W]\circ B\circ[P_2\otimes Id_U].$$

If $V$ is finite-dimensional, then

$$Tr_{V;U,W}(A\oplus B)=Tr_{V_1;U,W}(A)+Tr_{V_2;U,W}(B).$$

HINT. The proof proceeds exactly as in Proposition 2.12, using Theorem 2.29. ∎

EXERCISE 2.46. For $V=V_1\oplus V_2$ as above, and $K:V\otimes U\to V\otimes W$,

$$\begin{array}{rcl} Tr_{V;U,W}(K) & = & Tr_{V_1;U,W}([P_1\otimes Id_W]\circ K\circ[Q_1\otimes Id_U]) \\ & & +Tr_{V_2;U,W}([P_2\otimes Id_W]\circ K\circ[Q_2\otimes Id_U]). \end{array}$$

HINT. Using Theorem 2.29 and Lemma 1.36,

$$Tr_{V_i;U,W}([P_i\otimes Id_W]\circ K\circ[Q_i\otimes Id_U])=Tr_{V;U,W}([(Q_i\circ P_i)\otimes Id_W]\circ K).$$

The proof proceeds exactly as in Proposition 2.13. ∎

PROPOSITION 2.47. *For finite-dimensional $V_1$ and $V_2$, maps $A : V_1 \otimes U_1 \to V_2 \otimes W_2$, $B : V_2 \otimes U_2 \to V_1 \otimes W_1$, and switching maps as in the following diagrams,*

$$\begin{tikzcd}
V_1 \otimes U_2 \otimes U_1 \arrow[d, "s_1"] & V_2 \otimes U_2 \otimes U_1 \arrow[d, "{[B \otimes Id_{U_1}]}"] \\
V_1 \otimes U_1 \otimes U_2 \arrow[d, "{[A \otimes Id_{U_2}]}"] & V_1 \otimes W_1 \otimes U_1 \arrow[d, "s_3"] \\
V_2 \otimes W_2 \otimes U_2 \arrow[d, "s_2"] & V_1 \otimes U_1 \otimes W_1 \arrow[d, "{[A \otimes Id_{W_1}]}"] \\
V_2 \otimes U_2 \otimes W_2 \arrow[d, "{[B \otimes Id_{W_2}]}"] & V_2 \otimes W_2 \otimes W_1 \arrow[d, "s_4"] \\
V_1 \otimes W_1 \otimes W_2 & V_2 \otimes W_1 \otimes W_2
\end{tikzcd}$$

*the traces of the composites are equal:*

$$\begin{aligned}
& Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}([B \otimes Id_{W_2}] \circ s_2 \circ [A \otimes Id_{U_2}] \circ s_1) \\
= \quad & Tr_{V_2;U_2\otimes U_1,W_1\otimes W_2}(s_4 \circ [A \otimes Id_{W_1}] \circ s_3 \circ [B \otimes Id_{U_1}]).
\end{aligned}$$

PROOF. The canonical map

$$j : \mathrm{Hom}(V_2, V_1) \otimes \mathrm{Hom}(U_2, W_1) \to \mathrm{Hom}(V_2 \otimes U_2, V_1 \otimes W_1)$$

is invertible by the finite-dimensionality hypothesis and Claim 1.34. Also consider a map

$$Q_1^1 : \mathbb{K} \to \mathrm{End}(W_2) : 1 \mapsto Id_{W_2},$$

as in (2.4) (and again not necessarily from some direct sum as in Example 2.9), and analogously

$$\tilde{Q}_1^1 : \mathbb{K} \to \mathrm{End}(U_1) : 1 \mapsto Id_{U_2}.$$

The claim of the Proposition is that the lower block of this diagram is commutative, starting with $B \otimes 1$ in the upper right.

$$\begin{tikzcd}
\mathrm{Hom}(V_2,V_1)\otimes\mathrm{Hom}(U_2,W_1)\otimes\mathbb{K} \arrow[r, "{[j\otimes Id_{\mathbb{K}}]}"] \arrow[d, "{[j\otimes\tilde{Q}_1^1]}"'] & \mathrm{Hom}(V_2\otimes U_2,V_1\otimes W_1)\otimes\mathbb{K} \arrow[dl, "{[Id_{\mathrm{Hom}(V_2\otimes U_2,V_1\otimes W_1)}\otimes\tilde{Q}_1^1]}"'] \arrow[d, "{[Id_{\mathrm{Hom}(V_2\otimes U_2,V_1\otimes W_1)}\otimes Q_1^1]}"'] \\
\mathrm{Hom}(V_2\otimes U_2,V_1\otimes W_1)\otimes\mathrm{End}(U_1) \arrow[d, "j''"'] & \mathrm{Hom}(V_2\otimes U_2,V_1\otimes W_1)\otimes\mathrm{End}(W_2) \arrow[d, "j'"'] \\
\mathrm{Hom}(V_2\otimes U_2\otimes U_1,V_1\otimes W_1\otimes U_1) \arrow[d, "{\mathrm{Hom}(Id_{V_2\otimes U_2\otimes U_1},s_4\otimes[A\otimes Id_{W_1}]\circ s_3)}"] & \mathrm{Hom}(V_2\otimes U_2\otimes W_2,V_1\otimes W_1\otimes W_2) \arrow[d, "{\mathrm{Hom}(s_2\otimes[A\otimes Id_{U_2}]\circ s_1,Id_{V_1\otimes W_1\otimes W_2})}"'] \\
\mathrm{Hom}(V_2\otimes U_2\otimes U_1,V_2\otimes W_1\otimes W_2) \arrow[d, "Tr_{V_2;U_2\otimes U_1,W_1\otimes W_2}"] & \mathrm{Hom}(V_1\otimes U_2\otimes U_1,V_1\otimes W_1\otimes W_2) \arrow[dl, "Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}"] \\
\mathrm{Hom}(U_2\otimes U_1,W_1\otimes W_2) &
\end{tikzcd}$$

The upper triangle of the above diagram is commutative, so to prove the claim it is enough, by the invertibility of the upper arrow $[j \otimes Id_{\mathbb{K}}]$, to check the outside of the diagram, starting with $B_1 \otimes B_2 \otimes 1$ in the upper left, for $B_1 : V_2 \to V_1$, $B_2 : U_2 \to W_1$, so that the downward composite on the left side is equal to the composite path going clockwise around the right.

The following easily checked diagram shows how the $B_2$ factor commutes with $A$.

$$\begin{tikzcd}
V_1\otimes U_2\otimes U_1 \arrow[r, "{[Id_{V_1}\otimes[B_2\otimes Id_{U_1}]]}"] \arrow[d, "s_1"] & V_1\otimes W_1\otimes U_1 \arrow[d, "s_3"] \\
V_1\otimes U_1\otimes U_2 \arrow[r, "{[Id_{V_1\otimes U_1}\otimes B_2]}"] \arrow[d, "{[A\otimes Id_{U_2}]}"] & V_1\otimes U_1\otimes W_1 \arrow[d, "{[A\otimes Id_{W_1}]}"] \\
V_2\otimes W_2\otimes U_2 \arrow[r, "{[Id_{V_2\otimes W_2}\otimes B_2]}"] \arrow[d, "s_2"] & V_2\otimes W_2\otimes W_1 \arrow[d, "s_4"] \\
V_2\otimes U_2\otimes W_2 \arrow[r, "{[Id_{V_2}\otimes[B_2\otimes Id_{W_2}]]}"] & V_2\otimes W_1\otimes W_2
\end{tikzcd}$$

Using Theorem 2.29,

$$\begin{aligned}
& Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}([[B_1\otimes B_2]\otimes Id_{W_2}]\circ s_2\circ[A\otimes Id_{U_2}]\circ s_1) \\
= {} & Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}([B_1\otimes Id_{W_1\otimes W_2}]\circ[Id_{V_2}\otimes[B_2\otimes Id_{W_2}]]\circ s_2\circ[A\otimes Id_{U_2}]\circ s_1) \\
= {} & Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}([B_1\otimes Id_{W_1\otimes W_2}]\circ s_4\circ[A\otimes Id_{W_1}]\circ s_3\circ[Id_{V_1}\otimes[B_2\otimes Id_{U_1}]]) \\
= {} & Tr_{V_2;U_2\otimes U_1,W_1\otimes W_2}(s_4\circ[A\otimes Id_{W_1}]\circ s_3\circ[Id_{V_1}\otimes[B_2\otimes Id_{U_1}]]\circ[B_1\otimes Id_{U_2\otimes U_1}]) \\
= {} & Tr_{V_2;U_2\otimes U_1,W_1\otimes W_2}(s_4\circ[A\otimes Id_{W_1}]\circ s_3\circ[[B_1\otimes B_2]\otimes Id_{U_1}]).
\end{aligned}$$

∎

EXERCISE 2.48. Using various switching maps, Proposition 2.47 can be used to prove related identities. For example, given maps:

$$\begin{aligned} A' : U_1 \otimes V_1 &\to V_2 \otimes W_2 \\ B' : U_2 \otimes V_2 &\to V_1 \otimes W_1 \\ s_5 : V_1 \otimes U_2 \otimes U_1 &\to U_2 \otimes U_1 \otimes V_1 \\ s_6 : V_2 \otimes U_2 \otimes U_1 &\to U_1 \otimes U_2 \otimes V_2, \end{aligned}$$

the following identity can be proved as a consequence of Proposition 2.47:

$$\begin{aligned} & Tr_{V_1;U_2\otimes U_1,W_1\otimes W_2}([B'\otimes Id_{W_2}]\circ[Id_{U_2}\otimes A']\circ s_5) \\ = \;& Tr_{V_2;U_2\otimes U_1,W_1\otimes W_2}(s_4\circ[A'\otimes Id_{W_1}]\circ[Id_{U_1}\otimes B']\circ s_6). \end{aligned} \tag{2.7}$$

HINT. Let $A = A' \circ s$ and $B = B' \circ s$ for appropriate switching maps $s$. ∎

REMARK 2.49. Equation (2.7) is related to [PS] Proposition 2.7, on the "cyclicity" of the generalized trace.

## 2.3. Vector valued trace

In analogy with Definition 2.24, but with no space $U$, the "vector valued" or "$W$-valued" trace of a map $V \to V \otimes W$ should be an element of $W$. The results on the generalized trace have analogues in this case, but the construction uses different canonical maps.

### 2.3.1. Defining the vector valued trace.

The following Definition 2.50 applies to an arbitrary vector space $W$ and a finite-dimensional space $V$ to define the $W$-valued trace

$$Tr_{V;W} : \mathrm{Hom}(V, V\otimes W) \to W,$$

in terms of the previously defined (scalar) trace $Tr_V$, and canonical maps

$$\begin{aligned} n &: \mathrm{End}(V)\otimes W \to \mathrm{Hom}(V, V\otimes W) \\ j_1' &: \mathrm{End}(V)^*\otimes \mathrm{End}(W) \to \mathrm{Hom}(\mathrm{End}(V)\otimes W, \mathbb{K}\otimes W), \end{aligned}$$

where $j_1'$ is another canonical $j$ map in analogy with $j_1$ from Definition 2.24, and $n$ is invertible by Lemma 1.44.

DEFINITION 2.50. For finite-dimensional $V$,

$$\begin{aligned} Tr_{V;W} &= (\mathrm{Hom}(n^{-1}, l_W)\circ j_1')(Tr_V\otimes Id_W) \\ &= l_W\circ[Tr_V\otimes Id_W]\circ n^{-1}. \end{aligned}$$

EXAMPLE 2.51. A map of the form $n(A\otimes w) : V \to V\otimes W$, for finite-dimensional $V$, $A : V\to V$, and $w\in W$, has trace

$$Tr_{V;W}(n(A\otimes w)) = l_W((j_1'(Tr_V\otimes Id_W))(A\otimes w)) = Tr_V(A)\cdot w.$$

The vector valued trace is related to the generalized trace in two different ways by Theorem 2.52 and Theorem 2.53, which use the two different variants of the $q$ maps from Notation 1.49. Theorem 2.52 relates Definition 2.24 to the case of Definition 2.50 where $W$ is replaced by the vector space $\mathrm{Hom}(U, W)$.

THEOREM 2.52. *For finite-dimensional $V$, and a map $K : V \to V \otimes \mathrm{Hom}(U, W)$,*

$$Tr_{V;\mathrm{Hom}(U,W)}(K) = Tr_{V;U,W}(q(n' \circ K)).$$

PROOF. The maps

$$\begin{aligned} q : \mathrm{Hom}(V, \mathrm{Hom}(U, V \otimes W)) &\to \mathrm{Hom}(V \otimes U, V \otimes W), \\ n' : V \otimes \mathrm{Hom}(U, W) &\to \mathrm{Hom}(U, V \otimes W) \end{aligned}$$

are as in Definition 1.46 and Notation 1.41.

In the following diagram,

$$\begin{tikzcd} M_{11} \arrow[r, shift left, "\mathrm{Hom}(n^{-1}, l_1)"] \arrow[d, shift right, "\mathrm{Hom}(j_2^{-1}, l_1)"] & M_{12} \arrow[l, shift left, "\mathrm{Hom}(n, l_1^{-1})"] \\ M_{21} \arrow[u, shift right, "\mathrm{Hom}(j_2, l_1^{-1})"] \arrow[r, "\mathrm{Hom}(q, Id_{\mathrm{Hom}(U,W)})"] & M_{22} \arrow[u, "\mathrm{Hom}(\mathrm{Hom}(Id_V, n'), Id_{\mathrm{Hom}(U,W)})"'] \end{tikzcd}$$

the objects are

$$\begin{aligned} M_{11} &= \mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(U,W), \mathbb{K} \otimes \mathrm{Hom}(U,W)) \\ M_{21} &= \mathrm{Hom}(\mathrm{Hom}(V \otimes U, V \otimes W), \mathrm{Hom}(U,W)) \\ M_{12} &= \mathrm{Hom}(\mathrm{Hom}(V, V \otimes \mathrm{Hom}(U, W)), \mathrm{Hom}(U, W)) \\ M_{22} &= \mathrm{Hom}(\mathrm{Hom}(V, \mathrm{Hom}(U, V \otimes W)), \mathrm{Hom}(U, W)). \end{aligned}$$

The square is commutative by Lemma 1.6 and Lemma 1.52.

Definition 2.24 and Definition 2.50 are related in this case by $l_W = l_1$ and $j_1' = j_1$. The Theorem follows:

$$\begin{aligned} Tr_{V;\mathrm{Hom}(U,W)} &= (\mathrm{Hom}(n^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\ &= (\mathrm{Hom}(j_2^{-1} \circ q \circ \mathrm{Hom}(Id_V, n'), l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}) \\ &= \mathrm{Hom}(q \circ \mathrm{Hom}(Id_V, n'), Id_{\mathrm{Hom}(U,W)})(Tr_{V;U,W}). \qquad (2.8) \end{aligned}$$

Considering that $q$ and $n'$ are invertible by Lemma 1.47 and Lemma 1.44, line (2.8) can be written as:

$$Tr_{V;U,W} = Tr_{V;\mathrm{Hom}(U,W)} \circ \mathrm{Hom}(Id_V, (n')^{-1}) \circ q^{-1}. \qquad (2.9)$$

∎

THEOREM 2.53. *For finite-dimensional $V$, and a map $F : V \otimes U \to V \otimes W$,*

$$Tr_{V;W} \circ (q^{-1}(F)) = Tr_{V;U,W}(F).$$

PROOF. The map

$$q : \mathrm{Hom}(U, \mathrm{Hom}(V, V \otimes W)) \to \mathrm{Hom}(V \otimes U, V \otimes W) \qquad (2.10)$$

is as in (1.5) from Notation 1.49.

In the following diagram, the map $n$ is as in Definition 2.50, so that the downward composite in the right column is

$$\mathrm{Hom}(Id_U, Tr_{V;W}) : \mathrm{Hom}(U, \mathrm{Hom}(V, V \otimes W)) \to \mathrm{Hom}(U, W).$$

The downward composite in the left column is $Tr_{V;U,W}$ as in Definition 2.24. The other maps $n_1$ and $n_2$ are as indicated in the diagram. The claim can be re-written in a way analogous to Equation (2.9) from Theorem 2.52:

$$Tr_{V;U,W} = \mathrm{Hom}(Id_U, Tr_{V;W}) \circ q^{-1}, \qquad (2.11)$$

which will follow from the commutativity of the diagram.

$$\begin{tikzcd}
\mathrm{Hom}(V\otimes U, V\otimes W) & & \mathrm{Hom}(U,\mathrm{Hom}(V,V\otimes W)) \arrow[ll, "q"'] \\
\mathrm{End}(V)\otimes\mathrm{Hom}(U,W) \arrow[u, "j_2"] \arrow[rr, "n_1"'] \arrow[d, "{[Tr_V\otimes Id_{\mathrm{Hom}(U,W)}]}"'] & & \mathrm{Hom}(U,\mathrm{End}(V)\otimes W) \arrow[u, "{\mathrm{Hom}(Id_U,n)}"'] \arrow[d, "{\mathrm{Hom}(Id_U,[Tr_V\otimes Id_W])}"] \\
\mathbb{K}\otimes\mathrm{Hom}(U,W) \arrow[rr, "n_2"'] \arrow[d, "l_1"] & & \mathrm{Hom}(U,\mathbb{K}\otimes W) \arrow[dll, "{\mathrm{Hom}(Id_U,l_W)}"] \\
\mathrm{Hom}(U,W) & &
\end{tikzcd}$$

The upper square is commutative by a version of Lemma 1.52 with some re-ordering; we briefly re-state its Proof for this case: for $A\in\mathrm{End}(V)$, $B\in\mathrm{Hom}(U,W)$, $v\in V$, $u\in U$,

$$\begin{aligned}
(q\circ\mathrm{Hom}(Id_U,n)\circ n_1): A\otimes B &\mapsto q(n\circ(n_1(A\otimes B))):\\
v\otimes u &\mapsto ((n\circ(n_1(A\otimes B)))(u))(v)\\
&= (n(A\otimes(B(u))))(v)=(A(v))\otimes(B(u))\\
&= (j_2(A\otimes B))(v\otimes u).
\end{aligned}$$

The commutativity of the middle square uses Lemma 1.42 and $Id_{\mathrm{Hom}(U,W)} = \mathrm{Hom}(Id_U, Id_W)$, but not any special properties of $Tr_V$. The lower triangle is easily checked. ∎

Definition 2.50 is related to the scalar trace, as in Definition 2.3, when $W=\mathbb{K}$.

THEOREM 2.54. *For finite-dimensional $V$ and $H: V\to V\otimes\mathbb{K}$,*

$$Tr_V(l_V\circ H) = Tr_{V;\mathbb{K}}(H).$$

PROOF. Let $l_2:\mathrm{End}(V)^*\otimes\mathbb{K}\to\mathrm{End}(V)^*$ be another scalar multiplication map. The following diagram is commutative.

$$\begin{tikzcd}
\mathrm{End}(V)^*\otimes\mathbb{K}^* \arrow[d, "j_1'"] & & \mathrm{End}(V)^*\otimes\mathbb{K} \arrow[ll, "{[Id_{\mathrm{End}(V)^*}\otimes m]}"'] \arrow[dd, "l_2"] \\
\mathrm{Hom}(\mathrm{End}(V)\otimes\mathbb{K},\mathbb{K}\otimes\mathbb{K}) \arrow[d, shift left, "{\mathrm{Hom}(n^{-1},l_{\mathbb{K}})}"] & & \\
\mathrm{Hom}(V,V\otimes\mathbb{K})^* \arrow[u, shift left, "{\mathrm{Hom}(n,l_{\mathbb{K}}^{-1})}"] & & \mathrm{End}(V)^* \arrow[ll, "{\mathrm{Hom}(Id_V,l_V)^*}"]
\end{tikzcd}$$

For $\lambda$, $\mu\in\mathbb{K}$, $\Phi\in\mathrm{End}(V)^*$, $A\in\mathrm{End}(V)$,

$$\begin{aligned}
\Phi\otimes\lambda &\mapsto (j_1'\circ[Id_{\mathrm{End}(V)^*}\otimes m])(\Phi\otimes\lambda) = j_1'(\Phi\otimes(m(\lambda))):\\
A\otimes\mu &\mapsto (\Phi(A))\otimes(\mu\cdot\lambda),\\
\Phi\otimes\lambda &\mapsto (\mathrm{Hom}(n,l_{\mathbb{K}}^{-1})\circ\mathrm{Hom}(Id_V,l_V)^*\circ l_2)(\Phi\otimes\lambda)\\
&= l_{\mathbb{K}}^{-1}\circ((\lambda\cdot\Phi)\circ\mathrm{Hom}(Id_V,l_V))\circ n:\\
A\otimes\mu &\mapsto l_{\mathbb{K}}^{-1}((\lambda\cdot\Phi)(l_V\circ(n(A\otimes\mu)))) = 1\otimes(\lambda\cdot\Phi(\mu\cdot A)),
\end{aligned}$$

since $(l_V \circ (n(A \otimes \mu))) : v \mapsto l_V((A(v)) \otimes \mu) = (\mu \cdot A)(v)$. The Theorem follows from $[Id_{\mathrm{End}(V)^*} \otimes m](Tr_V \otimes 1) = Tr_V \otimes Id_{\mathbb{K}}$:

$$\begin{aligned}
\mathrm{Hom}(Id_V, l_V)^*(Tr_V) &= (\mathrm{Hom}(Id_V, l_V)^* \circ l_2)(Tr_V \otimes 1) \\
&= (\mathrm{Hom}(n^{-1}, l_{\mathbb{K}}) \circ j_1' \circ [Id_{\mathrm{End}(V)^*} \otimes m])(Tr_V \otimes 1) \\
&= Tr_{V;\mathbb{K}}.
\end{aligned}$$

∎

Definition 2.24 and Definition 2.50 are related in the case $U = \mathbb{K}$:

THEOREM 2.55. *For finite-dimensional $V$ and $H : V \to V \otimes W$,*

$$Tr_{V;\mathbb{K},W}(H \circ l_V) : 1 \mapsto Tr_{V;W}(H).$$

PROOF. The following diagram is commutative:

$$\begin{tikzcd}
\mathrm{End}(V)^* \otimes \mathrm{End}(\mathrm{Hom}(\mathbb{K},W)) \arrow[r, "a_1"] \arrow[d, "j_1"] & \mathrm{End}(V)^* \otimes \mathrm{End}(W) \arrow[d, "j_1'"] \\
\mathrm{Hom}(\mathrm{End}(V) \otimes \mathrm{Hom}(\mathbb{K},W), \mathbb{K} \otimes \mathrm{Hom}(\mathbb{K},W)) \arrow[r, "a_2"] \arrow[d, shift left, "\mathrm{Hom}(j_2^{-1}, l_1)"] & \mathrm{Hom}(\mathrm{End}(V) \otimes W, \mathbb{K} \otimes W) \arrow[d, shift left, "\mathrm{Hom}(n^{-1}, l_W)"] \\
\mathrm{Hom}(\mathrm{Hom}(V \otimes \mathbb{K}, V \otimes W), \mathrm{Hom}(\mathbb{K},W)) \arrow[r, "a_3"] \arrow[u, shift left, "\mathrm{Hom}(j_2, l_1^{-1})"] & \mathrm{Hom}(\mathrm{Hom}(V, V \otimes W), W) \arrow[u, shift left, "\mathrm{Hom}(n, l_W^{-1})"]
\end{tikzcd}$$

where the horizontal arrows are

$$\begin{aligned}
a_1 &= [Id_{\mathrm{End}(V)^*} \otimes \mathrm{Hom}(m, m^{-1})] \\
a_2 &= \mathrm{Hom}([Id_{\mathrm{End}(V)} \otimes m], [Id_{\mathbb{K}} \otimes m^{-1}]) \\
a_3 &= \mathrm{Hom}(\mathrm{Hom}(l_V, Id_{V \otimes W}), m^{-1}).
\end{aligned}$$

The statement of the Theorem becomes

$$a_3(Tr_{V;\mathbb{K},W}) = Tr_{V;W}.$$

The top square is commutative by Lemma 1.37. The lower square is commutative by Lemma 1.6, Lemma 1.38, and Lemma 1.43. The Theorem follows from $a_1(Tr_V \otimes Id_{\mathrm{Hom}(\mathbb{K},W)}) = Tr_V \otimes Id_W$:

$$\begin{aligned}
a_3(Tr_{V;\mathbb{K},W}) &= (a_3 \circ \mathrm{Hom}(j_2^{-1}, l_1) \circ j_1)(Tr_V \otimes Id_{\mathrm{Hom}(\mathbb{K},W)}) \\
&= (\mathrm{Hom}(n^{-1}, l_W) \circ j_1' \circ a_1)(Tr_V \otimes Id_{\mathrm{Hom}(\mathbb{K},W)}) \\
&= (\mathrm{Hom}(n^{-1}, l_W) \circ j_1')(Tr_V \otimes Id_W) \\
&= Tr_{V;W}.
\end{aligned}$$

∎

EXERCISE 2.56. The result of Theorem 2.27,

$$Tr_V(H) = (Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V))(1),$$

can be given a different proof using the vector valued trace as an intermediate step.

HINT. By Theorem 2.54 and Theorem 2.55,

$$Tr_V = \mathrm{Hom}(Id_V, l_V^{-1})^*(\mathrm{Hom}(\mathrm{Hom}(l_V, Id_{V \otimes \mathbb{K}}), m^{-1})(Tr_{V;\mathbb{K},\mathbb{K}})).$$

∎

COROLLARY 2.57. *For $H : \mathbb{K} \to \mathbb{K} \otimes W$, $Tr_{\mathbb{K};W}(H) = l_W(H(1))$.*

PROOF. Theorem 2.55 applies, with $l_{\mathbb{K}} : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$:

$$Tr_{\mathbb{K};W}(H) = (Tr_{\mathbb{K};\mathbb{K},W}(H \circ l_{\mathbb{K}}))(1).$$

By Theorem 2.26, this quantity is equal to

$$(\mathrm{Hom}(l_{\mathbb{K}}^{-1}, l_W)(H \circ l_{\mathbb{K}}))(1) = l_W(H(1)).$$

∎

### 2.3.2. Properties of the vector valued trace.

The following results on the $W$-valued trace are corollaries of results from Section 2.2. In most cases, Theorem 2.55 applies, leading to a straightforward calculation.

COROLLARY 2.58. *For finite-dimensional $V$ and $V'$, and $A : V \to V'$, $B : V' \to V \otimes W$,*

$$Tr_{V;W}(B \circ A) = Tr_{V';W}((j_W(A \otimes Id_W)) \circ B).$$

PROOF. Theorem 2.29 and Theorem 2.55 apply, using the map $l_{V'} : V' \otimes \mathbb{K} \to V'$, and the equation $A \circ l_V = l_{V'} \circ (j_{\mathbb{K}}(A \otimes Id_{\mathbb{K}}))$, a version of Lemma 1.38.

$$\begin{aligned}
Tr_{V;W}(B \circ A) &= (Tr_{V';\mathbb{K},W}(B \circ A \circ l_V))(1) \\
&= (Tr_{V;\mathbb{K},W}(B \circ l_{V'} \circ (j_{\mathbb{K}}(A \otimes Id_{\mathbb{K}}))))(1) \\
&= (Tr_{V';\mathbb{K},W}((j_W(A \otimes Id_W)) \circ B \circ l_{V'}))(1) \\
&= Tr_{V',W}((j_W(A \otimes Id_W)) \circ B).
\end{aligned}$$

∎

COROLLARY 2.59. *For finite-dimensional $V$, and $A : V \to V \otimes W$, $B : W \to W'$,*

$$Tr_{V;W'}([Id_V \otimes B] \circ A) = B(Tr_{V;W}(A)).$$

PROOF. By Theorem 2.30 and Theorem 2.55,

$$\begin{aligned}
Tr_{V;W'}([Id_V \otimes B] \circ A) &= (Tr_{V;\mathbb{K},W'}([Id_V \otimes B] \circ A \circ l_V))(1) \\
&= (B \circ (Tr_{V;\mathbb{K},W}(A \circ l_V)))(1) = B(Tr_{V;W}(A)).
\end{aligned}$$

∎

COROLLARY 2.60. *If $V$ and $V'$ are finite-dimensional then for any maps $A : V \to V'$, $B : W \to W'$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V, V \otimes W) \arrow[rr, "Tr_{V;W}"] & & W \arrow[dd, "B"] \\
\mathrm{Hom}(V', V \otimes W) \arrow[u, "\mathrm{Hom}(A, Id_{V\otimes W})"'] \arrow[d, "\mathrm{Hom}(Id_{V'}, [A\otimes B])"'] & & \\
\mathrm{Hom}(V', V' \otimes W') \arrow[rr, "Tr_{V';W'}"] & & W'
\end{tikzcd}$$

PROOF. This follows from Corollary 2.58 and Corollary 2.59. ∎

LEMMA 2.61. *For a direct sum $W = W_1 \oplus W_2$ with data $P_i$, $Q_i$, there is also a direct sum $\mathrm{Hom}(V, V \otimes W) = \mathrm{Hom}(V, V \otimes W_1) \oplus \mathrm{Hom}(V, V \otimes W_2)$, with projections $\mathrm{Hom}(Id_V, [Id_V \otimes P_i])$ and inclusions $\mathrm{Hom}(Id_V, [Id_V \otimes Q_i])$. If $V$ is finite-dimensional, then the map $Tr_{V;W} : \mathrm{Hom}(V, V \otimes W) \to W$ respects the direct sums, and each induced map is equal to a $W_i$-valued trace:*

$$Tr_{V;W_i} = P_i \circ Tr_{V;W} \circ \mathrm{Hom}(Id_V, [Id_V \otimes Q_i]) : \mathrm{Hom}(V, V \otimes W_i) \to W_i.$$

PROOF. The projections and inclusions are as in Example 1.81 and Example 1.82. The claims follow from Definition 1.88 and Corollary 2.59. ∎

COROLLARY 2.62. *For finite-dimensional $V$ and $V'$, and $A : V \otimes V' \to V \otimes V' \otimes W$,*

$$Tr_{V \otimes V';W}(A) = Tr_{V';W}(Tr_{V;V',V'\otimes W}(A)).$$

PROOF. Using Theorem 2.30, Theorem 2.38, and Theorem 2.55, and the scalar multiplication identity $l_{V\otimes V'} = [Id_V \otimes l_{V'}] : V \otimes V' \otimes \mathbb{K} \to V \otimes V'$,

$$\begin{aligned}
Tr_{V\otimes V';W}(A) &= (Tr_{V\otimes V';\mathbb{K},W}(A \circ l_{V\otimes V'}))(1)\\
&= (Tr_{V\otimes V';\mathbb{K},W}(A \circ [Id_V \otimes l_{V'}]))(1)\\
&= (Tr_{V';\mathbb{K},W}(Tr_{V;V'\otimes\mathbb{K},V'\otimes W}(A \circ [Id_V \otimes l_{V'}])))(1)\\
&= (Tr_{V';\mathbb{K},W}((Tr_{V;V',V'\otimes W}(A)) \circ l_{V'}))(1)\\
&= Tr_{V';W}(Tr_{V;V',V'\otimes W}(A)).
\end{aligned}$$

∎

EXERCISE 2.63. Let $V$ be finite-dimensional and denote by $n''$ the map

$$\begin{aligned}
n'' &: W \otimes \mathrm{Hom}(U, W') \to \mathrm{Hom}(U, W \otimes W')\\
&: w \otimes E \mapsto (u \mapsto w \otimes (E(u))).
\end{aligned}$$

For $A : V \to V \otimes W$, $B : U \to W'$,

$$Tr_{V;U,W\otimes W'}(j_0'(A \otimes B)) = n''((Tr_{V;W}(A)) \otimes B).$$

HINT. By Theorem 2.30, Theorem 2.55, Corollary 2.42, and the equations

$$(j_0'(A \otimes B)) \circ [Id_V \otimes l_U] = [A \otimes (B \circ l_U)] = [(A \circ l_V) \otimes B],$$

$$\begin{aligned}
(Tr_{V;U,W\otimes W'}(j_0'(A \otimes B))) \circ l_U &= Tr_{V;\mathbb{K}\otimes U,W\otimes W'}([A \otimes (B \circ l_U)])\\
&= Tr_{V;\mathbb{K}\otimes U,W\otimes W'}([(A \circ l_V) \otimes B])\\
&= j_4((Tr_{V;\mathbb{K},W}(A \circ l_V)) \otimes B)\\
\implies Tr_{V;U,W\otimes W'}(j_0'(A \otimes B)) &= (j_4((Tr_{V;\mathbb{K},W}(A \circ l_V)) \otimes B)) \circ l_U^{-1} :\\
u &\mapsto ((Tr_{V;\mathbb{K},W}(A \circ l_V))(1)) \otimes (B(u))\\
&= (Tr_{V;W}(A)) \otimes (B(u))\\
&= (n''((Tr_{V;W}(A)) \otimes B))(u).
\end{aligned}$$

∎

COROLLARY 2.64. *For finite-dimensional $V_1$ and $V_2$, and maps $A : V_1 \to V_1 \otimes W_1$, $B : V_2 \to V_2 \otimes W_2$, the following identity holds:*

$$Tr_{V_1\otimes V_2;W_1\otimes W_2}(s_1 \circ (j_3'(A \otimes B))) = (Tr_{V_1;W_1}(A)) \otimes (Tr_{V_2;W_2}(B)) \in W_1 \otimes W_2.$$

PROOF. The canonical $j_3$ map is as in Theorem 2.40,

$$j_3' : \mathrm{Hom}(V_1, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2, V_2 \otimes W_2) \to \mathrm{Hom}(V_1 \otimes V_2, V_1 \otimes V_2 \otimes W_1 \otimes W_2),$$

and $s_1$ is a switching map. In the following diagram,

$$\begin{tikzcd} M_{11} \arrow[r, "a_1"] \arrow[d, "j_3'"] & M_{12} \arrow[r, "j_3"] & M_{13} \arrow[d, "a_2"] \\ M_{21} \arrow[r, "a_3"] & M_{22} \arrow[r, "a_4"] & M_{23} \end{tikzcd}$$

the objects are

$$\begin{aligned} M_{11} &= \mathrm{Hom}(V_1, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2, V_2 \otimes W_2) \\ M_{21} &= \mathrm{Hom}(V_1 \otimes V_2, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{12} &= \mathrm{Hom}(V_1 \otimes \mathbb{K}, V_1 \otimes W_1) \otimes \mathrm{Hom}(V_2 \otimes \mathbb{K}, V_2 \otimes W_2) \\ M_{22} &= \mathrm{Hom}(V_1 \otimes V_2 \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{13} &= \mathrm{Hom}(V_1 \otimes \mathbb{K} \otimes V_2 \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2) \\ M_{23} &= \mathrm{Hom}(V_1 \otimes V_2 \otimes \mathbb{K} \otimes \mathbb{K}, V_1 \otimes W_1 \otimes V_2 \otimes W_2), \end{aligned}$$

and the arrows are

$$\begin{aligned} a_1 &= [\mathrm{Hom}(l_{V_1}, Id_{V_1\otimes W_1}) \otimes \mathrm{Hom}(l_{V_2}, Id_{V_1\otimes W_2})] \\ a_2 &= \mathrm{Hom}(s_2, Id_{V_1\otimes W_1\otimes V_2\otimes W_2}) \\ a_3 &= \mathrm{Hom}(l_{V_1\otimes V_2}, Id_{V_1\otimes W_1\otimes V_2\otimes W_2}) \\ a_4 &= \mathrm{Hom}([Id_{V_1\otimes V_2} \otimes l_{\mathbb{K}}], Id_{V_1\otimes W_1\otimes V_2\otimes W_2}). \end{aligned}$$

The diagram is commutative, by Lemma 1.37 and a scalar multiplication identity. By Theorem 2.55, Theorem 2.30, Theorem 2.40, and the diagram,

$$\begin{aligned} LHS = {} & (Tr_{V_1\otimes V_2;\mathbb{K},W_1\otimes W_2}(s_1 \circ (j_3'(A \otimes B)) \circ l_{V_1\otimes V_2}))(1) \\ = {} & (Tr_{V_1\otimes V_2;\mathbb{K},W_1\otimes W_2}(s_1 \circ (j_3((A \circ l_{V_1}) \otimes (B \circ l_{V_2}))) \circ s_2 \circ [Id_{V_1\otimes V_2} \otimes l_{\mathbb{K}}^{-1}]))(1) \\ = {} & ((Tr_{V_1\otimes V_2;\mathbb{K}\otimes\mathbb{K},W_1\otimes W_2}(s_1 \circ (j_3((A \circ l_{V_1}) \otimes (B \circ l_{V_2}))) \circ s_2)) \circ l_{\mathbb{K}}^{-1})(1) \\ = {} & (j_4((Tr_{V_1;\mathbb{K},W_1}(A \circ l_{V_1})) \otimes (Tr_{V_2;\mathbb{K},W_2}(B \circ l_{V_2}))))(1 \otimes 1) \\ = {} & (Tr_{V_1;W_1}(A)) \otimes (Tr_{V_2;W_2}(B)) = RHS. \quad \blacksquare \end{aligned}$$

EXAMPLE 2.65. If $L$ is finite-dimensional and $Ev_L : L^* \otimes L \to \mathbb{K}$ is invertible, as in Lemma 2.21, then $Tr_{L;W}$ is invertible, because it is a composite of invertible maps by Definition 2.50:

$$Tr_{L;W} = l_W \circ [Tr_L \otimes Id_W] \circ n^{-1} : \mathrm{Hom}(L, L \otimes W) \to W.$$

Also by Lemma 2.21 and Example 2.51, for any $w \in W$,

$$Tr_{L;W}(n(Id_L \otimes w)) = 1 \cdot w = w.$$

EXERCISE 2.66. For a direct sum $V = V_1 \oplus V_2$ as in Definition 1.77, and maps $A : V_1 \to V_1 \otimes W$, $B : V_2 \to V_2 \otimes W$, define $A \oplus B : V \to V \otimes W$, using the inclusions from Example 1.81:

$$A \oplus B = [Q_1 \otimes Id_W] \circ A \circ P_1 + [Q_2 \otimes Id_W] \circ B \circ P_2.$$

If $V$ is finite-dimensional, then

$$Tr_{V;W}(A \oplus B) = Tr_{V_1;W}(A) + Tr_{V_2;W}(B).$$

HINT. The proof proceeds exactly as in Proposition 2.12, using Corollary 2.58. ∎

EXERCISE 2.67. If $V$ is finite-dimensional and $V = V_1 \oplus V_2$, then for $K : V \to V \otimes W$,

$$Tr_{V;W}(K) = Tr_{V_1;W}([P_1 \otimes Id_W] \circ K \circ Q_1) + Tr_{V_2;W}([P_2 \otimes Id_W] \circ K \circ Q_2).$$

HINT. Using Corollary 2.58 and Lemma 1.36,

$$Tr_{V_i;W}([P_i \otimes Id_W] \circ K \circ Q_i) = Tr_{V;W}([(Q_i \circ P_i) \otimes Id_W] \circ K).$$

The proof proceeds exactly as in Proposition 2.13. ∎

## 2.4. Equivalence of alternative definitions

In [**JSV**] §3, the canonical trace of a map $F : V \otimes U \to V \otimes W$ is defined in terms of category theory. In the context and notation of these notes, the definition of [**JSV**] can be interpreted as saying that $Tr_{V;U,W}(F)$ is the following composite map from $U$ to $W$:

$$U \longrightarrow V^* \otimes V \otimes U \xrightarrow{[Id_{V^*} \otimes F]} V^* \otimes V \otimes W \xrightarrow{l_W \circ [Ev_V \otimes Id_W]} W, \tag{2.12}$$

where the first arrow is defined for $u \in U$ by:

$$u \mapsto (k^{-1}(Id_V)) \otimes u.$$

As mentioned after Theorem 2.10, this map could be expressed in terms of maps $l_U : \mathbb{K} \otimes U \to U$ and an inclusion $Q_1^1 : \mathbb{K} \to \mathrm{End}(V) : 1 \mapsto Id_V$ as in Example 2.9 and Equation (2.4).

NOTATION 2.68. For finite-dimensional $V$, $k : V^* \otimes V \to \mathrm{End}(V)$, a switching map $s : V^* \otimes V \to V \otimes V^*$, and the map $Q_1^1 : \mathbb{K} \to \mathrm{End}(V) : 1 \mapsto Id_V$, define $\eta_V : \mathbb{K} \to V \otimes V^*$ by:

$$\eta_V = s \circ k^{-1} \circ Q_1^1.$$

The switching map is included for later convenience in Theorem 2.96. The arrow in (2.12) can then be described as follows:

$$\begin{aligned} & [s^{-1} \otimes Id_U] \circ [\eta_V \otimes Id_U] \circ l_U^{-1} \\ = & [k^{-1} \otimes Id_U] \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1} : U \quad \to \quad V^* \otimes V \otimes U \\ & \qquad : u \quad \mapsto \quad (k^{-1}(Id_V)) \otimes u. \end{aligned} \tag{2.13}$$

The following Theorem shows that the formula (2.12) for $Tr_{V;U,W}(F)$ coincides with Definition 2.24. $V$ must be finite-dimensional, but $U$ and $W$ may be arbitrary.

THEOREM 2.69. *For finite-dimensional $V$, $F : V \otimes U \to V \otimes W$, and $u \in U$,*

$$(Tr_{V;U,W}(F))(u) = (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes F] \circ [k^{-1} \otimes Id_U])(Id_V \otimes u).$$

PROOF. The following diagram is commutative, where the top arrow is $a_1 = [Id_{\mathrm{End}(V^*)} \otimes j_2]$.

$$\begin{tikzcd}
\mathrm{End}(V^*) \otimes \mathrm{Hom}(V \otimes U, V \otimes W) \arrow[d, "j"] & \mathrm{End}(V^*) \otimes \mathrm{End}(V) \otimes \mathrm{Hom}(U, W) \arrow[l, "a_1"'] \arrow[d, "{[j \otimes Id_{\mathrm{Hom}(U,W)}]}"] \\
\mathrm{Hom}(V^* \otimes V \otimes U, V^* \otimes V \otimes W) \arrow[d, "{\mathrm{Hom}([k^{-1} \otimes Id_U], [Ev_V \otimes Id_W])}"] & \mathrm{End}(V^* \otimes V) \otimes \mathrm{Hom}(U, W) \arrow[l, "j"'] \arrow[d, "{[\mathrm{Hom}(k^{-1}, Ev_V) \otimes Id_{\mathrm{Hom}(U,W)}]}"] \\
\mathrm{Hom}(\mathrm{End}(V) \otimes U, \mathbb{K} \otimes W) \arrow[d, "{\mathrm{Hom}(Id_{\mathrm{End}(V) \otimes U}, l_W)}"] & \mathrm{End}(V)^* \otimes \mathrm{Hom}(U, W) \arrow[l, "j"'] \arrow[d, "{[(d_{\mathrm{End}(V)}(Id_V)) \otimes Id_{\mathrm{Hom}(U,W)}]}"] \\
\mathrm{Hom}(\mathrm{End}(V) \otimes U, W) \arrow[d, "{d_{\mathrm{End}(V) \otimes U, W}(Id_V \otimes u)}"] & \mathbb{K} \otimes \mathrm{Hom}(U, W) \arrow[d, "l_1"] \\
W & \mathrm{Hom}(U, W) \arrow[l, "d_{UW}(u)"']
\end{tikzcd}$$

The upper and lower squares are easy to check (the maps $d_{UW}$, $d_{\mathrm{End}(V) \otimes U, W}$ are as in Definition 1.13), and the middle square is commutative by Lemma 1.37. Starting with $Id_{V^*} \otimes F$ in the upper left corner, the RHS of the Theorem is the output of the composition in the left column. The top three arrows in the right column come from the construction in Theorem 2.10,

$$Tr_V(A) = ((d_{\mathrm{End}(V)}(Id_V)) \circ \mathrm{Hom}(k^{-1}, Ev_V) \circ j)(Id_{V^*} \otimes A),$$

so that the composite of $a_1^{-1} = [Id_{\mathrm{End}(V^*)} \otimes j_2^{-1}]$ with the right column of maps takes $Id_{V^*} \otimes F$ to $Tr_{V;U,W}(F) = l_1((j_1(Tr_V \otimes Id_{\mathrm{Hom}(U,W)}))(j_2^{-1}(F)))$. The lowest arrow plugs $u$ into $Tr_{V;U,W}(F)$, giving the LHS of the Theorem, so the equality follows directly from the commutativity of the diagram. ∎

COROLLARY 2.70. *For finite-dimensional $V$ and $A : V \to V \otimes W$,*

$$Tr_{V;W}(A) = (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes A] \circ k^{-1})(Id_V).$$

PROOF. By Theorem 2.69 and Theorem 2.55,

$$\begin{aligned}
LHS &= (Tr_{V;\mathbb{K},W}(A \circ l_V))(1) \\
&= (l_W \circ [Ev_V \otimes Id_W] \circ [Id_{V^*} \otimes (A \circ l_V)] \circ [k^{-1} \otimes Id_{\mathbb{K}}])(Id_V \otimes 1) \\
&= RHS. \quad ∎
\end{aligned}$$

This shows, in analogy with Theorem 2.10 and (2.12) from Theorem 2.69, that the $W$-valued trace of $A$ is the output of the distinguished element $k^{-1}(Id_V)$ under the composite map

$$V^* \otimes V \xrightarrow{[Id_{V^*} \otimes A]} V^* \otimes V \otimes W \xrightarrow{l_W \circ [Ev_V \otimes Id_W]} W. \tag{2.14}$$

So, Corollary 2.70 could be used as an alternative, but equivalent, definition of vector valued trace. This Section continues with some identities for the vector valued trace, some of which (Theorem 2.74, Corollary 2.88, Corollary 2.108) could also be used in alternative approaches to the definition of $Tr_{V;W}$.

### 2.4.1. A vector valued canonical evaluation.

DEFINITION 2.71. The distinguished element

$$Ev_{VW} \in \mathrm{Hom}(\mathrm{Hom}(V,W) \otimes V, W)$$

is the canonical evaluation map, defined by

$$Ev_{VW}(A \otimes v) = A(v).$$

In the $W = \mathbb{K}$ case, $Ev_{V\mathbb{K}}$ is the distinguished element $Ev_V \in (V^* \otimes V)^*$ from Definition 2.2. The scalar evaluation $Ev_V$ and vector valued evaluation $Ev_{VW}$ are related as follows.

LEMMA 2.72. *For any $V$ and $W$, let $s' : V \otimes W \to W \otimes V$ be a switching map. The following diagram is commutative.*

$$\begin{tikzcd}
V^* \otimes V \otimes W \arrow[rr, "{[Ev_V \otimes Id_W]}"'] \arrow[d, "{[Id_{V^*} \otimes s']}"] & & \mathbb{K} \otimes W \arrow[dd, "l"] \\
V^* \otimes W \otimes V \arrow[d, "{[k_{VW} \otimes Id_V]}"] & & \\
\mathrm{Hom}(V,W) \otimes V \arrow[rr, "Ev_{VW}"'] & & W
\end{tikzcd}$$

PROOF.

$$\begin{array}{rcl}
\phi \otimes v \otimes w & \mapsto & (l \circ [Ev_V \otimes Id_W])(\phi \otimes v \otimes w) \\
& = & \phi(v) \cdot w, \\
\phi \otimes v \otimes w & \mapsto & (Ev_{VW} \circ [k_{VW} \otimes Id_V] \circ [Id_{V^*} \otimes s'])(\phi \otimes v \otimes w) \\
& = & Ev_{VW}((k_{VW}(\phi \otimes w)) \otimes v) \\
& = & \phi(v) \cdot w. \quad \blacksquare
\end{array}$$

The canonical evaluation maps have the following naturality property.

LEMMA 2.73. *For any vector spaces $U$, $V$, $V'$, $W$, and any maps $G : V' \to V$, $B : U \to W$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V,U) \otimes V \arrow[rr, "Ev_{VU}"] & & U \arrow[dd, "B"] \\
\mathrm{Hom}(V,U) \otimes V' \arrow[u, "{[Id_{\mathrm{Hom}(U,V)} \otimes G]}"] \arrow[d, "{[\mathrm{Hom}(G,B) \otimes Id_{V'}]}"'] & & \\
\mathrm{Hom}(V',W) \otimes V' \arrow[rr, "Ev_{V'W}"] & & W
\end{tikzcd}$$

PROOF. For $A \in \mathrm{Hom}(V,U)$, $v \in V'$,

$$\begin{array}{rcl} & & B \circ Ev_{VU} \circ [Id_{\mathrm{Hom}(V,U)} \otimes G] : \\ A \otimes v & \mapsto & B(Ev_{VU}(A \otimes (G(v)))) = B(A(G(v))), \\ & & Ev_{V'W} \circ [\mathrm{Hom}(G,B) \otimes Id_{V'}] : \\ A \otimes v & \mapsto & Ev_{V'W}((B \circ A \circ G) \otimes v) = (B \circ A \circ G)(v). \end{array}$$

∎

THEOREM 2.74. *For a map*

$$n' : \mathrm{Hom}(V,W) \otimes V \to \mathrm{Hom}(V, V \otimes W),$$

*if $V$ is finite-dimensional, then*

$$Tr_{V;W} \circ n' = Ev_{VW}.$$

PROOF. The map $n'$ is as in Notation 1.41. The conclusion is equivalent to the formula $Tr_{V;W}(n'(A \otimes v)) = A(v)$, for $A \in \mathrm{Hom}(V,W)$ and $v \in V$.

The claimed equality $Tr_{V;W} \circ n' = Ev_{VW}$ appears in the lower triangle in the following diagram, so it follows from the commutativity of the rest of the diagram.

$$\begin{tikzcd}
\mathrm{End}(V) \otimes W \arrow[dr, "{[Tr_V \otimes Id_W]}"] \arrow[ddd, "n"] & & V^* \otimes V \otimes W \arrow[ll, "{[k_{VV} \otimes Id_W]}"'] \arrow[dl, "{[Ev_V \otimes Id_W]}"'] \arrow[dd, "{[Id_{V^*} \otimes s']}"] \arrow[dddll, "k_{V,V\otimes W}"'] \\
& \mathbb{K} \otimes W \arrow[ddd, "l"] & \\
& & V^* \otimes W \otimes V \arrow[d, "{[k_{VW} \otimes Id_V]}"] \\
\mathrm{Hom}(V, V \otimes W) \arrow[dr, "Tr_{V;W}"] & & \mathrm{Hom}(V,W) \otimes V \arrow[ll, "n'"'] \arrow[dl, "Ev_{VW}"'] \\
& W &
\end{tikzcd}$$

The front left square is Definition 2.50, and the upper triangle is Definition 2.3, together with Lemma 1.36. The front right square is exactly Lemma 2.72. The back square is commutative, where the back left triangle is exactly Lemma 1.65, and the back right triangle is a variation on Lemma 1.65, with an extra switching $s'$ and the differently ordered $n'$, checked by the following calculation.

$$\begin{array}{rcl} \phi \otimes v \otimes w & \mapsto & (n' \circ [k_{VW} \otimes Id_V] \circ [Id_{V^*} \otimes s'])(\phi \otimes v \otimes w) \\ & = & n'((k_{VW}(\phi \otimes w)) \otimes v) : \\ u & \mapsto & v \otimes (\phi(u) \cdot w) = (k_{V,V \otimes W}(\phi \otimes v \otimes w))(u). \end{array} \tag{2.15}$$

∎

Recall the canonical map $e^W_{VV} : \mathrm{End}(V) \to \mathrm{Hom}(\mathrm{Hom}(V,W) \otimes V, W)$ from Definition 1.56. In the following diagram (adapted from part of the diagram from

Lemma 2.5), the left triangle is commutative by Lemma 1.58.

$$\begin{tikzcd}
 & \mathrm{Hom}(\mathrm{Hom}(V,W)\otimes V, W) & \\
\mathrm{End}(V) \arrow[ur, "e^W_{VV}"] \arrow[d, "t^W_{VV}"'] & & \mathrm{Hom}(\mathrm{Hom}(V,V\otimes W),W) \arrow[ul, "\mathrm{Hom}(n',Id_W)"'] \\
\mathrm{End}(\mathrm{Hom}(V,W)) \arrow[uur, "q"'] & &
\end{tikzcd}$$

The three spaces on the left side of the diagram each contain a distinguished element, giving an analogue of Equation (2.1):

$$Ev_{VW} = q(Id_{\mathrm{Hom}(V,W)}) = q(t^W_{VV}(Id_V)) = e^W_{VV}(Id_V). \tag{2.16}$$

For finite-dimensional $V$, Theorem 2.74 gives:

$$\begin{aligned}
Tr_{V;W} &= Ev_{VW} \circ (n')^{-1} && (2.17)\\
&= ((\mathrm{Hom}(n', Id_W))^{-1} \circ q)(Id_{\mathrm{Hom}(V,W)}) && (2.18)\\
&= ((\mathrm{Hom}(n', Id_W))^{-1} \circ e^W_{VV})(Id_V).
\end{aligned}$$

where line (2.18) is an analogue of Equation (2.2) from Lemma 2.5.

Using Equation (2.17) as a definition for the vector valued trace allows some proofs to be simplified and avoids scalar multiplication. For example, the following result re-states Corollary 2.59 but gives a simpler proof.

Corollary 2.75. *For any $V$, $W$, $W'$, if $V$ is finite-dimensional and $B \in \mathrm{Hom}(W, W')$ then*

$$Tr_{V;W'} \circ \mathrm{Hom}(Id_V, [Id_V \otimes B]) = B \circ Tr_{V;W} : \mathrm{Hom}(V, V \otimes W) \to W'.$$

Proof. For $n'$ as in Theorem 2.74 and an analogous map $n''$, the downward composite in the left column of the following diagram is $Tr_{V;W} = Ev_{VW} \circ (n')^{-1}$, and in the right column is $Tr_{V;W'}$.

$$\begin{tikzcd}
\mathrm{Hom}(V, V\otimes W) \arrow[r, "\mathrm{Hom}(Id_V,[Id_V\otimes B])"] & \mathrm{Hom}(V, V\otimes W') \\
\mathrm{Hom}(V,W)\otimes V \arrow[u, "n'"] \arrow[r, "{[\mathrm{Hom}(Id_V,B)\otimes Id_V]}"] \arrow[d, "Ev_{VW}"'] & \mathrm{Hom}(V,W')\otimes V \arrow[u, "n''"'] \arrow[d, "Ev_{VW'}"] \\
W \arrow[r, "B"] & W'
\end{tikzcd}$$

The blocks are commutative; the upper by Lemma 1.42 and the lower by Lemma 2.73 (in the case $G = Id_V$). ∎

COROLLARY 2.76. *For finite-dimensional $V$, and maps*

$$\begin{array}{rcl} n' : \mathrm{Hom}(V,W)\otimes V & \to & \mathrm{Hom}(V,V\otimes W) \\ q : \mathrm{Hom}(U,\mathrm{Hom}(V,V\otimes W)) & \to & \mathrm{Hom}(V\otimes U, V\otimes W) \\ F : V\otimes U & \to & V\otimes W, \end{array}$$

*the following diagram is commutative.*

$$\begin{tikzcd} U \arrow[rr, "Tr_{V;U,W}(F)"] \arrow[drr, "(n')^{-1}\circ(q^{-1}(F))"'] & & W \\ & & \mathrm{Hom}(V,W)\otimes V \arrow[u, "Ev_{VW}"'] \end{tikzcd}$$

PROOF. The invertible map $q$ is as in Theorem 2.53, where Equation (2.11) states the commutativity of the upper triangle in the following diagram. The commutativity of the lower triangle follows from Lemma 1.6 and Theorem 2.74, and the claim follows.

$$\begin{tikzcd} \mathrm{Hom}(V\otimes U, V\otimes W) \arrow[d, "Tr_{V;U,W}"'] & \mathrm{Hom}(U,\mathrm{Hom}(V,V\otimes W)) \arrow[l, "q"'] \arrow[dl, "\mathrm{Hom}(Id_U, Tr_{V;W})"'] \\ \mathrm{Hom}(U,W) & \mathrm{Hom}(U,\mathrm{Hom}(V,W)\otimes V) \arrow[l, "\mathrm{Hom}(Id_U, Ev_{VW})"] \arrow[u, "\mathrm{Hom}(Id_U, n')"'] \end{tikzcd}$$

∎

THEOREM 2.77. *Denote*

$$n_1 : \mathrm{End}(V)\otimes U \to \mathrm{Hom}(V, V\otimes U).$$

*If $V$ is finite-dimensional then, for any $F : V\otimes U \to V\otimes W$ and $u\in U$,*

$$(Tr_{V;U,W}(F))(u) = Tr_{V;W}(F\circ(n_1(Id_V\otimes u))).$$

PROOF. Consider the following diagram.

$$\begin{tikzcd} U \arrow[r] & V^*\otimes V\otimes U \arrow[r, "{[Id_{V^*}\otimes F]}"] \arrow[d, "{[k\otimes Id_U]}"] \arrow[ddl, "k_{V,V\otimes U}"'] & V^*\otimes V\otimes W \arrow[r, "{[Ev_V\otimes Id_W]}"] \arrow[d, "{[Id_{V^*}\otimes s']}"] \arrow[ddl, "k_{V,V\otimes W}"'] & \mathbb{K}\otimes W \arrow[dd, "l_W"] \\ \mathbb{K}\otimes U \arrow[u, "l_U"] \arrow[r, "{[Q_1^1\otimes Id_U]}"] & \mathrm{End}(V)\otimes U \arrow[dl, "n_1"] & V^*\otimes W\otimes V \arrow[d, "{[k_{VW}\otimes Id_V]}"] & \\ \mathrm{Hom}(V, V\otimes U) \arrow[r, "\mathrm{Hom}(Id_V,F)"'] & \mathrm{Hom}(V,V\otimes W) & \mathrm{Hom}(V,W)\otimes V \arrow[l, "n'"] \arrow[r, "Ev_{VW}"'] & W \end{tikzcd}$$

The composition from $U$ to $W$ clockwise along the upper row gives $Tr_{V;U,W}(F)$ by Theorem 2.69. The left square is from (2.13), and the right block is Lemma 2.72. The center left triangle with the $n_1$ map is exactly Lemma 1.65, and the center right triangle with the $n'$ map is the variation on Lemma 1.65 copied from (2.15) in the Proof of Theorem 2.74. The center block is commutative by Lemma 1.62. The claim follows from Theorem 2.74:

$$LHS = Ev_{VW}((n')^{-1}(F\circ(n_1(Id_V\otimes u)))) = RHS.$$

∎

Although Corollary 2.76 and Theorem 2.77 have arrived by different approaches, their formulas for the generalized trace are closely related. For $F : V \otimes U \to V \otimes W$, from Corollary 2.76,

$$Tr_{V;U,W}(F) = Ev_{VW} \circ (n')^{-1} \circ (q^{-1}(F)).$$

From Theorem 2.77,

$$Tr_{V;U,W}(F) = Ev_{VW} \circ (n')^{-1} \circ \mathrm{Hom}(Id_V, F) \circ n_1 \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1}.$$

Using Lemma 1.47, the composite $\mathrm{Hom}(Id_V, F) \circ n_1 \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1}$ is equal to $q^{-1}(F)$:

$$\begin{aligned}
\mathrm{Hom}(Id_V, F) \circ n_1 \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1} : u &\mapsto F \circ (n_1(Id_V \otimes u)) : \\
v &\mapsto F(v \otimes u), \\
q^{-1}(F) : u &\mapsto (q^{-1}(F))(u) : \\
v &\mapsto F(v \otimes u).
\end{aligned} \tag{2.19}$$

Similarly for the vector valued trace formula, Corollary 2.70 and Theorem 2.74 are related by the following commutative diagram.

$$\begin{tikzcd}
V^* \otimes V \arrow[r, "{[Id_{V^*} \otimes A]}"] \arrow[d, "k"] & V^* \otimes V \otimes W \arrow[r, "{[Ev_V \otimes Id_W]}"] \arrow[d, "k_{V,V\otimes W}"] & \mathbb{K} \otimes W \arrow[r, "l_W"] & W \\
\mathrm{End}(V) \arrow[r, "{\mathrm{Hom}(Id_V, A)}"] & \mathrm{Hom}(V, V \otimes W) & \mathrm{Hom}(V, W) \otimes V \arrow[l, "n'"] \arrow[ur, "Ev_{VW}"] &
\end{tikzcd}$$

The left square is commutative by Lemma 1.62, and the right block is from the Proof of Theorem 2.77. Starting with $Id_V \in \mathrm{End}(V)$, the composition clockwise along the upper row gives $Tr_{V;W}(A) \in W$ as in (2.14) from Corollary 2.70, and counterclockwise along the lower row gives $Ev_{VW}((n')^{-1}(A))$, which also equals $Tr_{V;W}(A)$ by Theorem 2.74.

Lemma 2.78. *For any $U$, $V$, $W$, the following diagram is commutative.*

$$\begin{tikzcd}
U \otimes \mathrm{Hom}(V, W) \otimes V \arrow[r, "{[Id_U \otimes Ev_{VW}]}"] \arrow[d, "{[n \otimes Id_V]}"'] & U \otimes W \\
\mathrm{Hom}(V, U \otimes W) \otimes V \arrow[ur, "Ev_{V,U\otimes W}"'] &
\end{tikzcd}$$

Proof. Both paths take $u \otimes A \otimes v \in U \otimes \mathrm{Hom}(V, W) \otimes V$ to $u \otimes (A(v)) \in U \otimes W$. ∎

Theorem 2.79. *For any $V$, $U$, $W$, and $F : V \otimes U \to V \otimes W$, if $V$ is finite-dimensional then the $n$ maps in the following diagram are invertible:*

$$\begin{tikzcd}
V \otimes U \arrow[r, "F"] & V \otimes W \\
V \otimes \mathrm{Hom}(V, U) \otimes V \arrow[u, "{[Id_V \otimes Ev_{VU}]}"] \arrow[d, "{[n_2 \otimes Id_V]}"'] & V \otimes \mathrm{Hom}(V, W) \otimes V \arrow[u, "{[Id_V \otimes Ev_{VW}]}"'] \arrow[d, "{[n_2' \otimes Id_V]}"] \\
\mathrm{Hom}(V, V \otimes U) \otimes V \arrow[r, "{[\mathrm{Hom}(Id_V, F) \otimes Id_V]}"'] & \mathrm{Hom}(V, V \otimes W) \otimes V
\end{tikzcd}$$

*and the diagram is commutative, in the sense that*

$$\begin{aligned} & F \circ [Id_V \otimes Ev_{VU}] \circ [n_2 \otimes Id_V]^{-1} \\ = \quad & [Id_V \otimes Ev_{VW}] \circ [n_2' \otimes Id_V]^{-1} \circ [\mathrm{Hom}(Id_V, F) \otimes Id_V]. \end{aligned}$$

PROOF. The $n_2$, $n_2'$ maps are defined as they appear in the diagram, with subscript notation used to avoid duplication with the labels appearing elsewhere in this Section; they are invertible by Lemma 1.44.

By Lemma 2.78, the upward composite on the left, $[Id_V \otimes Ev_{VU}] \circ [n_2 \otimes Id_V]^{-1}$, is equal to $Ev_{V,V\otimes U}$, and similarly the upward composite on the right is equal to $Ev_{V,V\otimes W}$. The claim then follows from Lemma 2.73. ∎

In the following Theorem 2.80, diagram (2.20) is a generalization of the diagram from Theorem 2.79.

THEOREM 2.80. *Given maps*

$$\begin{aligned} F : V \otimes U &\to V \otimes W \\ a_1 : V \otimes \mathrm{Hom}(X, U) \otimes V &\to V \otimes U \\ a_2 : V \otimes \mathrm{Hom}(X, W) \otimes V &\to V \otimes W, \end{aligned}$$

*if $V$ is finite-dimensional then the following are equivalent.*

(1) *The diagram (2.20) is commutative, in the sense that*

$$\begin{aligned} & F \circ a_1 \circ [n_2 \otimes Id_V]^{-1} \\ = \quad & a_2 \circ [n_2' \otimes Id_V]^{-1} \circ [\mathrm{Hom}(Id_X, F) \otimes Id_V]. \end{aligned}$$

(2) *The diagram (2.21) is commutative, in the sense that*

$$\begin{aligned} & F \circ a_1 \circ [Id_V \otimes n_3]^{-1} \circ n_4^{-1} \\ = \quad & a_2 \circ [Id_V \otimes n_3']^{-1} \circ (n_4')^{-1} \circ \mathrm{Hom}(Id_X, [F \otimes Id_V]). \end{aligned}$$

$$\begin{tikzcd} V \otimes U \arrow[rr, "F"] & & V \otimes W \\ V \otimes \mathrm{Hom}(X,U) \otimes V \arrow[u, "a_1"] \arrow[d, "{[n_2 \otimes Id_V]}"'] & & V \otimes \mathrm{Hom}(X,W) \otimes V \arrow[u, "a_2"'] \arrow[d, "{[n_2' \otimes Id_V]}"] \\ \mathrm{Hom}(X, V \otimes U) \otimes V \arrow[rr, "{[\mathrm{Hom}(Id_X,F) \otimes Id_V]}"'] & & \mathrm{Hom}(X, V \otimes W) \otimes V \end{tikzcd} \tag{2.20}$$

$$\begin{tikzcd} V \otimes U \arrow[rr, "F"] & & V \otimes W \\ V \otimes \mathrm{Hom}(X,U) \otimes V \arrow[u, "a_1"] \arrow[d, "{[Id_V \otimes n_3]}"'] & & V \otimes \mathrm{Hom}(X,W) \otimes V \arrow[u, "a_2"'] \arrow[d, "{[Id_V \otimes n_3']}"] \\ V \otimes \mathrm{Hom}(X, U \otimes V) \arrow[d, "n_4"'] & & V \otimes \mathrm{Hom}(X, W \otimes V) \arrow[d, "n_4'"] \\ \mathrm{Hom}(X, V \otimes U \otimes V) \arrow[rr, "{\mathrm{Hom}(Id_X,[F \otimes Id_V])}"'] & & \mathrm{Hom}(X, V \otimes W \otimes V) \end{tikzcd} \tag{2.21}$$

PROOF. The four $n$ maps are defined as they appear in the diagram; they are invertible by Lemma 1.44. The diagrams have the same arrows $a_1$, $a_2$, $F$, so showing the composite maps counter-clockwise from $V \otimes \mathrm{Hom}(X,U) \otimes V$ to $V \otimes \mathrm{Hom}(X,W) \otimes V$ are equal to each other is enough to establish the claimed equivalence. The left arrow in the following diagram is copied from the left column of (2.20) and the upper and right arrows match the left column of (2.21). Introducing the lower arrow $n_5$ gives a commutative diagram by Lemma 1.45.

$$\begin{tikzcd} V \otimes \mathrm{Hom}(X,U) \otimes V \arrow[r, "{[Id_V \otimes n_3]}"'] \arrow[d, "{[n_2 \otimes Id_V]}"'] & V \otimes \mathrm{Hom}(X, U \otimes V) \arrow[d, "n_4"'] \\ \mathrm{Hom}(X, V \otimes U) \otimes V \arrow[r, "n_5"'] & \mathrm{Hom}(X, V \otimes U \otimes V) \end{tikzcd}$$

With an analogous diagram for right column arrows of (2.20) and (2.21), replacing $U$ with $W$ and introducing $n_5' : \mathrm{Hom}(X, V \otimes W) \otimes V \to \mathrm{Hom}(X, V \otimes W \otimes V)$, Lemma 1.45 gives the equation $n_4' \circ [Id_V \otimes n_3'] = n_5' \circ [n_2' \otimes Id_V]$. The equality follows,

$$\begin{aligned} & [n_2' \otimes Id_V]^{-1} \circ [\mathrm{Hom}(Id_X, F) \otimes Id_V] \circ [n_2 \otimes Id_V] \\ = \; & [Id_V \otimes n_3']^{-1} \circ (n_4')^{-1} \circ n_5' \circ [\mathrm{Hom}(Id_X, F) \otimes Id_V] \circ n_5^{-1} \circ n_4 \circ [Id_V \otimes n_3] \\ (2.22) = \; & [Id_V \otimes n_3']^{-1} \circ (n_4')^{-1} \circ \mathrm{Hom}(Id_X, [F \otimes Id_V]) \circ n_4 \circ [Id_V \otimes n_3], \end{aligned}$$

line (2.22) using Lemma 1.42. ∎

LEMMA 2.81. *For a switching map $s : V \otimes V \to V \otimes V$, if $k : V^* \otimes V \to \mathrm{End}(V)$ is invertible, then the following map:*

$$v \mapsto (l_V \circ [Ev_V \otimes Id_V] \circ [Id_{V^*} \otimes s] \circ [k^{-1} \otimes Id_V])(Id_V \otimes v)$$

*is equal to the identity map $Id_V$.*

PROOF. For the special case of Lemma 2.72 with $W = V$, $s' = s$, the above composite map is $Ev_{VV}$, so the given expression is

$$v \mapsto Ev_{VV}(Id_V \otimes v) = Id_V(v) = v.$$

∎

EXAMPLE 2.82. If $V$ is finite-dimensional, then the generalized trace of the switching map $s : V \otimes V \to V \otimes V$ is:

$$Tr_{V;V,V}(s) = Id_V, \tag{2.23}$$

by the formula from Theorem 2.69 and Lemma 2.81:

$$Tr_{V;V,V}(s) : v \mapsto (l_V \circ [Ev_V \otimes Id_V] \circ [Id_{V^*} \otimes s] \circ [k^{-1} \otimes Id_V])(Id_V \otimes v) = v.$$

REMARK 2.83. Equation (2.23) is related to the “yanking” property of [**JSV**].

The following Lemma is analogous to Lemma 2.81.

LEMMA 2.84. *For a switching involution*

$$s'' : V^* \otimes V \otimes V^* \to V^* \otimes V \otimes V^* : \phi \otimes v \otimes \psi \mapsto \psi \otimes v \otimes \phi,$$

*if $k : V^* \otimes V \to \mathrm{End}(V)$ is invertible, then the following map:*

$$\phi \mapsto (l_{V^*} \circ [Ev_V \otimes Id_V] \circ s'' \circ [k^{-1} \otimes Id_{V^*}])(Id_V \otimes \phi)$$

*is equal to the identity map $Id_{V^*}$.*

PROOF. The following diagram is commutative; the calculation is similar that in the Proof of Lemma 2.72. Abbreviate $t = t_{VV}$ as in Lemma 2.5.

$$\begin{tikzcd}
V^* \otimes V \otimes V^* \arrow[rr, "{[Ev_V \otimes Id_{V^*}]}"'] \arrow[d, "s''"] & & \mathbb{K} \otimes V^* \arrow[ddd, "l_{V^*}"] \\
V^* \otimes V \otimes V^* \arrow[d, "{[k \otimes Id_{V^*}]}"] & & \\
\mathrm{End}(V) \otimes V^* \arrow[d, "{[t \otimes Id_{V^*}]}"] & & \\
\mathrm{End}(V^*) \otimes V^* \arrow[rr, "Ev_{V^*V^*}"'] & & V^*
\end{tikzcd}$$

Starting with $Id_V \otimes \phi$, the commutativity of the diagram and the existence of $k^{-1}$ give:

$$\begin{aligned}
(2.24) \qquad & & (l_{V^*} \circ [Ev_V \otimes Id_V] \circ s'' \circ [k^{-1} \otimes Id_{V^*}])(Id_V \otimes \phi) \\
& = & Ev_{V^*V^*}((t(Id_V)) \otimes \phi) = Id_V^*(\phi) = \phi.
\end{aligned}$$

∎

THEOREM 2.85. *If $k : V^* \otimes V \to \mathrm{End}(V)$ is invertible, then $d_V : V \to V^{**}$ is invertible.*

PROOF. The following map, temporarily denoted $B : V^{**} \to V$, is an inverse:

$$B : \Phi \mapsto l_V([\Phi \otimes Id_V](k^{-1}(Id_V))).$$

For any $V$, $W$, and $v \in V$, the following diagram is commutative (the two paths $V^* \otimes W \to W$ are equal compositions).

$$\begin{tikzcd}
V^* \otimes W \arrow[r, "k_{VW}"] \arrow[d, "{[(d_V(v)) \otimes Id_W]}"'] & \mathrm{Hom}(V,W) & \mathrm{Hom}(V,W) \otimes \mathbb{K} \arrow[l, "l"'] \arrow[d, "{[Id_{\mathrm{Hom}(V,W)} \otimes (m(v))]}"] \\
\mathbb{K} \otimes W \arrow[r, "l_W"] & W & \mathrm{Hom}(V,W) \otimes V \arrow[l, "Ev_{VW}"']
\end{tikzcd}$$

In the case $W = V$, starting with $Id_V$ in the top middle gives the following equality:

$$\begin{aligned}
(B \circ d_V)(v) &= l_V([(d_V(v)) \otimes Id_V](k^{-1}(Id_V))) \\
&= Ev_{VV}(Id_V \otimes v) = v.
\end{aligned}$$

To check the composite in the other order, in the second diagram, $s'''$ is another switching map as indicated in the diagram, $\eta_V$ and $s$ are as in (2.13), the block is commutative, and the composition in the left column acts as the identity map, by

(2.24) from Lemma 2.84.

$$
\begin{tikzcd}
V^* \arrow[d, "l_{V^*}^{-1}"] & \\
\mathbb{K} \otimes V^* \arrow[d, "{[\eta_V \otimes Id_{V^*}]}"] & \\
V \otimes V^* \otimes V^* \arrow[d, "{[s \otimes Id_{V^*}]}"] & \\
V^* \otimes V \otimes V^* \arrow[r, "{[[\Phi \otimes Id_V] \otimes Id_{V^*}]}"] \arrow[d, "s''"] & \mathbb{K} \otimes V \otimes V^* \arrow[d, "s'''"] \\
V^* \otimes V \otimes V^* \arrow[d, "{[Ev_V \otimes Id_{V^*}]}"] & V^* \otimes \mathbb{K} \otimes V \arrow[d, "{[Id_{V^*} \otimes l_V]}"] \\
\mathbb{K} \otimes V^* \arrow[d, "l_{V^*}"] & V^* \otimes V \arrow[d, "Ev_V"] \\
V^* \arrow[r, "\Phi"] & \mathbb{K}
\end{tikzcd}
$$

The conclusion is:

$$
\begin{aligned}
\Phi(\phi) &= (Ev_V \circ [Id_{V^*} \otimes l_V] \circ s''' \circ [[\Phi \otimes Id_V] \otimes Id_{V^*}])((k^{-1}(Id_V)) \otimes \phi) \\
&= Ev_V(\phi \otimes (l_V([\Phi \otimes Id_V](k^{-1}(Id_V))))) \\
&= \phi(l_V([\Phi \otimes Id_V](k^{-1}(Id_V)))) \\
&= \phi(B(\Phi)) = ((d_V \circ B)(\Phi))(\phi).
\end{aligned}
$$

■

Proposition 2.86. *Given $U$, $V$, $W$, a map $G : U \to \mathrm{Hom}(V, W)$, and the canonical map*

$$q : \mathrm{Hom}(U, \mathrm{Hom}(V, W)) \to \mathrm{Hom}(U \otimes V, W),$$

*if $V$ is finite-dimensional then*

$$q(G) = Tr_{V;W} \circ n' \circ [G \otimes Id_V]. \tag{2.25}$$

Proof. $q$ is as in Definition 1.46, and $n'$ is as in Theorem 2.74, which gives, for $u \otimes v \in U \otimes V$,

$$
\begin{aligned}
(Tr_{V;W} \circ n' \circ [G \otimes Id_V])(u \otimes v) &= (Ev_{VW} \circ [G \otimes Id_V])(u \otimes v) \\
&= (G(u))(v) \\
&= (q(G))(u \otimes v).
\end{aligned}
$$

■

Equation (2.25) from Proposition 2.86 can also be re-written, for $F = q(G) \in \mathrm{Hom}(U \otimes V, W)$,

$$F = Ev_{VW} \circ [(q^{-1}(F)) \otimes Id_V] = Tr_{V;W} \circ n' \circ [(q^{-1}(F)) \otimes Id_V].$$

THEOREM 2.87. *For vector spaces $V$ and $Z$, let $s_1 : V \otimes V \to V \otimes V$ be the switching map and denote*

$$n : \mathrm{End}(V) \otimes V \otimes Z \to \mathrm{Hom}(V, V \otimes V \otimes Z).$$

*If $V$ is finite-dimensional then for any $B \in V \otimes Z$,*

$$Tr_{V;V\otimes Z}([s_1 \otimes Id_Z] \circ (n(Id_V \otimes B))) = B.$$

PROOF. The following diagram is commutative, the lower triangle by Lemma 1.65 and the upper block in a variation on Lemma 1.65 that is straightforward to check.

$$\begin{tikzcd}
\mathrm{End}(V) \otimes V \otimes Z \arrow[d, "n"'] & V^* \otimes V \otimes V \otimes Z \arrow[l, "{[k_{VV} \otimes Id_{V\otimes Z}]}"'] \arrow[d, "{[[Id_{V^*} \otimes s_1] \otimes Id_Z]}"] \\
\mathrm{Hom}(V, V \otimes V \otimes Z) \arrow[d, "{\mathrm{Hom}(Id_V, [s_1 \otimes Id_Z])}"'] & V^* \otimes V \otimes V \otimes Z \arrow[dl, "k_{V,V\otimes V\otimes Z}"] \arrow[d, "{[k_{VV} \otimes Id_{V\otimes Z}]}"] \\
\mathrm{Hom}(V, V \otimes V \otimes Z) & \mathrm{End}(V) \otimes V \otimes Z \arrow[l, "n"]
\end{tikzcd}$$

So the LHS of the claim is:

$$\begin{aligned}
& Tr_{V,V\otimes Z}([s_1 \otimes Id_Z] \circ (n(Id_V \otimes B))) \\
= \quad & (Tr_{V;V\otimes Z} \circ \mathrm{Hom}(Id_V, [s_1 \otimes Id_Z]) \circ n)(Id_V \otimes B) \\
= \quad & (Tr_{V;V\otimes Z} \circ n \circ [k_{VV} \otimes Id_{V\otimes Z}] \\
& \circ [[Id_{V^*} \otimes s_1] \otimes Id_Z] \circ [k_{VV}^{-1} \otimes Id_{V\otimes Z}])(Id_V \otimes B).
\end{aligned}$$

Let $s_3 : V \otimes Z \to Z \otimes V$ be another switching, and let $s_2$ and $s_4$ be the switchings as indicated:

$$s_2 = [Id_{V^*} \otimes s_4] : V^* \otimes V \otimes (V \otimes Z) \to V^* \otimes (V \otimes Z) \otimes V. \tag{2.26}$$

Using $W = V \otimes Z$ and this $s_4$ in the role of $s'$ from the diagram from the Proof of Theorem 2.74, the commutativity of that diagram continues the chain of equalities:

$$\begin{aligned}
= \quad & (Ev_{V,V\otimes Z} \circ [k_{V,V\otimes Z} \otimes Id_V] \circ [Id_{V^*} \otimes s_4] \\
& \circ [[Id_{V^*} \otimes s_1] \otimes Id_Z] \circ [k_{VV}^{-1} \otimes Id_{V\otimes Z}])(Id_V \otimes B) \\
= \quad & (Ev_{V,V\otimes Z} \circ [k_{V,V\otimes Z} \otimes Id_V] \circ [Id_{V^*\otimes V} \circ s_3] \circ [k_{VV}^{-1} \otimes Id_{V\otimes Z}])(Id_V \otimes B) \\
(2.27) = \quad & (Ev_{V,V\otimes Z} \circ [k_{V,V\otimes Z} \otimes Id_V] \circ [k_{VV}^{-1} \otimes s_3])(Id_V \otimes B).
\end{aligned}$$

The commutativity of the following diagram, using Lemma 1.36, Lemma 1.65, and a variation on Lemma 2.78,

$$\begin{tikzcd}
V^* \otimes V \otimes Z \otimes V \arrow[r, "{[k_{V,V\otimes Z} \otimes Id_V]}"] \arrow[d, "{[k_{VV} \otimes Id_{Z\otimes V}]}"'] & \mathrm{Hom}(V, V \otimes Z) \otimes V \arrow[dd, "Ev_{V,V\otimes Z}"] \\
\mathrm{End}(V) \otimes Z \otimes V \arrow[ur, "{[n_2 \otimes Id_V]}"'] & \\
\mathrm{End}(V) \otimes V \otimes Z \arrow[u, "{[Id_{\mathrm{End}(V)} \otimes s_3]}"] \arrow[r, "{[Ev_{V,V} \otimes Id_Z]}"] & V \otimes Z
\end{tikzcd}$$

brings line (2.27) to the conclusion:

$$= \quad [Ev_{V,V} \otimes Id_Z](Id_V \otimes B) = B.$$

∎

The switching $s_3 : V \otimes Z \to Z \otimes V$ from Theorem 2.87 appears in the following Corollaries, which are related to constructions in [**Stolz-Teichner**].

COROLLARY 2.88. *Given vector spaces $V$, $W$, and a map $A : V \to V \otimes W$, if $V$ is finite-dimensional and there exist a vector space $Z$ and a factorization of the form*

$$A = [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1},$$

*for $l : \mathbb{K} \otimes V \to V$, $B_1 : \mathbb{K} \to V \otimes Z$, and $B_2 : Z \otimes V \to W$, then*

$$Tr_{V;W}(A) = (B_2 \circ s_3 \circ B_1)(1).$$

PROOF. It is straightforward to check that the following diagram is commutative. The $s_1$ is the switching from Theorem 2.87, and temporarily denote $V_1 = V_2 = V$, to keep track of switching, so that as in Equation (2.26), $s_4 : V_1 \otimes (V_2 \otimes Z) \to (V_2 \otimes Z) \otimes V_1$.

$$\begin{tikzcd}
V_1 \arrow[dd, "n(Id_V\otimes(B_1(1)))"'] & \mathbb{K}\otimes V_1 \arrow[l, "l"'] \arrow[d, "{[B_1\otimes Id_V]}"] \\
 & V_2\otimes Z\otimes V_1 \\
V_1\otimes V_2\otimes Z \arrow[ur, "s_4"] \arrow[dr, "{[s_1\otimes Id_Z]}"'] & \\
 & V_2\otimes V_1\otimes Z \arrow[uu, "{[Id_V\otimes s_3]}"']
\end{tikzcd}$$

The following equalities use the commutativity of the diagram, Corollary 2.59 (or Corollary 2.75) twice, and Theorem 2.87 applied to $B = B_1(1) \in V \otimes Z$.

$$\begin{aligned}
Tr_{V;W}(A) &= Tr_{V;W}([Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1}) \\
&= B_2(Tr_{V;Z\otimes V}([B_1 \otimes Id_V] \circ l^{-1})) \\
&= B_2(Tr_{V;Z\otimes V}([Id_V \otimes s_3] \circ [s_1 \otimes Id_Z] \circ (n(Id_V \otimes (B_1(1)))))) \\
&= (B_2 \circ s_3)(Tr_{V;V\otimes Z}([s_1 \otimes Id_Z] \circ (n(Id_V \otimes (B_1(1)))))) \\
&= (B_2 \circ s_3)(B_1(1)).
\end{aligned}$$

∎

COROLLARY 2.89. *Given a vector space $V$ and a map $A : V \to V$, if $V$ is finite-dimensional and there exist a vector space $Z$ and a factorization of the form*

$$A = l_V \circ [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ l^{-1},$$

*for $l_V : V \otimes \mathbb{K} \to V$, $l : \mathbb{K} \otimes V \to V$, $B_1 : \mathbb{K} \to V \otimes Z$, and $B_2 : Z \otimes V \to \mathbb{K}$, then*

$$Tr_V(A) = (B_2 \circ s_3 \circ B_1)(1).$$

PROOF. Theorem 2.54 and Corollary 2.88 apply. ∎

Corollary 2.90. *Given vector spaces $V$, $W$, and a map $A : V \otimes U \to V \otimes W$, if $V$ is finite-dimensional and there exist a vector space $Z$ and a factorization of the form*

$$A = [Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ s_5,$$

*for $s_5 : V \otimes U \to U \otimes V$, $B_1 : U \to V \otimes Z$, and $B_2 : Z \otimes V \to W$, then*

$$Tr_{V;U,W}(A) = B_2 \circ s_3 \circ B_1.$$

Proof. Theorem 2.30 applies:

$$\begin{aligned} Tr_{V;U,W}(A) &= Tr_{V;U,W}([Id_V \otimes B_2] \circ [B_1 \otimes Id_V] \circ s_5) \\ &= B_2 \circ (Tr_{V;U,Z\otimes V}([B_1 \otimes Id_V] \circ s_5)), \end{aligned}$$

so to prove the claim it is enough to check

$$Tr_{V;U,Z\otimes V}([B_1 \otimes Id_V] \circ s_5) = s_3 \circ B_1 \tag{2.28}$$

for $B_1 \in \mathrm{Hom}(U, V \otimes Z)$. In the following diagram, temporarily denote by $a_1$ the following map,

$$\begin{aligned} a_1 &= j \circ [Id_{\mathrm{Hom}(U,V\otimes Z)} \otimes Q_1^1] \circ l^{-1}_{\mathrm{Hom}(U,V\otimes Z)} : \\ B_1 &\mapsto [B_1 \otimes Id_V], \end{aligned}$$

for $Q_1^1 : \mathbb{K} \to \mathrm{End}(V)$ as in Example 2.9.

$$\begin{tikzcd} V \otimes \mathrm{Hom}(U,Z) \arrow[r, "s_3'"] \arrow[d, "n''"'] & \mathrm{Hom}(U,Z) \otimes V \arrow[d, "n_1"] \\ \mathrm{Hom}(U, V \otimes Z) \arrow[r, "\mathrm{Hom}(Id_V, s_3)"'] \arrow[d, "a_1"'] & \mathrm{Hom}(U, Z \otimes V) \\ \mathrm{Hom}(U \otimes V, V \otimes Z \otimes V) \arrow[d, "\mathrm{Hom}(s_5, Id_{V\otimes Z\otimes V})"'] & \mathrm{Hom}(V, V \otimes \mathrm{Hom}(U, Z \otimes V)) \arrow[u, "Tr_{V;\mathrm{Hom}(U,Z\otimes V)}"'] \arrow[d, "\mathrm{Hom}(Id_V, n')"] \\ \mathrm{Hom}(V \otimes U, V \otimes Z \otimes V) \arrow[uur, "Tr_{V;U,Z\otimes V}"] & \mathrm{Hom}(V, \mathrm{Hom}(U, V \otimes Z \otimes V)) \arrow[l, "q"] \end{tikzcd}$$

The maps $n''$, $n'$, $n_1$, as in Definition 1.40 and Notation 1.41, are all invertible, by the finite-dimensionality of $V$ and Lemma 1.44. The commutativity of the top block, with the switching maps, is easily checked. The commutativity of the lower right triangle is Theorem 2.52. So, the claim of (2.28) is that the lower left triangle is commutative, and this will follow from showing that the outer part of the diagram is commutative, starting with $v_0 \otimes B_3 \in V \otimes \mathrm{Hom}(U, Z)$.

For $v \in V$, $u \in U$, the following maps are equal. The first step uses the formula for the inverse of the canonical map $q$ from Lemma 1.47.

$$\begin{aligned} (q^{-1}((a_1(n''(v_0 \otimes B_3))) \circ s_5))(v) : u &\mapsto ((a_1(n''(v_0 \otimes B_3))) \circ s_5)(v \otimes u) \\ &= [(n''(v_0 \otimes B_3)) \otimes Id_V](u \otimes v) \\ &= v_0 \otimes (B_3(u)) \otimes v, \\ n'([Id_V \otimes n_1](v_0 \otimes B_3 \otimes v)) : u &\mapsto v_0 \otimes ((n_1(B_3 \otimes v))(u)) \\ &= v_0 \otimes (B_3(u)) \otimes v. \end{aligned} \tag{2.29}$$

Denote, as in Definition 1.20,

$$m(v_0 \otimes B_3) : \mathbb{K} \to V \otimes \mathrm{Hom}(U,Z) : 1 \mapsto v_0 \otimes B_3,$$

so that

$$v_0 \otimes B_3 \otimes v = ([(m(v_0 \otimes B_3)) \otimes Id_V] \circ l^{-1})(v),$$

and from (2.29),

$$q^{-1}((a_1(n''(v_0 \otimes B_3))) \circ s_5) = n' \circ [Id_V \otimes n_1] \circ [(m(v_0 \otimes B_3)) \otimes Id_V] \circ l^{-1}.$$

The conclusion uses Corollary 2.88:

$$\begin{array}{rcl} v_0 \otimes B_3 & \mapsto & Tr_{V;\mathrm{Hom}(U,Z\otimes V)}((n')^{-1} \circ (q^{-1}((a_1(n''(v_0 \otimes B_3))) \circ s_5))) \\ & = & Tr_{V;\mathrm{Hom}(U,Z\otimes V)}([Id_V \otimes n_1] \circ [(m(v_0 \otimes B_3)) \otimes Id_V] \circ l^{-1}) \\ & = & (n_1 \circ s_3' \circ (m(v_0 \otimes B_3)))(1) = (n_1 \circ s_3')(v_0 \otimes B_3). \end{array}$$

∎

EXAMPLE 2.91. Consider $V = V_1 = V_2$ and $s_4 : V_1 \otimes (V_2 \otimes Z) \to (V_2 \otimes Z) \otimes V_1$ as in Equation (2.26) and the Proof of Corollary 2.88. Then

$$Tr_{V;V\otimes Z,Z\otimes V}(s_4) = s_3.$$

This is a special case of Corollary 2.90, with $U = V \otimes Z$, $W = Z \otimes V$, $s_5 = s_4$, $B_1 = Id_{V\otimes Z}$, and $B_2 = Id_{Z\otimes V}$.

EXAMPLE 2.92. Using Theorem 2.30 and Example 2.91,

$$\begin{array}{rcl} Tr_{V;Z\otimes V,V\otimes Z}(s_4^{-1}) & = & Tr_{V;Z\otimes V,V\otimes Z}([Id_V \otimes s_3^{-1}] \circ s_4 \circ [Id_V \otimes s_3^{-1}]) \\ & = & s_3^{-1} \circ (Tr_{V;V\otimes Z,Z\otimes V}(s_4)) \circ s_3^{-1} \\ & = & s_3^{-1}. \end{array}$$

EXAMPLE 2.93. Formula (2.23) from Example 2.82 also follows as a special case; for the switching map $s_1 : V \otimes V \to V \otimes V$ as in Theorem 2.87, $Tr_{V;V,V}(s_1) = Id_V$. This is the case of Corollary 2.90 with $U = V$, $Z = \mathbb{K}$, $B_1 = l_V^{-1}$, $B_2 = l$, $A = s_1 = s_5 = [Id_V \otimes l] \circ [l_V^{-1} \otimes Id_V] \circ s_5$, and $s_3 : V \otimes \mathbb{K} \to \mathbb{K} \otimes V$, so

$$Tr_{V;V,V}(s_5) = l \circ s_3 \circ l_V^{-1} = Id_V.$$

THEOREM 2.94. *For finite-dimensional $V$ and $U$, a switching map $s : W \otimes U \to U \otimes W$, and $A : V \otimes U \to V \otimes W$,*

$$Tr_{V;U,W}(A) = Tr_{V\otimes U;U,W}([Id_V \otimes s] \circ [A \otimes Id_U]).$$

PROOF. Using Theorem 2.38, Theorem 2.30, Corollary 2.42, an easily checked equality relating the switching maps $s$ and $s' : U \otimes U \to U \otimes U$, Theorem 2.30 again, and finally Example 2.82,

$$\begin{array}{rcl} RHS & = & Tr_{U;U,W}(Tr_{V;U\otimes U,U\otimes W}([Id_V \otimes s] \circ [A \otimes Id_U])) \\ & = & Tr_{U;U,W}(s \circ (Tr_{V;U\otimes U,W\otimes U}([A \otimes Id_U]))) \\ & = & Tr_{U;U,W}(s \circ [(Tr_{V;U,W}(A)) \otimes Id_U]) \\ & = & Tr_{U;U,W}([Id_U \otimes (Tr_{V;U,W}(A))] \circ s') \\ & = & (Tr_{V;U,W}(A)) \circ (Tr_{U;U,U}(s')) \\ & = & Tr_{V;U,W}(A). \end{array}$$

∎

EXERCISE 2.95. For finite-dimensional $V$ and $W$, a switching map $s : W \otimes U \to U \otimes W$, and $A : V \otimes U \to V \otimes W$,

$$Tr_{V;U,W}(A) = Tr_{V\otimes W;U,W}([A \otimes Id_W] \otimes [Id_V \otimes s]).$$

HINT. The steps are analogous to the steps in the Proof of Theorem 2.94. If $U$ and $W$ are both finite-dimensional, then the equality of the RHS of this equation with the RHS from Theorem 2.94 follows directly from Theorem 2.29. ∎

### 2.4.2. Coevaluation and dualizability.

THEOREM 2.96. *For finite-dimensional $V$, $\eta_V : \mathbb{K} \to V \otimes V^*$ as in Notation 2.68, and scalar multiplication maps $l_V : \mathbb{K} \otimes V \to V$, $l_{V^*} : \mathbb{K} \otimes V^* \to V^*$, $l_1 : V \otimes \mathbb{K} \to V$, $l_2 : V^* \otimes \mathbb{K} \to V^*$,*

$$l_1 \circ [Id_V \otimes Ev_V] \circ [\eta_V \otimes Id_V] \circ l_V^{-1} = Id_V,$$

*and*

$$l_{V^*} \circ [Ev_V \otimes Id_{V^*}] \circ [Id_{V^*} \otimes \eta_V] \circ l_2^{-1} = Id_{V^*}.$$

PROOF. In the following two diagrams, $V = V_1 = V_2 = V_3$ — the subscripts are added just to track the action of the switchings and other canonical maps. In the first diagram, the upper left square uses the formula (2.13) with $k^{-1}$ from Notation 2.68, and is commutative by Lemma 1.36. The $s_5$ notation in the right half is from Corollary 2.90 and Example 2.93. The first claim is that the lower left part of the diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(V_2, V_1) \otimes V_3 & V_2^* \otimes V_1 \otimes V_3 \arrow[l, "{[k \otimes Id_V]}"'] \arrow[r, "{[Id_{V^*} \otimes s_5]}"] \arrow[d, "{[s \otimes Id_V]}"] & V_2^* \otimes V_3 \otimes V_1 \arrow[d, "{[Ev_V \otimes Id_V]}"'] \\
\mathbb{K} \otimes V_3 \arrow[u, "{[Q_1^1 \otimes Id_V]}"] \arrow[r, "{[\eta_V \otimes Id_V]}"] \arrow[ddr, "l_V"] & V_1 \otimes V_2^* \otimes V_3 \arrow[d, "{[Id_V \otimes Ev_V]}"] & \mathbb{K} \otimes V_1 \arrow[ddl, "l_V"'] \\
 & V_1 \otimes \mathbb{K} \arrow[d, "l_1"] & \\
 & V &
\end{tikzcd}$$

The commutativity of the right half of the diagram is easy to check. The first claim follows from checking that the identity map is equal to the composite of maps starting at $V$ and going clockwise. Lemma 2.81 applies.

$$\begin{aligned}
& l_1 \circ [Id_V \otimes Ev_V] \circ [\eta_V \otimes Id_V] \circ l_V^{-1} \\
(2.30) \quad = \; & l_V \circ [Ev_V \otimes Id_V] \circ [Id_{V^*} \otimes s_5] \circ [k^{-1} \otimes Id_V] \circ [Q_1^1 \otimes Id_V] \circ l_V^{-1} \\
= \; & Id_V.
\end{aligned}$$

The expression (2.30) is also equal to $Tr_{V;V,V}(s_5) = Id_V$ as in Examples 2.82 and 2.93.

For the second claim, consider the second diagram, where $s''$ is the switching involution from Lemma 2.84 and $s''''$ is another switching map so that the upper block is easily seen to be commutative.

$$\begin{tikzcd}
& \mathrm{Hom}(V_2,V_1)\otimes V_3^* & V_2^*\otimes V_1\otimes V_3^* \arrow[l, "{[k\otimes Id_{V^*}]}"'] \arrow[ddl, "s''"] \\
V_3^*\otimes \mathrm{Hom}(V_2,V_1) \arrow[ur, "s''''"] & V_3^*\otimes V_2^*\otimes V_1 \arrow[l, "{[Id_{V^*}\otimes k]}"'] \arrow[d, "{[Id_{V^*}\otimes s]}"'] & \\
V_3^*\otimes \mathbb{K} \arrow[u, "{[Id_{V^*}\otimes Q_1^1]}"'] \arrow[r, "{[Id_{V^*}\otimes \eta_V]}"] \arrow[ddr, "l_2"] & V_3^*\otimes V_1\otimes V_2^* \arrow[d, "{[Ev_V\otimes Id_{V^*}]}"'] & \\
& \mathbb{K}\otimes V_2^* \arrow[d, "l_{V^*}"] & \\
& V^* &
\end{tikzcd}$$

As in the first diagram, the definition of $\eta_V$ is used in the left square, and the second claim is that the lower left part of the second diagram is commutative. The calculation is again to check the clockwise composition, and Lemma 2.84 applies.

$$\begin{aligned}
& l_{V^*}\circ[Ev_V\otimes Id_{V^*}]\circ[Id_{V^*}\otimes\eta_V]\circ l_2^{-1} \\
= \; & l_{V^*}\circ[Ev_V\otimes Id_{V^*}]\circ s''\circ[k^{-1}\otimes Id_{V^*}]\circ s''''\circ[Id_{V^*}\otimes Q_1^1]\circ l_2^{-1} \\
= \; & l_{V^*}\circ[Ev_V\otimes Id_{V^*}]\circ s''\circ[k^{-1}\otimes Id_{V^*}]\circ[Q_1^1\otimes Id_{V^*}]\circ l_{V^*}^{-1} \\
= \; & Id_{V^*}.
\end{aligned}$$

∎

DEFINITION 2.97. A vector space $V$ is <u>dualizable</u> means: there exists $(D,\epsilon,\eta)$, where $D$ is a vector space, and $\epsilon: D\otimes V\to\mathbb{K}$ and $\eta:\mathbb{K}\to V\otimes D$ are linear maps such that the following diagrams (involving various scalar multiplication maps) are commutative.

$$\begin{tikzcd}
\mathbb{K}\otimes V \arrow[r, "{[\eta\otimes Id_V]}"] \arrow[d] & V\otimes D\otimes V \arrow[d, "{[Id_V\otimes\epsilon]}"] \\
V & V\otimes\mathbb{K} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
D\otimes\mathbb{K} \arrow[r, "{[Id_D\otimes\eta]}"] \arrow[d] & D\otimes V\otimes D \arrow[d, "{[\epsilon\otimes Id_D]}"] \\
D & \mathbb{K}\otimes D \arrow[l]
\end{tikzcd}$$

EXAMPLE 2.98. Given $V$ as in Theorem 2.96, the space $D=V^*$ and the maps $\epsilon=Ev_V$ and $\eta=\eta_V=s\circ k^{-1}\circ Q_1^1$ satisfy the identities from Definition 2.97.

REMARK 2.99. In category theory and other generalizations of this construction ([**Stolz-Teichner**], [**PS**]), $\eta$ is called a <u>coevaluation</u> map. A more general notion, with left and right duals, is considered by [**Maltsiniotis**].

Lemma 2.100. *If $V$ is dualizable, with duality data $(D, \epsilon, \eta)$, then there is an invertible map $D \to V^*$.*

Proof. It is equivalent, by Example 1.28 and Lemma 1.22, to show there is an invertible map $\mathrm{Hom}(\mathbb{K}, D) \to \mathrm{Hom}(\mathbb{K} \otimes V, \mathbb{K})$. Denote:

$$\begin{array}{rcl} A : \mathrm{Hom}(\mathbb{K}, D) & \to & \mathrm{Hom}(\mathbb{K} \otimes V, \mathbb{K}) \\ \delta & \mapsto & (\lambda \otimes v \mapsto \epsilon((\delta(\lambda)) \otimes v)), \\ B : \mathrm{Hom}(\mathbb{K} \otimes V, \mathbb{K}) & \to & \mathrm{Hom}(\mathbb{K}, D) \\ \phi & \mapsto & (\lambda \mapsto (l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta)(\lambda)), \end{array} \tag{2.31}$$

where $l$ denotes various scalar multiplications. The following diagrams are commutative, where unlabeled arrows are scalar multiplications or their inverses.

$$\begin{tikzcd} \mathbb{K} \arrow[r, "\delta"] \arrow[d, "\eta"] & D \arrow[d] \\ V \otimes D \arrow[d] & D \otimes \mathbb{K} \arrow[d, "{[Id_D \otimes \eta]}"] \\ \mathbb{K} \otimes V \otimes D \arrow[r, "{[[\delta \otimes Id_V] \otimes Id_D]}"] \arrow[dr, "{[(A(\delta)) \otimes Id_D]}"'] & D \otimes V \otimes D \arrow[d, "{[\epsilon \otimes Id_D]}"] \\ & \mathbb{K} \otimes D \arrow[d] \\ & D \end{tikzcd} \qquad \begin{tikzcd} \mathbb{K} \otimes V \arrow[d, "{[\eta \otimes Id_V]}"] & \\ V \otimes D \otimes V \arrow[r] \arrow[d, "{[Id_V \otimes \epsilon]}"] & \mathbb{K} \otimes V \otimes D \otimes V \arrow[d, "{[[\phi \otimes Id_D] \otimes Id_V]}"] \\ V \otimes \mathbb{K} \arrow[d] & \mathbb{K} \otimes D \otimes V \arrow[d] \\ V \arrow[d] & D \otimes V \arrow[d, "\epsilon"] \\ \mathbb{K} \otimes V \arrow[r, "\phi"] & \mathbb{K} \end{tikzcd}$$

In the left diagram, the top square is easily checked and the lower triangle uses the formula for $A$. The composition in the right column gives the identity map for $D$ by Definition 2.97, so $Id_D \circ \delta = B(A(\delta)) : \mathbb{K} \to D$.

In the right diagram, the left column gives the identity map for $\mathbb{K} \otimes V$ by Definition 2.97. For $\lambda \otimes v \in \mathbb{K} \otimes V$,

$$\begin{array}{rcl} (A \circ B)(\phi) : \lambda \otimes v & \mapsto & \epsilon(((B(\phi))(\lambda)) \otimes v) \\ & = & \epsilon(((l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta)(\lambda)) \otimes v) \\ & = & (\epsilon \circ [(l \circ [\phi \otimes Id_D] \circ l^{-1} \circ \eta) \otimes Id_V])(\lambda \otimes v) \\ & = & (\phi \circ Id_{\mathbb{K} \otimes V})(\lambda \otimes v). \end{array}$$

∎

Lemma 2.101. *If $V$ is dualizable, with two triples of duality data: $(D_1, \epsilon_1, \eta_1)$ and $(D_2, \epsilon_2, \eta_2)$, then the map $a_{12} : D_1 \to D_2$,*

$$D_1 \longrightarrow D_1 \otimes \mathbb{K} \xrightarrow{[Id_{D_1} \otimes \eta_2]} D_1 \otimes V \otimes D_2 \xrightarrow{[\epsilon_1 \otimes Id_{D_2}]} \mathbb{K} \otimes D_2 \longrightarrow D_2$$

*has inverse given by the map $a_{21} : D_2 \to D_1$,*

$$D_2 \longrightarrow D_2 \otimes \mathbb{K} \xrightarrow{[Id_{D_2} \otimes \eta_1]} D_2 \otimes V \otimes D_1 \xrightarrow{[\epsilon_2 \otimes Id_{D_1}]} \mathbb{K} \otimes D_1 \longrightarrow D_1$$

*and $a_{12}$ satisfies the identities $[Id_V \otimes a_{12}] \circ \eta_1 = \eta_2$ and $\epsilon_2 \circ [a_{12} \otimes Id_V] = \epsilon_1$.*

PROOF. Some of the arrows in the following diagram are left unlabeled, but they involve only identity maps, scalar multiplications and their inverses, and the given $\eta_1, \eta_2, \epsilon_1, \epsilon_2$ maps.

$$\begin{tikzcd}
D_1 \arrow[d] & & & \\
D_1 \otimes \mathbb{K} \arrow[rr] \arrow[dd] & & D_1 \otimes V \otimes D_2 \arrow[d] \arrow[dr] & \\
& D_1 \otimes \mathbb{K} \otimes V \otimes D_1 \arrow[d] & D_1 \otimes V \otimes D_2 \otimes \mathbb{K} \arrow[dl] & \mathbb{K} \otimes D_2 \arrow[d] \\
D_1 \otimes V \otimes D_1 \arrow[ur] \arrow[dd] & D_1 \otimes V \otimes D_2 \otimes V \otimes D_1 \arrow[d] \arrow[dr] & & D_2 \arrow[d] \\
& D_1 \otimes V \otimes \mathbb{K} \otimes D_1 \arrow[ul] & \mathbb{K} \otimes D_2 \otimes V \otimes D_1 \arrow[d] & D_2 \otimes \mathbb{K} \arrow[dl] \\
\mathbb{K} \otimes D_1 \arrow[d] & & D_2 \otimes V \otimes D_1 \arrow[ll] & \\
D_1 & & &
\end{tikzcd}$$

The composition in the left column gives the identity map $D_1 \to D_1$, and the middle left block is commutative, involving the composite $[Id_V \otimes \epsilon_2] \circ [\eta_2 \otimes Id_V]$. The commutativity of the upper, right, and lower blocks is easy to check. The composite $D_1 \to D_2 \to D_1$ clockwise from the top is equal to $a_{21} \circ a_{12}$, and the commutativity of the diagram establishes the claim that $a_{21} \circ a_{12} = Id_{D_1}$; checking the inverse in the other order follows from relabeling the subscripts.

For the identity $[Id_V \otimes a_{12}] \circ \eta_1 = \eta_2$, consider the following diagram.

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "\eta_2"] \arrow[dd, "\eta_1"'] & V \otimes D_2 \arrow[r] & \mathbb{K} \otimes V \otimes D_2 \arrow[dd, "{[[\eta_1 \otimes Id_V] \otimes Id_{D_2}]}"'] \\
& V \otimes \mathbb{K} \otimes D_2 \arrow[u] & \\
V \otimes D_1 \arrow[uur, "{[Id_V \otimes a_{12}]}"] \arrow[r] & V \otimes D_1 \otimes \mathbb{K} \arrow[r, "{[Id_V \otimes [Id_{D_1} \otimes \eta_2]]}"'] & V \otimes D_1 \otimes V \otimes D_2 \arrow[ul, "{[Id_V \otimes [\epsilon_1 \otimes Id_{D_2}]]}"']
\end{tikzcd}$$

The lower block uses the definition of $a_{12}$. The right block involves $\eta_1$ and $\epsilon_1$ so that one of the identities from Definition 2.97 applies. The claim is that the left triangle is commutative, and this follows from the easily checked commutativity of the outer rectangle.

Similarly for the identity $\epsilon_2 \circ [a_{12} \otimes Id_V] = \epsilon_1$, consider the following diagram.

$$\begin{tikzcd}
D_1 \otimes V \otimes D_2 \otimes V \arrow[rr, "{[Id_{D_1} \otimes [Id_V \otimes \epsilon_2]]}"] \arrow[dd, "{[[\epsilon_1 \otimes Id_{D_2}] \otimes Id_V]}"] & & D_1 \otimes V \otimes \mathbb{K} \arrow[r] & D_1 \otimes V \arrow[dl] \arrow[ddl, "{[a_{12} \otimes Id_V]}"'] \arrow[dd, "\epsilon_1"] \\
& & D_1 \otimes \mathbb{K} \otimes V \arrow[ull, "{[[Id_{D_1} \otimes \eta_2] \otimes Id_V]}"] & \\
\mathbb{K} \otimes D_2 \otimes V \arrow[rr] & & D_2 \otimes V \arrow[r, "\epsilon_2"'] & \mathbb{K}
\end{tikzcd}$$

The left block uses the definition of $a_{12}$. The top block involves $\eta_2$ and $\epsilon_2$ so that one of the identities from Definition 2.97 applies. The claim is that the right triangle is commutative, and this follows from the easily checked commutativity of the outer rectangle. ∎

In the case $D = V^*$ from Example 2.98, the maps from Lemma 2.100 and 2.101 agree (up to composition with trivial invertible maps as in the following Exercise) and so they are canonical.

Exercise 2.102. Applying Lemma 2.100 to the triple $(V^*, Ev_V, \eta_V)$ from Example 2.98 gives a map $A$ such that the left diagram is commutative. If $V$ is also dualizable with $(D_2, \epsilon_2, \eta_2)$, then the maps $B$ from Lemma 2.100 and $a_{12}$ from Lemma 2.101 make the right diagram commutative.

$$\begin{tikzcd}
V^* \arrow[r, "Id_{V^*}"] \arrow[d, "m"] & V^* \arrow[d, "{\mathrm{Hom}(l, Id_{\mathbb{K}})}"] \\
\mathrm{Hom}(\mathbb{K}, V^*) \arrow[r, "A"] & \mathrm{Hom}(\mathbb{K} \otimes V, \mathbb{K})
\end{tikzcd}
\qquad
\begin{tikzcd}
D_2 \arrow[d, "m"] & V^* \arrow[l, "a_{12}"'] \arrow[d, "{\mathrm{Hom}(l, Id_{\mathbb{K}})}"] \\
\mathrm{Hom}(\mathbb{K}, D_2) & \mathrm{Hom}(\mathbb{K} \otimes V, \mathbb{K}) \arrow[l, "B"']
\end{tikzcd}$$

Hint. The first claim is left as an exercise. For the second claim, consider $\phi \in V^*$, $\lambda \in \mathbb{K}$; the following quantities agree, showing the right diagram is commutative.

$$\begin{array}{rcl}
(m \circ a_{12})(\phi) : \lambda & \mapsto & (m(a_{12}(\phi)))(\lambda) = \lambda \cdot a_{12}(\phi) \\
& = & \lambda \cdot (l \circ [Ev_V \otimes Id_{D_2}] \circ [Id_{V^*} \otimes \eta_2] \circ l^{-1})(\phi) \\
& = & \lambda \cdot (l \circ [Ev_V \otimes Id_{D_2}])(\phi \otimes (\eta_2(1))), \\
(B \circ \mathrm{Hom}(l, Id_{\mathbb{K}}))(\phi) : \lambda & \mapsto & (B(\phi \circ l))(\lambda) \\
& = & (l \circ [(\phi \circ l) \otimes Id_{D_2}] \circ l^{-1} \circ \eta_2)(\lambda) \\
& = & (l \circ [(\phi \circ l) \otimes Id_{D_2}])(1 \otimes (\eta_2(\lambda))) \\
& = & (l \circ [\phi \otimes Id_{D_2}])(\eta_2(\lambda)).
\end{array}$$

∎

LEMMA 2.103. *If $V$ is dualizable, with $(D, \epsilon, \eta)$, then $D$ is dualizable, with duality data $(V, \epsilon \circ s, s \circ \eta)$, where $s : V \otimes D \to D \otimes V$ is a switching map.*

PROOF. In the following diagram, $V = V_1 = V_2$.

$$\begin{tikzcd}
V_1 \otimes \mathbb{K} \arrow[r, "{[Id_V \otimes \eta]}"] \arrow[d] & V_1 \otimes V_2 \otimes D \arrow[r, "{[Id_V \otimes s]}"] \arrow[d, "s_1"] & V_1 \otimes D \otimes V_2 \arrow[d, "{[s \otimes Id_V]}"] \\
\mathbb{K} \otimes V_1 \arrow[r, "{[\eta \otimes Id_V]}"] \arrow[d] & V_2 \otimes D \otimes V_1 \arrow[d, "{[Id_V \otimes \epsilon]}"] & D \otimes V_1 \otimes V_2 \arrow[l, "s_2"'] \arrow[d, "{[\epsilon \otimes Id_V]}"] \\
V & V_2 \otimes \mathbb{K} \arrow[l] & \mathbb{K} \otimes V_2 \arrow[l]
\end{tikzcd}$$

Unlabeled arrows are obvious switching or scalar multiplication. The $s_1$, $s_2$ switchings are as indicated by the subscripts. The lower left square is commutative, by the first identity from Definition 2.97, and the other small squares are easy to check, so the large square is commutative, which is the second identity for $(V, \epsilon \circ s, s \circ \eta)$ from Definition 2.97 applied to $D$.

Similarly, in the following diagram, $D = D_1 = D_2$.

$$\begin{tikzcd}
\mathbb{K} \otimes D_1 \arrow[r, "{[\eta \otimes Id_D]}"] \arrow[d] & V \otimes D_2 \otimes D_1 \arrow[r, "{[s \otimes Id_D]}"] \arrow[d, "s'_1"] & D_2 \otimes V \otimes D_1 \arrow[d, "{[Id_D \otimes s]}"] \\
D_1 \otimes \mathbb{K} \arrow[r, "{[Id_D \otimes \eta]}"] \arrow[d] & D_1 \otimes V \otimes D_2 \arrow[d, "{[\epsilon \otimes Id_D]}"] & D_2 \otimes D_1 \otimes V \arrow[l, "s'_2"'] \arrow[d, "{[Id_D \otimes \epsilon]}"] \\
D & \mathbb{K} \otimes D_2 \arrow[l] & D \otimes \mathbb{K} \arrow[l]
\end{tikzcd}$$

Again, the lower left square is commutative by hypothesis, and the commutativity of the large square is the first identity for $(V, \epsilon \circ s, s \circ \eta)$ from Definition 2.97 applied to $D$. ∎

LEMMA 2.104. *If $V$ is dualizable, then $d_V$ is invertible.*

PROOF. Let $a_1 : D \to V^*$ be the invertible map from Lemma 2.100, defined in terms of $\epsilon$ and $A_1 = A$ from (2.31). The transposes of these maps appear in the right square of the diagram.

By Lemma 2.103, $D$ is also dualizable, with an evaluation map $\epsilon \circ s$, which defines $A_2$ as in (2.31) and an invertible map $a_2 : V \to D^*$ from Lemma 2.100 again. These maps appear in the top square of the diagram.

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{K}, V) \arrow[r, "A_2"] & (\mathbb{K} \otimes D)^* & \\
V \arrow[u, "m_V"] \arrow[r, "a_2"] \arrow[dr, "d_V"'] & D^* \arrow[u, "l_D^*"'] & (\mathrm{Hom}(\mathbb{K}, D))^* \arrow[l, "m_D^*"'] \\
 & V^{**} \arrow[u, "a_1^*"] & (\mathbb{K} \otimes V)^{**} \arrow[l, "l_V^{**}"'] \arrow[u, "A_1^*"']
\end{tikzcd}$$

The two squares in the diagram are commutative by construction. The following calculation checks that $a_1^* \circ d_V : V \to D^*$ is equal to $a_2$.

$$\begin{array}{rcl} l_D^* \circ a_1^* \circ d_V & = & l_D^* \circ m_D^* \circ A_1^* \circ (l_V^{**})^{-1} \circ d_V \\ & = & ((l_V^{-1})^* \circ A_1 \circ m_D \circ l_D)^* \circ d_V : \\ v & \mapsto & (d_V(v)) \circ ((l_V^{-1})^* \circ A_1 \circ m_D \circ l_D) : \\ \lambda \otimes u & \mapsto & (d_V(v))((A_1(m_D(\lambda \cdot u))) \circ l_V^{-1}) \\ & = & (A_1(m_D(\lambda \cdot u)))(1 \otimes v) \\ & = & \epsilon(((m_D(\lambda \cdot u))(1)) \otimes v) \\ & = & \epsilon((\lambda \cdot u) \otimes v), \\ A_2 \circ m_V : v & \mapsto & A_2(m_V(v)) : \\ \lambda \otimes u & \mapsto & (\epsilon \circ s)(((m_V(v))(\lambda)) \otimes u) \\ & = & (\epsilon \circ s)((\lambda \cdot v) \otimes u) \\ & = & \epsilon(u \otimes (\lambda \cdot v)). \end{array}$$

It follows that $d_V = (a_1^*)^{-1} \circ a_2$ is invertible. ∎

THEOREM 2.105. *Given $V$, the following are equivalent.*

(1) $k : V^* \otimes V \to \mathrm{End}(V)$ *is invertible.*
(2) $V$ *is dualizable.*
(3) $d : V \to V^{**}$ *is invertible.*
(4) $V$ *is finite-dimensional.*

PROOF. The Proof of Theorem 2.96 only used the property that $k$ is invertible to show that $V$ is dualizable, with $D = V^*$, $\epsilon = Ev_V$, and $\eta = \eta_V = s \circ k^{-1} \circ Q_1^1$; this is the implication (1) $\implies$ (2). Lemma 2.104 just showed (2) $\implies$ (3), and Theorem 2.85 showed directly that (1) $\implies$ (3). The implication (3) $\implies$ (4) was stated without proof in Claim 1.16 and the implication (4) $\implies$ (1) was stated in Lemma 1.64, which was proved using Claim 1.34. ∎

REMARK 2.106. In the special case where $(D, \epsilon, \eta) = (V^*, Ev_V, \eta_V)$ from Example 2.98, the map $a_1$ from Lemma 2.104 is $Id_{V^*}$ as in Exercise 2.102, and the map $a_2$ is exactly $d_V$. This shows that Lemma 2.100 (establishing that $A_2$ has an inverse, $B$) is related to Theorem 2.85 (showing that $d_V$ has an inverse); the second diagram from the Proof of Theorem 2.85 is similar to the right diagram from the Proof of Lemma 2.100.

The following result is a generalization of Theorem 2.69.

THEOREM 2.107. *If $V$ is dualizable, with any duality data $(D_2, \epsilon_2, \eta_2)$, and $s_2 : V \otimes D_2 \to D_2 \otimes V$ is the switching map, then for any $F : V \otimes U \to V \otimes W$,*

$$(Tr_{V;U,W}(F))(u) = (l_W \circ [\epsilon_2 \otimes Id_W] \circ [Id_{D_2} \otimes F] \circ [(s_2 \circ \eta_2) \otimes Id_U] \circ l_U^{-1})(u).$$

PROOF. By Theorem 2.105, $V$ must be finite-dimensional, so the trace exists. By Theorem 2.96 (the Proof of which uses Theorem 2.69) and Example 2.98, there is another triple $(D_1, \epsilon_1, \eta_1) = (V^*, Ev_V, \eta_V)$ satisfying Definition 2.97. There is an invertible map $a_{12} : V^* \to D_2$ by Lemma 2.101. Consider the following diagram.

$$\begin{tikzcd}
 & & & W \\
\mathrm{End}(V)\otimes U & V^*\otimes V\otimes U \arrow[l, "{[k\otimes Id_U]}"'] \arrow[r, "{[Id_{V^*}\otimes F]}"] \arrow[d, "{[s\otimes Id_U]}"] & V^*\otimes V\otimes W \arrow[r, "{[Ev_V\otimes Id_W]}"] \arrow[ddr, "{[[a_{12}\otimes Id_V]\otimes Id_W]}"'] & \mathbb{K}\otimes W \arrow[u, "l_W"'] \\
\mathbb{K}\otimes U \arrow[u, "{[Q_1^1\otimes Id_U]}"] \arrow[r, "{[\eta_V\otimes Id_U]}"] \arrow[dr, "{[\eta_2\otimes Id_U]}"'] & V\otimes V^*\otimes U \arrow[d, "{[[Id_V\otimes a_{12}]\otimes Id_U]}"] & & \\
U \arrow[u, "l_U^{-1}"] & V\otimes D_2\otimes U \arrow[r, "{[s_2\otimes Id_U]}"'] & D_2\otimes V\otimes U \arrow[r, "{[Id_{D_2}\otimes F]}"'] & D_2\otimes V\otimes W \arrow[uu, "{[\epsilon_2\otimes Id_W]}"']
\end{tikzcd}$$

The composition from $U$ to $W$ along the top row gives $Tr_{V;U,W}(F)$ by Theorem 2.69. The left square is from Theorem 2.96 and the left and right triangles are commutative by Lemma 2.101. The RHS of the Theorem is the path from $U$ to $W$ along the lowest row, so the claimed equality follows from the easily checked commutativity of the middle block. ∎

Corollary 2.108. *If $V$ is dualizable, with any duality data $(D, \epsilon, \eta)$, and $s : V \otimes D \to D \otimes V$ is the switching map, then for any $A : V \to V \otimes W$,*

$$Tr_{V;W}(A) = (l_W \circ [\epsilon \otimes Id_W] \circ [Id_D \otimes A] \circ s \circ \eta)(1).$$

Proof. This follows from Theorem 2.107 in the same way that Corollary 2.70 follows from Theorem 2.69. By Theorem 2.55,

$$\begin{aligned}
LHS &= (Tr_{V;\mathbb{K},W}(A \circ l_V))(1) \\
&= (l_W \circ [\epsilon \otimes Id_W] \circ [Id_D \otimes (A \circ l_V)] \circ [(s \circ \eta) \otimes Id_{\mathbb{K}}] \circ l_{\mathbb{K}}^{-1})(1) \\
&= RHS.
\end{aligned}$$

∎

This generalizes Corollary 2.70 by showing that, for any duality data $(D, \epsilon, \eta)$, the $W$-valued trace of $A$ is the output of 1 under the composite map

$$\mathbb{K} \xrightarrow{\eta} V \otimes D \xrightarrow{s} D \otimes V \xrightarrow{[Id_D \otimes A]} D \otimes V \otimes W \xrightarrow{l_W \circ [\epsilon \otimes Id_W]} W.$$

Corollary 2.109. *If $V$ is dualizable, with any duality data $(D, \epsilon, \eta)$, and $s : V \otimes D \to D \otimes V$ is the switching map, then for any $A : V \to V$,*

$$Tr_V(A) = (\epsilon \circ [Id_D \otimes A] \circ s \circ \eta)(1).$$

Proof. By Theorem 2.54 and Corollary 2.108,

$$\begin{aligned}
LHS &= Tr_{V;\mathbb{K}}(l_V^{-1} \circ A) \\
&= (l_{\mathbb{K}} \circ [\epsilon \otimes Id_{\mathbb{K}}] \circ [Id_D \otimes (l_V^{-1} \circ A)] \circ s \circ \eta)(1) \\
&= RHS.
\end{aligned}$$

∎

This generalizes Theorem 2.10 and the map $\mathbb{K} \to \mathbb{K}$ from (2.6) by showing that for any $(D, \epsilon, \eta)$, the trace of $A \in \mathrm{End}(V)$ is the output of 1 under the composite map

$$\mathbb{K} \xrightarrow{\eta} V \otimes D \xrightarrow{s} D \otimes V \xrightarrow{[Id_D \otimes A]} D \otimes V \xrightarrow{\epsilon} \mathbb{K}.$$

As mentioned at the beginning of this Section, the above results can be used as definitions of the trace: Theorem 2.107 for $Tr_{V;U,W}$, Corollary 2.108 for $Tr_{V;W}$, and Corollary 2.109 for $Tr_V$. The proofs that these formulas are equivalent to the definitions from the previous Sections show that the trace can be calculated using any choice of duality data $(D, \epsilon, \eta)$, and that the output does not depend on the choice. Using this approach can also lead to simpler proofs of some properties of the trace.

EXERCISE 2.110. The result of Theorem 2.27,

$$Tr_V(H) = (Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V))(1),$$

can be given a different (and simpler) proof using a choice of duality $(D, \epsilon, \eta)$.

HINT. In the following diagram,

$$\begin{tikzcd}
V \otimes D \otimes \mathbb{K} \arrow[r, "{[s \otimes Id_{\mathbb{K}}]}"] & D \otimes V \otimes \mathbb{K} \arrow[rr, "{[Id_D \otimes (l_V^{-1} \circ H \circ l_V)]}"] \arrow[ddr, "{[Id_D \otimes l_V]}"] & & D \otimes V \otimes \mathbb{K} \arrow[dd, "{[Id_D \otimes l_V]}"'] \arrow[dr, "{[\epsilon \otimes Id_{\mathbb{K}}]}"] & \\
\mathbb{K} \otimes \mathbb{K} \arrow[u, "{[\eta \otimes Id_{\mathbb{K}}]}"'] \arrow[d, "l_1"'] & & & & \mathbb{K} \otimes \mathbb{K} \arrow[d, "l_1"] \\
\mathbb{K} \arrow[r, "\eta"] & V \otimes D \arrow[r, "s"] & D \otimes V \arrow[r, "{[Id_D \otimes H]}"] & D \otimes V \arrow[r, "\epsilon"] & \mathbb{K}
\end{tikzcd}$$

the lower row is copied from Corollary 2.109, corresponding to $Tr_V(H) \cdot Id_{\mathbb{K}}$ as in (2.6) from Theorem 2.27. The clockwise path from $\mathbb{K}$ to $\mathbb{K}$ is $Tr_{V;\mathbb{K},\mathbb{K}}(l_V^{-1} \circ H \circ l_V)$, by Theorem 2.107, in the case $U = W = \mathbb{K}$. The middle block in the diagram is commutative by Lemma 1.36, and the left and right blocks by versions of Lemma 1.38, and the claim follows from the commutativity of the diagram. $\blacksquare$

REMARK 2.111. Theorem 2.27, Exercise 2.56, and Exercise 2.110 give some details omitted from [$\mathbf{C}_2$] Example 2.13.

BIG EXERCISE 2.112. Theorem 2.30, and some of the other results of Section 2.2.2, can be proved starting with Theorem 2.107 as a definition of the generalized trace. $\blacksquare$

EXERCISE 2.113. For any spaces $D$, $U$, $V$, the arrows in this diagram are invertible.

$$\begin{tikzcd}
\mathrm{Hom}(U \otimes D, \mathrm{Hom}(V, U)) \arrow[r, "q"] & \mathrm{Hom}(U \otimes D \otimes V, U) \\
\mathrm{Hom}(D \otimes U, \mathrm{Hom}(V, U)) \arrow[u, "{\mathrm{Hom}(s, Id_{\mathrm{Hom}(V,U)})}"] & \mathrm{Hom}(U \otimes D \otimes V, U \otimes \mathbb{K}) \arrow[u, "{\mathrm{Hom}(Id_{U \otimes D \otimes V}, l_U)}"']
\end{tikzcd}$$

If $V$ is dualizable with data $(D, \epsilon, \eta)$, then on the right, there is a distinguished element $[Id_U \otimes \epsilon] \in \mathrm{Hom}(U \otimes D \otimes V, U \otimes \mathbb{K})$. In the case $(D, \epsilon, \eta) = (V^*, Ev_V, \eta_V)$ from Example 2.98, there is a distinguished element $k_{VU} \in \mathrm{Hom}(V^* \otimes U, \mathrm{Hom}(V, U))$ on

the left, and the two elements are related by the composition of arrows in the path.
∎

Big Exercise 2.114. For a dualizable space $V$, define (as in [**PS**] §2) the mate of a map $A : V \otimes U \to V \otimes W$ with respect to duality data $(D, \epsilon, \eta)$ as the map $A^m : D \otimes U \to D \otimes W$ given by the composition in the following diagram.

$$\begin{tikzcd}
D \otimes V \otimes U \otimes D \arrow[r, "{[Id_D \otimes [A \otimes Id_D]]}"] & D \otimes V \otimes W \otimes D \arrow[d, "{[\epsilon \otimes Id_{W \otimes D}]}"] \\
D \otimes U \otimes V \otimes D \arrow[u, "{[Id_D \otimes [s \otimes Id_D]]}"] & \mathbb{K} \otimes W \otimes D \arrow[d, "l"] \\
D \otimes U \otimes \mathbb{K} \arrow[u, "{[Id_{D \otimes U} \otimes \eta]}"] & W \otimes D \arrow[d, "s"] \\
D \otimes U \arrow[u, "l^{-1}"] \arrow[r, "A^m"] & D \otimes W
\end{tikzcd}$$

Then (as in [**PS**] §7),

$$Tr_{D;U,W}(A^m) = Tr_{V;U,W}(A).$$

In particular, LHS does not depend on the choice of $(D, \epsilon, \eta)$.

Hint. By Lemma 2.103, $D$ is dualizable, so by Theorem 2.107, LHS exists. The formula from Theorem 2.107 does not depend on the choice of duality data for $D$; it is convenient to choose to use $(V, \epsilon \circ s, s \circ \eta)$ from Lemma 2.103. In the following diagram, the lower middle block uses maps from the above definition of $A^m$, including

$$a = [Id_V \otimes [Id_D \otimes [A \otimes Id_D]]].$$

$$\begin{tikzcd}
V \otimes D \otimes U \arrow[r, "{[s \otimes Id_U]}"] & D \otimes V \otimes U \arrow[r, "{[Id_D \otimes A]}"] \arrow[d] & D \otimes V \otimes W \arrow[r, "{[\epsilon \otimes Id_W]}"] & \mathbb{K} \otimes W \arrow[ddd, "l"] \\
\mathbb{K} \otimes U \arrow[u, "{[\eta \otimes Id_U]}"] & D \otimes \mathbb{K} \otimes V \otimes U \arrow[d] & \mathbb{K} \otimes D \otimes V \otimes W \arrow[u] & \\
 & D \otimes V \otimes D \otimes V \otimes U \arrow[d, "s_0"] & D \otimes V \otimes D \otimes V \otimes W \arrow[u] & \\
U \arrow[uu] \arrow[dd] & V \otimes D \otimes V \otimes U \otimes D \arrow[r, "a"] & V \otimes D \otimes V \otimes W \otimes D \arrow[u] \arrow[d] & W \\
 & V \otimes D \otimes U \otimes V \otimes D \arrow[u] & V \otimes \mathbb{K} \otimes W \otimes D \arrow[d] & \\
\mathbb{K} \otimes U \arrow[d, "{[(s \circ \eta) \otimes Id_U]}"] & V \otimes D \otimes U \otimes \mathbb{K} \arrow[u] & V \otimes W \otimes D \arrow[d] & \\
D \otimes V \otimes U \arrow[r, "{[s^{-1} \otimes Id_U]}"] & V \otimes D \otimes U \arrow[r, "{[Id_V \otimes A^m]}"] \arrow[u] & V \otimes D \otimes W \arrow[r, "{[(\epsilon \otimes s) \otimes Id_W]}"] & \mathbb{K} \otimes W \arrow[uuu, "l"]
\end{tikzcd}$$

By Theorem 2.107, the path from $U$ to $W$ along the top row is $Tr_{V;U,W}(A)$, and from $U$ to $W$ along the lowest row is $Tr_{D;U,W}(A^m)$. The claimed equality follows from the commutativity of the diagram. The maps in the top middle block are as in the next diagram, with notation $D = D_1 = D_2$ and $V = V_1 = V_2 = V_3$ to indicate various switching maps.

$$\begin{tikzcd}
D_2 \otimes V_2 \otimes U \arrow[r, "{[Id_D \otimes A]}"] & D_1 \otimes V_3 \otimes W \\
D_2 \otimes \mathbb{K} \otimes V_2 \otimes U \arrow[u, "{[l \otimes Id_{V \otimes U}]}"] \arrow[ur, "{[l \otimes A]}"] \arrow[d, "{[[Id_D \otimes \eta] \otimes Id_{V \otimes U}]}"] & \mathbb{K} \otimes D_1 \otimes V_3 \otimes W \arrow[u, "{[l \otimes Id_W]}"'] \\
D_2 \otimes V_1 \otimes D_1 \otimes V_2 \otimes U \arrow[r, "{[Id_{D \otimes V \otimes D} \otimes A]}"'] \arrow[ur, "{[[\epsilon \otimes Id_D] \otimes A]}"] \arrow[d, "s_0"] & D_2 \otimes V_1 \otimes D_1 \otimes V_3 \otimes W \arrow[u, "{[\epsilon \otimes Id_{D \otimes V \otimes W}]}"'] \arrow[d, "s"] \\
V_1 \otimes D_1 \otimes V_2 \otimes U \otimes D_2 \arrow[r, "a"] & V_1 \otimes D_1 \otimes V_3 \otimes W \otimes D_2
\end{tikzcd}$$

The above diagram is commutative; the middle block involving both $\eta$ and $\epsilon$ uses one of the properties from Definition 2.97. Using these maps as specified in the above right column, it is also easy to check that the right block from the big diagram is commutative.

To check that the left block from the big diagram is commutative, note that the path going up from $U$ to $D \otimes V \otimes U$ takes input $u$ to output $((s \circ \eta)(1)) \otimes u$, and the downward path also takes $u$ to $((s \circ \eta)(1)) \otimes u$. So, it is enough to check that $((s \circ \eta)(1)) \otimes u$ has the same output along the two paths leading to $V \otimes D \otimes V \otimes U \otimes D$. This is shown in the next diagram, where the numbering $V = V_1 = V_2$ and $D = D_1 = D_2$ is chosen to match the left column of the previous diagram.

$$\begin{tikzcd}
 & & V_1 \otimes D_1 \otimes V_2 \otimes U \otimes D_2 \arrow[dl, "{[Id_V \otimes [Id_D \otimes [s \otimes Id_D]]]}", "{[Id_{V \otimes D} \otimes s]}"'] \\
D_2 \otimes V_1 \otimes D_1 \otimes V_2 \otimes U \arrow[urr, "s_0"] & V_1 \otimes D_1 \otimes D_2 \otimes V_2 \otimes U \arrow[l, "{[s \otimes Id_{V \otimes U}]}"] & V_1 \otimes D_1 \otimes U \otimes V_2 \otimes D_2 \arrow[u] \arrow[dl, "{[Id_{V \otimes D \otimes U} \otimes s]}"] \\
 & V_1 \otimes D_1 \otimes U \otimes D_2 \otimes V_2 \arrow[u, "{[Id_{V \otimes D} \otimes s_1^{-1}]}"] & \\
D \otimes \mathbb{K} \otimes V \otimes U \arrow[uu, "{[[Id_D \otimes \eta] \otimes Id_{V \otimes U}]}"] \arrow[dd, "{[l \otimes Id_{V \otimes U}]}"] & \mathbb{K} \otimes D \otimes V \otimes U \arrow[l, "{[s \otimes Id_{V \otimes U}]}"] \arrow[u, "{[\eta \otimes s_1]}"] & V \otimes D \otimes U \otimes \mathbb{K} \arrow[uu, "{[Id_V \otimes [Id_{D \otimes V} \otimes \eta]]}"'] \arrow[dl, "s"] \arrow[dd, "{[Id_V \otimes l]}"'] \\
 & \mathbb{K} \otimes V \otimes D \otimes U \arrow[u, "{[[Id_{\mathbb{K}} \otimes s] \otimes Id_U]}"] \arrow[dr, "l"] & \\
D \otimes V \otimes U & & V \otimes D \otimes U \arrow[ll, "{[s \otimes Id_U]}"]
\end{tikzcd}$$

All the blocks in the last diagram are commutative, except for the center right block. However, starting with $(\eta(1)) \otimes u \in V \otimes D \otimes U$ in the lower right corner, all paths leading upward to $V \otimes D \otimes V \otimes U \otimes D$ give the same output. The upward right column, and the clockwise path around the left side, correspond, respectively, to the lower half and the upper half of the left block in the big diagram. ∎

CHAPTER 3

# Bilinear Forms

As a special case of Definition 1.23, consider a bilinear function $V \times V \rightsquigarrow \mathbb{K}$, which takes as input an ordered pair of elements of a vector space $V$ and gives as output an element of $\mathbb{K}$ (a scalar), so that it is $\mathbb{K}$-linear in either input when the other is fixed. Definition 3.1 encodes this idea in a convenient way using linear maps, as described in Example 1.55. This Chapter examines the trace of a bilinear form on a finite-dimensional space $V$, with respect to a metric $g$ on $V$.

## 3.1. Symmetric bilinear forms

Definition 3.1. A bilinear form $h$ on a vector space $V$ is a $\mathbb{K}$-linear map $h : V \to V^*$.

For vectors $u$, $v \in V$, a bilinear form $h$ acts on $u$ to give an element of the dual, $h(u) \in V^*$, which then acts on $v$ to give $(h(u))(v) \in \mathbb{K}$.

Definition 3.2. The transpose map, $T_V \in \mathrm{End}(\mathrm{Hom}(V, V^*))$, is defined by $T_V = \mathrm{Hom}(d_V, Id_{V^*}) \circ t_{VV^*} : h \mapsto h^* \circ d_V$.

Lemma 3.3. *$T_V$ is an involution.*

Proof. The effect of the map $T_V$ is to switch the two inputs:

$$((T_V(h))(u))(v) = ((h^* \circ d_V)(u))(v) = (d_V(u))(h(v)) = (h(v))(u), \tag{3.1}$$

so the claim is obvious from Equation (3.1). This is also a corollary of Lemma 4.4 from Section 4.1, which considers some other approaches to bilinear forms and the definition of transpose, using different spaces and canonical maps. ∎

Definition 3.4. A bilinear form $h$ is symmetric means: $h = T_V(h)$. $h$ is antisymmetric means: $h = -T_V(h)$.

If $h$ is symmetric, then $(h(u))(v) = (h(v))(u)$, and if $h$ is antisymmetric, then $(h(u))(v) = -(h(v))(u)$.

Notation 3.5. If $\frac{1}{2} \in \mathbb{K}$, then the involution $T_V$ on $\mathrm{Hom}(V, V^*)$ produces, as in Lemma 1.119, a direct sum of the subspaces of symmetric and antisymmetric forms on $V$, denoted $\mathrm{Hom}(V, V^*) = Sym(V) \oplus Alt(V)$.

In particular, any bilinear form $h$ is canonically the sum of a symmetric form and an antisymmetric form,

$$h = \frac{1}{2} \cdot (h + T_V(h)) + \frac{1}{2} \cdot (h - T_V(h)). \tag{3.2}$$

Lemma 3.6. *If* $\frac{1}{2} \in \mathbb{K}$*, then the canonical map* $k_{UU^*} : U^* \otimes U^* \to \mathrm{Hom}(U, U^*)$ *respects the direct sums:*

$$k_{UU^*} : S^2(U^*) \oplus \Lambda^2(U^*) \to Sym(U) \oplus Alt(U).$$

Proof. The direct sums are from Example 1.124, produced by the involution $s$ on $U^* \otimes U^*$, and Notation 3.5, produced by the involution $T_U$. It is easily checked that the following diagram is commutative.

$$\begin{tikzcd} U^* \otimes U^* \arrow[r, "s"] \arrow[d, "k_{UU^*}"] & U^* \otimes U^* \arrow[d, "k_{UU^*}"] \\ \mathrm{Hom}(U, U^*) \arrow[r, "T_U"] & \mathrm{Hom}(U, U^*) \end{tikzcd}$$

So $k_{UU^*}$ respects the direct sums by Lemma 1.126. ∎

Definition 3.7. For a map $H : U \to V$, and a bilinear form $h : V \to V^*$, the map $H^* \circ h \circ H$ is a bilinear form on $U$, called the pullback of $h$.

Lemma 3.8. *For any vector spaces* $U$*,* $V$*, and any map* $H : U \to V$*, the following diagram is commutative.*

$$\begin{tikzcd} \mathrm{Hom}(V, V^*) \arrow[r, "T_V"] \arrow[d, "\mathrm{Hom}(H,H^*)"'] & \mathrm{Hom}(V, V^*) \arrow[d, "\mathrm{Hom}(H,H^*)"] \\ \mathrm{Hom}(U, U^*) \arrow[r, "T_U"] & \mathrm{Hom}(U, U^*) \end{tikzcd}$$

*If, further,* $\frac{1}{2} \in \mathbb{K}$*, then the map* $h \mapsto H^* \circ h \circ H$ *respects the direct sums:*

$$\mathrm{Hom}(H, H^*) : Sym(V) \oplus Alt(V) \to Sym(U) \oplus Alt(U).$$

Proof. Using Lemma 1.6 and Lemma 1.14, the transpose of the pullback of a bilinear form $h : V \to V^*$ is the pullback of the transpose:

$$\begin{aligned} T_U(H^* \circ h \circ H) &= (H^* \circ h \circ H)^* \circ d_U = H^* \circ h^* \circ H^{**} \circ d_U \\ &= H^* \circ h^* \circ d_V \circ H = H^* \circ (T_V(h)) \circ H. \end{aligned}$$

The claim about the direct sums from Notation 3.5 follows from Lemma 1.126. ∎

So, if $h \in Sym(V)$, then its pullback satisfies $H^* \circ h \circ H \in Sym(U)$. The pullback of an antisymmetric form is similarly antisymmetric

Notation 3.9. If an arbitrary vector space $V$ is a direct sum of $V_1$ and $V_2$, as in Definition 1.77, and $h_1 : V_1 \to V_1^*$, $h_2 : V_2 \to V_2^*$, then

$$P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2 : V \to V^* \tag{3.3}$$

will be called the direct sum $h_1 \oplus h_2$ of the bilinear forms $h_1$ and $h_2$.

The expression (3.3) is the same construction as in Lemma 1.86, applied to the direct sum $V^* = V_1^* \oplus V_2^*$ from Example 1.84.

THEOREM 3.10. $T_V(h_1 \oplus h_2) = (T_{V_1}(h_1)) \oplus (T_{V_2}(h_2))$.

PROOF.

$$\begin{aligned} LHS &= (P_1^* \circ h_1 \circ P_1)^* \circ d_V + (P_2^* \circ h_2 \circ P_2)^* \circ d_V \\ &= P_1^* \circ h_1^* \circ P_1^{**} \circ d_V + P_2^* \circ h_2^* \circ P_2^{**} \circ d_V \\ &= P_1^* \circ h_1^* \circ d_{V_1} \circ P_1 + P_2^* \circ h_2^* \circ d_{V_2} \circ P_2 \\ &= P_1^* \circ (T_{V_1}(h_1)) \circ P_1 + P_2^* \circ (T_{V_2}(h_2)) \circ P_2 = RHS, \end{aligned}$$

using Lemma 1.6 and Lemma 1.14. ∎

It follows that the direct sum of symmetric forms is symmetric, and that the direct sum of antisymmetric forms is antisymmetric.

The following Lemma will be convenient in some of the theorems about the tensor product of symmetric forms.

LEMMA 3.11. ([B] §II.4.4) *For $A : U_1 \to U_2$ and $B : V_1 \to V_2$, the following diagram is commutative.*

$$\begin{tikzcd} U_2^* \otimes V_2^* \arrow[r, "{[A^* \otimes B^*]}"] \arrow[d, "j"'] & U_1^* \otimes V_1^* \arrow[d, "j"] \\ \mathrm{Hom}(U_2 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) \arrow[r, "{\mathrm{Hom}([A \otimes B], Id_{\mathbb{K} \otimes \mathbb{K}})}"] \arrow[d, "{\mathrm{Hom}(Id_{U_2 \otimes V_2}, l)}"'] & \mathrm{Hom}(U_1 \otimes V_1, \mathbb{K} \otimes \mathbb{K}) \arrow[d, "{\mathrm{Hom}(Id_{U_1 \otimes V_1}, l)}"] \\ (U_2 \otimes V_2)^* \arrow[r, "{[A \otimes B]^*}"] & (U_1 \otimes V_1)^* \end{tikzcd}$$

PROOF. The scalar multiplication $\mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ is denoted $l$. The top square is commutative by Lemma 1.37, and the lower one by Lemma 1.6. ∎

THEOREM 3.12. *If $h_1 : U \to U^*$ and $h_2 : V \to V^*$, then*

$$\mathrm{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2] : (U \otimes V) \to (U \otimes V)^*$$

*is a bilinear form such that*

$$T_{U \otimes V}(\mathrm{Hom}(Id_{U \otimes V}, l) \circ j \circ [h_1 \otimes h_2])$$

*is equal to*

$$\mathrm{Hom}(Id_{U \otimes V}, l) \circ j \circ [(T_U(h_1)) \otimes (T_V(h_2))].$$

PROOF. First, for any $U$, $V$, the following diagram is commutative:

$$\begin{tikzcd} U \otimes V \arrow[r, "d_{U \otimes V}"] \arrow[d, "{[d_U \otimes d_V]}"] & (U \otimes V)^{**} \arrow[r, "{\mathrm{Hom}(Id_{U \otimes V}, l)^*}"] & \mathrm{Hom}(U \otimes V, \mathbb{K} \otimes \mathbb{K})^* \arrow[d, "j^*"] \\ U^{**} \otimes V^{**} \arrow[r, "j"] & \mathrm{Hom}(U^* \otimes V^*, \mathbb{K} \otimes \mathbb{K}) \arrow[r, "{\mathrm{Hom}(Id_{U^* \otimes V^*}, l)}"] & (U^* \otimes V^*)^* \end{tikzcd}$$

$$\begin{aligned}
u\otimes v &\mapsto (j^*\circ\mathrm{Hom}(Id_{U\otimes V},l)^*\circ d_{U\otimes V})(u\otimes v):\\
\phi\otimes\xi &\mapsto (d_{U\otimes V}(u\otimes v))(\mathrm{Hom}(Id_{U\otimes V},l)([\phi\otimes\xi]))\\
&= l([\phi\otimes\xi](u\otimes v))\\
&= \phi(u)\cdot\xi(v),\\
u\otimes v &\mapsto (\mathrm{Hom}(Id_{U^*\otimes V},l)\circ j\circ[d_U\otimes d_V])(u\otimes v)\\
&= l\circ[(d_U(u))\otimes(d_V(v))]:\\
\phi\otimes\xi &\mapsto \phi(u)\cdot\xi(v).
\end{aligned}$$

Note that the bottom row of the diagram is one of the columns of the diagram in Lemma 3.11 in the case $U_2=U^*$, $V_2=V^*$. The statement of the Theorem follows, using the above commutativity and Lemma 3.11.

$$\begin{aligned}
T_{U\otimes V}(h_1\otimes h_2) &= (\mathrm{Hom}(Id_{U\otimes V},l)\circ j\circ[h_1\otimes h_2])^*\circ d_{U\otimes V}\\
&= [h_1\otimes h_2]^*\circ j^*\circ\mathrm{Hom}(Id_{U\otimes V},l)^*\circ d_{U\otimes V}\\
&= [h_1\otimes h_2]^*\circ\mathrm{Hom}(Id_{U^*\otimes V^*},l)\circ j\circ[d_U\otimes d_V]\\
&= \mathrm{Hom}(Id_{U\otimes V},l)\circ j\circ[h_1^*\otimes h_2^*]\circ[d_U\otimes d_V]\\
&= \mathrm{Hom}(Id_{U\otimes V},l)\circ j\circ[(T_U(h_1))\otimes(T_V(h_2))].
\end{aligned}$$

∎

NOTATION 3.13. The bilinear form $\mathrm{Hom}(Id_{U\otimes V},l)\circ j\circ[h_1\otimes h_2]$ from the above Theorem will be called the tensor product of bilinear forms, and denoted $\{h_1\otimes h_2\}$, in analogy with the brackets from Notation 1.35. As defined, the tensor product bilinear form acts as

$$(\{h_1\otimes h_2\}(u_1\otimes v_1))(u_2\otimes v_2)=(h_1(u_1))(u_2)\cdot(h_2(v_1))(v_2).$$

When $h_1$ and $h_2$ are symmetric forms, it is clear from this formula that $\{h_1\otimes h_2\}$ is also symmetric, but the above proof, using Definition 3.4, makes explicit the roles of the symmetry and the scalar multiplication. It also follows that the tensor product of antisymmetric forms is symmetric.

There is a distributive law for the direct sum and tensor product of bilinear forms. Let $V$ be a direct sum of $V_1$ and $V_2$, and recall, from Example 1.81, that $V\otimes U$ is a direct sum of $V_1\otimes U$ and $V_2\otimes U$, with projection maps $[P_i\otimes Id_U]$.

THEOREM 3.14. *For bilinear forms $h_1$, $h_2$, $g$ on arbitrary vector spaces $V_1$, $V_2$, $U$, the following bilinear forms on $V\otimes U$ are equal:*

$$\{(h_1\oplus h_2)\otimes g\}=\{h_1\otimes g\}\oplus\{h_2\otimes g\}.$$

PROOF. Unraveling the definitions, and applying Lemma 3.11 and Lemma 1.36 gives the claimed equality:

$$\begin{aligned}
RHS &= [P_1\otimes Id_U]^*\circ\mathrm{Hom}(Id_{V_1\otimes U},l)\circ j\circ[h_1\otimes g]\circ[P_1\otimes Id_U]\\
&\quad +[P_2\otimes Id_U]^*\circ\mathrm{Hom}(Id_{V_2\otimes U},l)\circ j\circ[h_2\otimes g]\circ[P_2\otimes Id_U]\\
&= \mathrm{Hom}(Id_{V\otimes U},l)\circ j\circ[P_1^*\otimes Id_U]\circ[h_1\otimes g]\circ[P_1\otimes Id_U]\\
&\quad +\mathrm{Hom}(Id_{V\otimes U},l)\circ j\circ[P_2^*\otimes Id_U]\circ[h_2\otimes g]\circ[P_2\otimes Id_U]\\
&= \mathrm{Hom}(Id_{V\otimes U},l)\circ j\circ[(P_1^*\circ h_1\circ P_1+P_2^*\circ h_2\circ P_2)\otimes g]\\
&= LHS.
\end{aligned}$$

∎

## 3.2. Metrics

THEOREM 3.15. *Given a bilinear form $g : V \to V^*$, if $g$ is symmetric or antisymmetric then the following are equivalent.*

1. *$V$ is finite-dimensional and there exists $P : V^* \to V$ such that $P \circ g = Id_V$.*
2. *$V$ is finite-dimensional and there exists $Q : V^* \to V$ such that $g \circ Q = Id_{V^*}$.*
3. *$g$ is invertible.*

PROOF. Let $g$ be symmetric; the antisymmetric case is similar.

If $g$ is invertible, then from $g^* \circ d_V = g$ and Lemma 1.12,

$$(g^{-1})^* \circ g = (g^*)^{-1} \circ g = d_V$$

is invertible, so $V$ is finite-dimensional by Claim 1.16, which implies (1) and (2).

Assuming (1) and using Claim 1.16,

$$Id_{V^*} = Id_V^* = (P \circ g)^* = g^* \circ P^* = g \circ d_V^{-1} \circ P^*,$$

so $g$ has a left inverse and a right inverse, and (3) follows, as in Exercise 0.54. Similarly, assuming (2),

$$Id_{V^{**}} = Id_{V^*}^* = (g \circ Q)^* = Q^* \circ g^* = Q^* \circ g \circ d_V^{-1},$$

so $d_V = Q^* \circ g \implies Id_V = d_V^{-1} \circ Q^* \circ g$, and $g$ has a right inverse and a left inverse. ∎

The invertibility condition in Theorem 3.15 implies a non-degeneracy property: for each non-zero $v \in V$, there exists a vector $u \in V$ so that $(g(v))(u) \neq 0$.

DEFINITION 3.16. A metric $g$ on $V$ is a symmetric, invertible map $g : V \to V^*$.

By Theorem 3.15, a metric exists only on finite-dimensional vector spaces.

THEOREM 3.17. *Given a metric $g$ on $V$, the bilinear form $d_V \circ g^{-1} : V^* \to V^{**}$ is a metric on $V^*$.*

PROOF. To show $d_V \circ g^{-1}$ is symmetric, use the definition of $T_{V^*}$ and Lemma 1.17:

$$T_{V^*}(d_V \circ g^{-1}) = (d_V \circ g^{-1})^* \circ d_{V^*} = (g^{-1})^* \circ d_V^* \circ d_{V^*} = (g^*)^{-1} = d_V \circ g^{-1}.$$

The last step uses the symmetry of $g$. This map is invertible by Theorem 3.15 and Claim 1.16. ∎

The bilinear form $d_V \circ g^{-1}$ could be called the metric induced by $g$ on $V^*$, or the dual metric. It acts on elements $\phi$, $\xi \in V^*$ as

$$((d_V \circ g^{-1})(\phi))(\xi) = \xi(g^{-1}(\phi)).$$

This construction can be iterated to define metrics on $V^{**}$, etc.

COROLLARY 3.18. *If $g_1$ is a metric on $V_1$ and $g_2$ is a metric on $V_2$, then $g_1 \oplus g_2$ is a metric on $V = V_1 \oplus V_2$.*

PROOF. The direct sum $g_1 \oplus g_2$ as in Notation 3.9 is symmetric by Theorem 3.10, and is invertible by Lemma 1.86. Specifically, the inclusion maps $Q_1$, $Q_2$ are used to construct an inverse to the expression (3.3):

$$\begin{aligned}(Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) \circ (P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2) &= Id_V, \\ (P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2) \circ (Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) &= Id_{V^*}.\end{aligned} \tag{3.4}$$

∎

COROLLARY 3.19. *If $g_1$ and $g_2$ are metrics on $U$ and $V$, then $\{g_1 \otimes g_2\}$ is a metric on $U \otimes V$.*

PROOF. The bilinear form

$$\{g_1 \otimes g_2\} = \mathrm{Hom}(Id_{U\otimes V}, l) \circ j \circ [g_1 \otimes g_2]$$

as in Notation 3.13 is symmetric by Theorem 3.12, $j$ is invertible by the finite-dimensionality (Claim 1.34), and the inverse of $\mathrm{Hom}(Id_{U\otimes V}, l) \circ j \circ [g_1 \otimes g_2] : U \otimes V \to (U \otimes V)^*$ is

$$[g_1^{-1} \otimes g_2^{-1}] \circ j^{-1} \circ \mathrm{Hom}(Id_{U\otimes V}, l^{-1}).$$

∎

EXERCISE 3.20. If $h_1$, $h_2$, and $g$ are metrics on $V_1$, $V_2$, and $U$, and $h = h_1 \oplus h_2$ is the direct sum bilinear form on $V = V_1 \oplus V_2$, then the induced tensor product metric $\{h \otimes g\}$ on $V \otimes U$ coincides with the induced direct sum metric on $(V_1 \otimes U) \oplus (V_2 \otimes U)$, as in Theorem 3.14. ∎

## 3.3. Isometries

EXAMPLE 3.21. If $h$ is a metric on $V$, and $H : U \to V$ is invertible, then the pullback $H^* \circ h \circ H$ (as in Definition 3.7) is a metric on $U$, since it is symmetric by Lemma 3.8, and has inverse $H^{-1} \circ h^{-1} \circ (H^*)^{-1}$.

REMARK 3.22. The pullback of a metric $h$ by an arbitrary linear map $H$ need not be a metric, for example, the case where $H$ is the inclusion of a lightlike line in Minkowski space. See also Definition 3.106.

DEFINITION 3.23. A $\mathbb{K}$-linear map $H : U \to V$ is an isometry, with respect to metrics $g$ on $U$ and $h$ on $V$, means: $H$ is invertible, and $g = H^* \circ h \circ H$, so the diagram is commutative.

$$\begin{tikzcd} U \arrow[r, "H"] \arrow[d, "g"] & V \arrow[d, "h"] \\ U^* & V^* \arrow[l, "H^*"] \end{tikzcd}$$

This means that the metric $g$ is equal to the pullback of $h$ by $H$, and that for elements of $U$,

$$(g(u_1))(u_2) = (h(H(u_1)))(H(u_2)).$$

It follows immediately from the definition that the composite of isometries is an isometry, that the inverse of an isometry is an isometry, and that $Id_V$ and $-Id_V$ are isometries.

REMARK 3.24. The equation $g = H^* \circ h \circ H$ does not itself require that $H^{-1}$ exists, and one could consider non-surjective "isometric embeddings," but invertibility will be assumed as part of Definition 3.23, just for convenience.

EXERCISE 3.25. If $h : V \to V^*$ and $H : U \to V$, and $H^* \circ h \circ H : U \to U^*$ is invertible, then $H$ is a linear monomorphism. ∎

THEOREM 3.26. *Any metric $g : U \to U^*$ is an isometry with respect to itself, $g$, and the dual metric, $d_U \circ g^{-1}$.*

PROOF. The pullback by $g$ of the dual metric is

$$g^* \circ d_U \circ g^{-1} \circ g = g,$$

by the symmetry of $g$. ∎

THEOREM 3.27. *Given a metric $g$ on $U$, $d_U : U \to U^{**}$ is an isometry with respect to $g$ and the dual of the dual metric $d_{U^*} \circ (d_U \circ g^{-1})^{-1} = d_{U^*} \circ g \circ d_U^{-1}$ on $U^{**}$.*

PROOF. By the identity $d_U^* \circ d_{U^*} = Id_{U^*}$ from Lemma 1.17,

$$g = d_U^* \circ d_{U^*} \circ g \circ d_U^{-1} \circ d_U.$$

∎

THEOREM 3.28. *Given metrics $g_1$, $g_2$, $h_1$, and $h_2$ on $U_1$, $U_2$, $V_1$, and $V_2$, if $A : U_1 \to U_2$ and $B : V_1 \to V_2$ are isometries, then $[A \otimes B] : U_1 \otimes V_1 \to U_2 \otimes V_2$ is an isometry with respect to the induced metrics.*

PROOF. The statement of the Theorem is that

$$\{g_1 \otimes h_1\} = [A \otimes B]^* \circ \{g_2 \otimes h_2\} \circ [A \otimes B].$$

The RHS can be expanded, and then Lemma 3.11 applies:

$$\begin{aligned} RHS &= [A \otimes B]^* \circ \mathrm{Hom}(Id_{U_2 \otimes V_2}, l) \circ j \circ [g_2 \otimes h_2] \circ [A \otimes B] \\ &= \mathrm{Hom}(Id_{U_1 \otimes V_1}, l) \circ j \circ [A^* \otimes B^*] \circ [g_2 \otimes h_2] \circ [A \otimes B] \\ &= \mathrm{Hom}(Id_{U_1 \otimes V_1}, l) \circ j \circ [g_1 \otimes h_1] = LHS. \end{aligned}$$

The last step uses Lemma 1.36 and $g_1 = A^* \circ g_2 \circ A$, $h_1 = B^* \circ h_2 \circ B$. ∎

LEMMA 3.29. *For vector spaces $U_1$, $U_2$, $V_1$, $V_2$, and maps $F : V_1 \to V_2^*$, $E : U_1^* \to U_2$, if $V_1$ is finite-dimensional then the following diagram is commutative.*

$$\begin{tikzcd} U_1^* \otimes V_1 \arrow[r, "p_{U_1 V_1}"] \arrow[d, "{[(d_{U_2} \circ E) \otimes F]}"] & V_1^{**} \otimes U_1^* \arrow[d, "{[(F \circ d_{V_1}^{-1}) \otimes E]}"] \\ U_2^{**} \otimes V_2^* & V_2^* \otimes U_2 \arrow[l, "p_{V_2 U_2}"] \end{tikzcd}$$

PROOF. The $p$ maps are as in Notation 1.72.

$$\begin{aligned} \phi \otimes v &\mapsto (p_{V_2 U_2} \circ [(F \circ d_{V_1}^{-1}) \otimes E] \circ p_{U_1 V_1})(\phi \otimes v) \\ &= (p_{V_2 U_2} \circ [(F \circ d_{V_1}^{-1}) \otimes E])((d_{V_1}(v)) \otimes \phi) \\ &= p_{V_2 U_2}((F(v)) \otimes (E(\phi))) \\ &= (d_{U_2}(E(\phi))) \otimes (F(v)) = [(d_{U_2} \circ E) \otimes F](\phi \otimes v). \end{aligned}$$

∎

THEOREM 3.30. *Given metrics $g$ and $h$ on $U$ and $V$, the canonical map $f_{UV}$ : $U^* \otimes V \to (V^* \otimes U)^*$ is an isometry with respect to the induced metrics.*

PROOF. The diagram is commutative, where the compositions in the left and right columns define the induced metrics.

$$\begin{tikzcd}
U^* \otimes V \arrow[r, "f_{UV}"] \arrow[dd, "{[(d_U \circ g^{-1}) \otimes h]}"'] \arrow[ddr, "p_{UV}"] & (V^* \otimes U)^* \arrow[d, shift left, "{\mathrm{Hom}(Id_{V^* \otimes U}, l^{-1})}"] \\
 & \mathrm{Hom}(V^* \otimes U, \mathbb{K} \otimes \mathbb{K}) \arrow[u, shift left, "{\mathrm{Hom}(Id_{V^* \otimes U}, l)}"] \arrow[d, shift left, "j^{-1}"] \\
U^{**} \otimes V^* \arrow[d, "j"'] & V^{**} \otimes U^* \arrow[u, shift left, "j"] \arrow[d, shift left, "{[(h \circ d_V^{-1}) \otimes g^{-1}]}"] \\
\mathrm{Hom}(U^* \otimes V, \mathbb{K} \otimes \mathbb{K}) \arrow[d, "{\mathrm{Hom}(Id_{U^* \otimes V}, l)}"'] & V^* \otimes U \arrow[u, shift left, "{[(d_V \circ h^{-1}) \otimes g]}"] \arrow[ul, "p_{VU}"] \arrow[dl, "f_{VU}"] \arrow[d, "d_{V^* \otimes U}"] \\
(U^* \otimes V)^* & (V^* \otimes U)^{**} \arrow[l, "f_{UV}^*"]
\end{tikzcd}$$

The lower triangle is commutative by Lemma 1.71. The two blocks with $f$ and $p$ maps are commutative by Lemma 1.74, and the block in the middle is commutative by Lemma 3.29. ∎

LEMMA 3.31. *Given metrics $g$ and $h$ on $U$ and $V$, let $U = U_1 \oplus U_2$, with direct sum data $Q_i$, $P_i$, and let $V = V_1 \oplus V_2$, with data $Q_i'$, $P_i'$. Suppose that for $i = 1$ or 2, the bilinear form $(Q_i')^* \circ h \circ Q_i'$ is a metric on $V_i$. If $H : U \to V$ is an isometry that respects the direct sums, then the bilinear form $Q_i^* \circ g \circ Q_i$ is a metric on $U_i$, and the induced map $P_i' \circ H \circ Q_i : U_i \to V_i$ is an isometry.*

PROOF. The induced map $P_i' \circ H \circ Q_i$ is invertible, as in Lemma 1.89. The following calculation (which uses the property that $H$ respects the direct sums) shows that the bilinear form $Q_i^* \circ g \circ Q_i$ is equal to the pullback of $(Q_i')^* \circ h \circ Q_i'$ by the map $P_i' \circ H \circ Q_i$, so it is a metric as in Example 3.21, and $P_i' \circ H \circ Q_i$ is an isometry, by Definition 3.23.

$$\begin{aligned}
& (P_i' \circ H \circ Q_i)^* \circ ((Q_i')^* \circ h \circ Q_i') \circ (P_i' \circ H \circ Q_i) \\
= \; & Q_i^* \circ H^* \circ (P_i')^* \circ (Q_i')^* \circ h \circ Q_i' \circ P_i' \circ H \circ Q_i \\
= \; & (Q_i' \circ P_i' \circ H \circ Q_i)^* \circ h \circ Q_i' \circ P_i' \circ H \circ Q_i \\
= \; & (H \circ Q_i \circ P_i \circ Q_i)^* \circ h \circ H \circ Q_i \circ P_i \circ Q_i \\
= \; & Q_i^* \circ H^* \circ h \circ H \circ Q_i \\
= \; & Q_i^* \circ g \circ Q_i.
\end{aligned}$$

∎

## 3.4. Trace with respect to a metric

DEFINITION 3.32. With respect to a metric $g$ on $V$, the trace of a bilinear form $h$ on $V$ is defined by

$$Tr_g(h) = Tr_V(g^{-1} \circ h).$$

By Lemma 2.6, this is the same as $Tr_{V^*}(h \circ g^{-1})$, and another way to write the definition is

$$Tr_g = \mathrm{Hom}(Id_V, g^{-1})^*(Tr_V) \in \mathrm{Hom}(V, V^*)^*.$$

THEOREM 3.33. *Given a metric $g$ on $V$, if $h$ is any bilinear form on $V$, then $Tr_g(T_V(h)) = Tr_g(h)$.*

PROOF.

$$\begin{aligned} Tr_g(h^* \circ d_V) &= Tr_{V^*}(h^* \circ d_V \circ g^{-1}) = Tr_{V^*}(h^* \circ (g^{-1})^*) \\ &= Tr_{V^*}((g^{-1} \circ h)^*) = Tr_V(g^{-1} \circ h) = Tr_g(h), \end{aligned}$$

using the symmetry of $g$ and Lemma 2.5. ∎

COROLLARY 3.34. *If $\frac{1}{2} \in \mathbb{K}$, then the trace of an antisymmetric form is 0 with respect to any metric $g$.* ∎

THEOREM 3.35. *Given a metric $g$ on $V$, if $Tr_V(Id_V) \neq 0$, then $\mathrm{Hom}(V, V^*) = \mathbb{K} \oplus \ker(Tr_g)$.*

PROOF. Since $Tr_g(g) = Tr_V(Id_V) \neq 0$, Lemmas 1.100 and 1.101 apply. For any $h : V \to V^*$ there is a canonical decomposition of $h$ into two terms: one that is a scalar multiple of $g$ and the other that has trace zero with respect to $g$:

$$h = \frac{Tr_g(h)}{Tr_V(Id_V)} \cdot g + \left( h - \frac{Tr_g(h)}{Tr_V(Id_V)} \cdot g \right).$$

∎

COROLLARY 3.36. *Given a metric $g$ on $V$, if both $\frac{1}{2} \in \mathbb{K}$ and $Tr_V(Id_V) \neq 0$, then $\mathrm{Hom}(V, V^*)$ admits a direct sum $\mathbb{K} \oplus Sym_0(V, g) \oplus Alt(V)$, where $Sym_0(V, g)$ is the kernel of the restriction of $Tr_g$ to $Sym(V)$.*

PROOF. Using Theorem 3.33, Theorem 1.125 applies. The canonical decomposition of any bilinear form $h$ into three terms, corresponding to (1.13) with $w = v = g$, is:

$$h = \frac{Tr_g(h)}{Tr_V(Id_V)} \cdot g + \left( \frac{1}{2}(h + T_V(h)) - \frac{Tr_g(h)}{Tr_V(Id_V)} \cdot g \right) + \frac{1}{2}(h - T_V(h)).$$

∎

PROPOSITION 3.37. *Given a metric $g$ on $V$, the trace is "invariant under pullback," that is, for an invertible map $H : U \to V$,*

$$Tr_{H^* \circ g \circ H}(H^* \circ h \circ H) = Tr_g(h).$$

PROOF.

$$\begin{aligned} Tr_{H^* \circ g \circ H}(H^* \circ h \circ H) &= Tr_U(H^{-1} \circ g^{-1} \circ (H^*)^{-1} \circ H^* \circ h \circ H) \\ &= Tr_U(H^{-1} \circ g^{-1} \circ h \circ H) \\ &= Tr_V(g^{-1} \circ h) = Tr_g(h), \end{aligned}$$

by Lemma 2.6. ∎

PROPOSITION 3.38. *Given metrics $g_1$, $g_2$ on $V_1$, $V_2$, if $V = V_1 \oplus V_2$, then for any bilinear forms $h_1 : V_1 \to V_1^*$, $h_2 : V_2 \to V_2^*$,*

$$Tr_{g_1 \oplus g_2}(h_1 \oplus h_2) = Tr_{g_1}(h_1) + Tr_{g_2}(h_2).$$

PROOF. Using the formula (3.4) for $(g_1 \oplus g_2)^{-1}$ from Corollary 3.18, and Lemma 2.6,

$$\begin{aligned}
LHS &= Tr_V((Q_1 \circ g_1^{-1} \circ Q_1^* + Q_2 \circ g_2^{-1} \circ Q_2^*) \circ (P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2)) \\
&= Tr_V(Q_1 \circ g_1^{-1} \circ h_1 \circ P_1 + Q_2 \circ g_2^{-1} \circ h_2 \circ P_2) \\
&= Tr_{V_1}(P_1 \circ Q_1 \circ g_1^{-1} \circ h_1) + Tr_{V_2}(P_2 \circ Q_2 \circ g_2^{-1} \circ h_2) = RHS.
\end{aligned}$$

∎

PROPOSITION 3.39. *Given metrics $g$ and $h$ on $U$ and $V$, for bilinear forms $E : U \to U^*$ and $F : V \to V^*$,*

$$Tr_{\{g \otimes h\}}(\{E \otimes F\}) = Tr_g(E) \cdot Tr_h(F).$$

PROOF. Using the formula from Corollary 3.19, there is a convenient cancellation, and then Corollary 2.36 applies:

$$\begin{aligned}
Tr_{\{g \otimes h\}}(\{E \otimes F\}) &= Tr_{U \otimes V}([g^{-1} \otimes h^{-1}] \circ j^{-1} \circ \mathrm{Hom}(Id_{U \otimes V}, l^{-1}) \\
&\quad \circ \mathrm{Hom}(Id_{U \otimes V}, l) \circ j \circ [E \otimes F]) \\
&= Tr_{U \otimes V}(j_2((g^{-1} \circ E) \otimes (h^{-1} \circ F))) \\
&= Tr_g(E) \cdot Tr_h(F).
\end{aligned}$$

∎

## 3.5. The induced metric on Hom$(U, V)$

DEFINITION 3.40. Given metrics $g$ and $h$ on $U$ and $V$, define a bilinear form $b$ on $\mathrm{Hom}(U, V)$, acting on elements $A$, $B : U \to V$ as:

$$(b(B))(A) = Tr_V(A \circ g^{-1} \circ B^* \circ h).$$

$b$ can be written as a composite:

$$b = \mathrm{Hom}(Id_{\mathrm{Hom}(U,V)}, Tr_V) \circ t^V_{VU} \circ \mathrm{Hom}(h, g^{-1}) \circ t_{UV},$$

using the generalized transpose $t^V_{VU}$ from Definition 1.7. By Lemma 2.6, $b$ can also be written as a trace with respect to $g$:

$$(b(B))(A) = Tr_U(g^{-1} \circ B^* \circ h \circ A) = Tr_g(B^* \circ h \circ A).$$

THEOREM 3.41. *Given metrics $g$ and $h$ on $U$ and $V$, the induced tensor product metric on $U^* \otimes V$ is equal to the pullback of the bilinear form $b$ by the canonical map $k_{UV}$.*

PROOF. The diagram is commutative, where the composition in the left column defines the induced metric (as in Theorem 3.30), and the composition in the right column defines the bilinear form $b$.

$$\begin{tikzcd}
U^* \otimes V \arrow[rr, "k_{UV}"] \arrow[dd, "{[(d_U \circ g^{-1}) \otimes h]}"'] \arrow[dr, "p_{UV}"] & & \mathrm{Hom}(U, V) \arrow[d, "t_{UV}"] \\
& V^{**} \otimes U^* \arrow[r, "k_{V^*U^*}"] \arrow[d, "{[(h \circ d_V^{-1}) \otimes g^{-1}]}"] & \mathrm{Hom}(V^*, U^*) \arrow[d, "{\mathrm{Hom}(h, g^{-1})}"] \\
U^{**} \otimes V^* \arrow[d, "j"'] & V^* \otimes U \arrow[l, "p_{VU}"] \arrow[r, "k_{VU}"] \arrow[ddl, "f_{VU}"] & \mathrm{Hom}(V, U) \arrow[d, "t^V_{VU}"] \arrow[ddll, "e_{VU}"'] \\
\mathrm{Hom}(U^* \otimes V, \mathbb{K} \otimes \mathbb{K}) \arrow[d, "{\mathrm{Hom}(Id_{U^* \otimes V}, l)}"'] & & \mathrm{Hom}(\mathrm{Hom}(U, V), \mathrm{End}(V)) \arrow[d, "{\mathrm{Hom}(Id_{\mathrm{Hom}(U,V)}, Tr_V)}"] \\
(U^* \otimes V)^* & & \mathrm{Hom}(U, V)^* \arrow[ll, "k^*_{UV}"]
\end{tikzcd}$$

The three squares in the upper half of the diagram are commutative, by Lemma 3.29, Lemma 1.75, and Lemma 1.62 (with $h \circ d_V^{-1} = h^*$ because $h$ is symmetric). The left triangle in the lower half is commutative by Lemma 1.74, and the middle triangle is just the definition $f_{VU} = e_{VU} \circ k_{VU}$ from Notation 1.69. Checking the lower right triangle, starting with $D \in \mathrm{Hom}(V, U)$, uses $(k_{UV}(\phi \otimes v)) \circ D = k_{VV}((D^*(\phi)) \otimes v)$, which follows from Lemma 1.62:

$$\mathrm{Hom}(D, Id_V) \circ k_{UV} = k_{VV} \circ [D^* \otimes Id_V],$$

and the definition of the trace (Definition 2.24):

$$\begin{aligned} D &\mapsto (k_{UV}^* \circ \mathrm{Hom}(Id_{\mathrm{Hom}(U,V)}, Tr_V) \circ t_{VU}^V)(D) \\ &= Tr_V \circ (t_{VU}^V(D)) \circ k_{UV} : \\ \phi \otimes v &\mapsto Tr_V((t_{VU}^V(D))(k_{UV}(\phi \otimes v))) \\ &= Tr_V((k_{UV}(\phi \otimes v)) \circ D) \\ &= ((k_{VV}^{-1})^*(Ev_V))(k_{VV}((D^*(\phi)) \otimes v)) \\ &= Ev_V((\phi \circ D) \otimes v) \\ &= \phi(D(v)) = (e_{VU}(D))(\phi \otimes v). \end{aligned}$$

∎

In particular, the pullback $k_{UV}^* \circ b \circ k_{UV} = \{(d_U \circ g^{-1}) \otimes h\}$ from Theorem 3.41 acts on $\phi, \psi \in U^*$ and $v, w \in V$ as:

$$\begin{aligned} \psi(g^{-1}(\phi)) \cdot (h(v))(w) &= (\{(d_U \circ g^{-1}) \otimes h\}(\phi \otimes v))(\psi \otimes w) \\ &= (b(k_{UV}(\phi \otimes v)))(k_{UV}(\psi \otimes w)) \\ &= Tr_U(g^{-1} \circ (k_{UV}(\phi \otimes v))^* \circ h \circ (k_{UV}(\psi \otimes w))). \end{aligned} \tag{3.5}$$

COROLLARY 3.42. *Given metrics $g$ and $h$ on $U$ and $V$, $b$ is a metric on* $\mathrm{Hom}(U, V)$.

PROOF. This follows from Example 3.21, where $k_{UV}^{-1}$ is the invertible map relating the metric on $U^* \otimes V$ to the bilinear form $b$, proving that $b$ is symmetric and invertible, and $k_{UV}$ and $k_{UV}^{-1}$ are isometries. ∎

COROLLARY 3.43. *Given metrics $g$ and $h$ on $U$ and $V$, the canonical map* $e_{UV} : \mathrm{Hom}(U, V) \to (V^* \otimes U)^*$ *is an isometry with respect to the induced metrics.*

PROOF. This follows from Theorem 3.30, since $e_{UV} = f_{UV} \circ k_{UV}^{-1}$. ∎

REMARK 3.44. Historically, the $b$ metric involving the trace has been called the "Hilbert-Schmidt" metric; we will just refer to it as the metric on $\mathrm{Hom}(U, V)$ induced by metrics on $U$ and $V$ and will usually use the $b$ notation. The relationship between the metric $b$ and the tensor product metric seems to be well-known, although possibly not in this generality. A special case of a Hermitian version of Theorem 3.41 appears in [**Bhatia**] §I.4, and a positive definite version for endomorphisms in [**G$_2$**] §III.4. Matrix versions of Theorem 3.41 appear in [**Neudecker**], [**L**], and [**HJ**] §4.2.

THEOREM 3.45. *If $A : U_2 \to U_1$ is an isometry with respect to metrics $g_2$, $g_1$, and $B : V_1 \to V_2$ is an isometry with respect to metrics $h_1$, $h_2$, then* $\mathrm{Hom}(A, B) : \mathrm{Hom}(U_1, V_1) \to \mathrm{Hom}(U_2, V_2)$ *is an isometry with respect to the induced metrics.*

PROOF. The hypotheses are $h_1 = B^* \circ h_2 \circ B$, and $g_2 = A^* \circ g_1 \circ A$. For $E$, $F \in \mathrm{Hom}(U_1, V_2)$, the pullback of the induced metric on $\mathrm{Hom}(U_2, V_2)$ is

$$\begin{aligned} (b(B \circ F \circ A))(B \circ E \circ A) &= Tr_{V_2}(B \circ E \circ A \circ g_2^{-1} \circ A^* \circ F^* \circ B^* \circ h_2) \\ &= Tr_{V_1}(E \circ g_1^{-1} \circ F^* \circ B^* \circ h_2 \circ B) \\ &= Tr_{V_1}(E \circ g_1^{-1} \circ F^* \circ h_1). \end{aligned}$$

∎

THEOREM 3.46. *With respect to the $b$ metrics induced by $g_1$, $g_2$, $h_1$, $h_2$ on $U_1$, $U_2$, $V_1$, $V_2$, the map $j : \mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2) \to \mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2)$ is an isometry.*

PROOF. For $A_1$, $B_1 : U_1 \to V_1$, $A_2$, $B_2 : U_2 \to V_2$, the statement of the Theorem is that the tensor product metric and pullback metric are equal:

$$(b(B_1))(A_1) \cdot (b(B_2))(A_2) = (b(j(B_1 \otimes B_2)))(j(A_1 \otimes A_2)).$$

Computing the RHS, using the metrics $\{g_1 \otimes g_2\}$, and $\{h_1 \otimes h_2\}$, gives:

$$\begin{aligned} RHS &= Tr_{V_1 \otimes V_2}([A_1 \otimes A_2] \circ [g_1^{-1} \otimes g_2^{-1}] \circ j^{-1} \circ \mathrm{Hom}(Id_{U_1 \otimes U_2}, l^{-1}) \\ &\quad \circ [B_1 \otimes B_2]^* \circ \mathrm{Hom}(Id_{V_1 \otimes V_2}, l) \circ j \circ [h_1 \otimes h_2]) \\ &= Tr_{V_1 \otimes V_2}([A_1 \otimes A_2] \circ [g_1^{-1} \otimes g_2^{-1}] \circ [B_1^* \otimes B_2^*] \circ [h_1 \otimes h_2]) \\ &= Tr_{V_1 \otimes V_2}([(A_1 \circ g_1^{-1} \circ B_1^* \circ h_1) \otimes (A_2 \circ g_2^{-1} \circ B_2^* \circ h_2)]) \\ &= Tr_{V_1}(A_1 \circ g_1^{-1} \circ B_1^* \circ h_1) \cdot Tr_{V_2}(A_2 \circ g_2^{-1} \circ B_2^* \circ h_2). \end{aligned}$$

The first step uses Lemma 3.11, and the second step uses Lemma 1.36, and finally Corollary 2.36 gives a product of traces equal to LHS. ∎

THEOREM 3.47. *Given metrics $g$ and $h$ on $U$ and $V$, the map $t_{UV} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(V^*, U^*)$ is an isometry with respect to the induced metrics.*

PROOF. Calculating the pullback, for $A$, $B \in \mathrm{Hom}(U, V)$ gives

$$\begin{aligned} (b(B^*))(A^*) &= Tr_{U^*}(A^* \circ (d_V \circ h^{-1})^{-1} \circ B^{**} \circ d_U \circ g^{-1}) \\ &= Tr_{U^*}(A^* \circ h \circ d_V^{-1} \circ B^{**} \circ d_U \circ g^{-1}) \\ &= Tr_{U^*}(A^* \circ h \circ B \circ g^{-1}) \\ &= Tr_V(B \circ g^{-1} \circ A^* \circ h) \\ &= (b(B))(A). \end{aligned}$$

∎

COROLLARY 3.48. *Given a metric $g$ on $V$, $T_V : \mathrm{Hom}(V, V^*) \to \mathrm{Hom}(V, V^*)$ is an isometry with respect to the induced $b$ metric.*

PROOF. By Definition 3.2, $T_V = \mathrm{Hom}(d_V, Id_{V^*}) \circ t_{VV^*}$, which is a composition of isometries, by Theorem 3.27, Theorem 3.45, and Theorem 3.47. ∎

## 3.6. Orthogonal direct sums

DEFINITION 3.49. A direct sum $U = U_1 \oplus U_2 \oplus \cdots \oplus U_\nu$, with inclusion maps $Q_i$, is <u>orthogonal</u> with respect to a metric $g$ on $U$ means: $Q_I^* \circ g \circ Q_i = 0_{\mathrm{Hom}(U_i, U_I^*)}$ for $i \neq I$.

Equivalently, the direct sum is orthogonal if and only if $g : U \to U^*$ respects the direct sums (as in Definition 1.88), where the direct sum structure on $U^*$ is as in Example 1.84.

LEMMA 3.50. *Given $U$ with a metric $g$, if $U = U_1 \oplus U_2$ and $U = U_1' \oplus U_2'$ are equivalent direct sums, and one is orthogonal with respect to $g$, then so is the other.*

PROOF. This follows from Lemma 1.98. ∎

EXAMPLE 3.51. Given metrics $g_1$, $g_2$ on $U_1$, $U_2$, if $U = U_1 \oplus U_2$, then the direct sum is orthogonal with respect to the induced metric from Corollary 3.18, $g = g_1 \oplus g_2 = P_1^* \circ g_1 \circ P_1 + P_2^* \circ g_2 \circ P_2$.

THEOREM 3.52. *Given a metric $g$ on $U$, if $Tr_U(Id_U) \neq 0$, then any direct sum $\mathrm{End}(U) = \mathbb{K} \oplus \mathrm{End}_0(U)$ from Example 2.9 is orthogonal with respect to the $b$ metric induced by $g$.*

PROOF. As noted in Lemma 1.100 and Example 2.9, any such direct sum is technically not unique, but equivalent to any other choice, so the non-uniqueness does not affect the claimed orthogonality by Lemma 3.50.

If $A \in \mathrm{End}_0(U) = \ker(Tr_U)$, then the $b$ metric applied to $A$ and any element of the line spanned by $Id_U$ is $Tr_U(A \circ g^{-1} \circ (\lambda \cdot Id_U)^* \circ g) = \lambda \cdot Tr_U(A) = 0$. ∎

THEOREM 3.53. *Given a metric $g$ on $U$, if $Tr_U(Id_U) \neq 0$, then the direct sum $\mathrm{Hom}(U, U^*) = \mathbb{K} \oplus \ker(Tr_g)$ from Theorem 3.35 is orthogonal with respect to the induced metric.*

PROOF. Such a direct sum is as in Lemmas 1.100 and 1.101.

If $E \in \ker(Tr_g)$, then the $b$ metric applied to $E$ and any scalar multiple of $g$ is $Tr_{U^*}(E \circ g^{-1} \circ (\lambda \cdot g)^* \circ d_U \circ g^{-1}) = \lambda \cdot Tr_{U^*}(E \circ g^{-1}) = 0$. ∎

LEMMA 3.54. *Given a metric $g$ on $U$, if $U = U_1 \oplus U_2$ is an orthogonal direct sum with respect to $g$, then the involution on $U$ from Example 1.122, $K = Q_1 \circ P_1 - Q_2 \circ P_2$, and similarly $-K$, are isometries with respect to $g$.*

PROOF. Using the orthogonality,

$$\begin{aligned}(\pm K)^* \circ g \circ (\pm K) &= (Q_1 \circ P_1 - Q_2 \circ P_2)^* \circ g \circ (Q_1 \circ P_1 - Q_2 \circ P_2) \\ &= (Q_1 \circ P_1 + Q_2 \circ P_2)^* \circ g \circ (Q_1 \circ P_1 + Q_2 \circ P_2) = g.\end{aligned}$$

∎

LEMMA 3.55. *Given a metric $g$ on $U$, if $\frac{1}{2} \in \mathbb{K}$ and $K \in \mathrm{End}(U)$ is an involution and an isometry with respect to $g$, then the direct sum produced by $K$, as in Lemma 1.119, is orthogonal.*

PROOF. Using the isometry property, $K^* \circ g \circ K = g$, so using the involution property, $K^* \circ g = g \circ K$. To check that $g$ respects the direct sum, as in Definition 1.88, use $Q_i \circ P_i = \frac{1}{2} \cdot (Id_U \pm K)$ as in Lemma 1.119:

$$g \circ Q_i \circ P_i = g \circ \frac{1}{2} \cdot (Id_U \pm K) = \frac{1}{2} \cdot (Id_{U^*} \pm K^*) \circ g = (Q_i \circ P_i)^* \circ g = P_i^* \circ Q_i^* \circ g.$$

∎

THEOREM 3.56. *Given a metric $g$ on $U$, if $\frac{1}{2} \in \mathbb{K}$, then the direct sum*

$$\mathrm{Hom}(U, U^*) = Sym(U) \oplus Alt(U)$$

*is orthogonal with respect to the induced metric.*

PROOF. This follows from Lemma 3.55, Lemma 3.3, and Corollary 3.48. ∎

COROLLARY 3.57. *Given a metric $g$ on $U$, If $\frac{1}{2} \in \mathbb{K}$ and $Tr_U(Id_U) \neq 0$, then the direct sum*

$$\mathrm{Hom}(U, U^*) = \mathbb{K} \oplus Sym_0(U, g) \oplus Alt(U)$$

*is orthogonal with respect to the induced metric.* ∎

There is a converse to the construction of Example 3.51: if a direct sum is orthogonal with respect to a given metric $g$, then metrics are induced on the summands.

THEOREM 3.58. *Given a metric $g$ on $U$ and a direct sum $U = U_1 \oplus U_2$ with projections and inclusions $P_i$, $Q_i$, if the direct sum is orthogonal with respect to $g$, then each of the maps $g_i = Q_i^* \circ g \circ Q_i : U_i \to U_i^*$ is a metric, and $g = g_1 \oplus g_2$.*

PROOF. The pullback $g_i$ has inverse $P_i \circ g^{-1} \circ P_i^*$ by Lemma 1.89, and is symmetric by Lemma 3.8, so it is a metric. Since $g$ respects the direct sums, $P_i^* \circ Q_i^* \circ g = g \circ Q_i \circ P_i$, so using the definition of direct sum of bilinear forms,

$$\begin{aligned} g_1 \oplus g_2 &= P_1^* \circ Q_1^* \circ g \circ Q_1 \circ P_1 + P_2^* \circ Q_2^* \circ g \circ Q_2 \circ P_2 \\ &= g \circ (Q_1 \circ P_1 + Q_2 \circ P_2) = g. \end{aligned}$$

∎

EXAMPLE 3.59. Theorem 3.58, applied to the above direct sums, demonstrates that under suitable hypotheses, a metric $g$ on $U$ induces metrics on $\mathrm{End}_0(U)$, $\ker(Tr_g)$, $Sym(U)$, $Sym_0(U, g)$, and $Alt(U)$.

THEOREM 3.60. *Given metrics $g$ and $h$ on $U$ and $V$, if $U = U_1 \oplus U_2$ is an orthogonal direct sum with respect to $g$, then the direct sum $U \otimes V = (U_1 \otimes V) \oplus (U_2 \otimes V)$, as in Example 1.81, is orthogonal with respect to the tensor product metric $\{g \otimes h\}$, and the metric on $U_i \otimes V$ induced by the direct sum coincides with $\{g_i \otimes h\}$.*

PROOF. Using Lemma 3.11, and the inclusion maps $[Q_i \otimes Id_V] : U_i \otimes V \to U \otimes V$,

$$\begin{aligned} & [Q_I \otimes Id_V]^* \circ \mathrm{Hom}(Id_{U \otimes V}, l) \circ j \circ [g \otimes h] \circ [Q_i \otimes Id_V] \\ = \;& \mathrm{Hom}(Id_{U_I \otimes V}, l) \circ j \circ [Q_I^* \otimes Id_V^*] \circ [g \otimes h] \circ [Q_i \otimes Id_V] \\ = \;& \mathrm{Hom}(Id_{U_I \otimes V}, l) \circ j \circ [(Q_I^* \circ g \circ Q_i) \otimes h]. \end{aligned}$$

For $i \neq I$, the result is zero, showing the direct sum is orthogonal, and for $i = I$, the calculation shows that the tensor product of the induced metric $g_i = Q_i^* \circ g \circ Q_i$ and $h$ is equal to the metric induced by $\{g \otimes h\}$ and $[Q_i \otimes Id_V]$ on $U_i \otimes V$. ∎

THEOREM 3.61. *Given metrics $g$ and $h$ on $U$ and $V$, if $V = V_1 \oplus V_2$, with direct sum data $Q_i'$, $P_i'$, is an orthogonal direct sum with respect to $h$, and $U = U_1 \oplus U_2$, and $H : U \to V$ is an isometry with respect to $g$ and $h$ which respects the direct sums, then the direct sum $U_1 \oplus U_2$ is orthogonal with respect to $g$, and $P_i' \circ H \circ Q_i : U_i \to V_i$ is an isometry with respect to the induced metrics.*

PROOF. It is straightforward to check that $H^* : V^* \to U^*$ respects the direct sums. It follows that $g = H^* \circ h \circ H$ is a composite of maps that respect the direct sums, so $U_1 \oplus U_2$ is orthogonal with respect to $g$. The induced metrics on $U_i$ and $V_i$ are as in Theorem 3.58, and the last claim is a special case of Lemma 3.31. ∎

## 3.7. Topics and applications

The following facts about the trace, metrics, and direct sums are left as exercises; their proofs are short and lend themselves to the methods and notation of the previous Sections. Some of the results generalize well-known properties of metrics on real vector spaces that appear in topics in geometry, algebra, or applications. A few of the results, labeled Lemmas, will be needed later.

### 3.7.1. Foundations of geometry.

PROPOSITION 3.62. *Let $U$ and $V$ be vector spaces, and let $h : V \to V^*$ be an invertible $\mathbb{K}$-linear map. Suppose $H$ is just a function with domain $U$ and target $V$, which is not necessarily linear, but which is right cancellable. If there is some $\mathbb{K}$-linear map $g : U \to U^*$ so that*

$$((h \circ H)(u)) \circ H = g(u)$$

*for all $u \in U$, then $H$ is $\mathbb{K}$-linear.*

PROOF. The right cancellable property in the category of sets is as in Exercise 6.18: $A \circ H = B \circ H \implies A = B$, for any, not necessarily linear, functions $A$ and $B$.

For $\mathbb{K}$-linearity, two equations must hold. First, for any $\lambda \in \mathbb{K}$, $u \in U$,

$$\begin{aligned} ((h \circ H)(\lambda \cdot u)) \circ H = g(\lambda \cdot u) &= \lambda \cdot g(u) = \lambda \cdot ((h \circ H)(u)) \circ H \\ \implies (h \circ H)(\lambda \cdot u) &= \lambda \cdot (h \circ H)(u) \\ \implies h(H(\lambda \cdot u)) &= h(\lambda \cdot H(u)) \\ \implies H(\lambda \cdot u) &= \lambda \cdot H(u). \end{aligned}$$

Second, for any $u_1$, $u_2 \in U$,

$$\begin{aligned} ((h \circ H)(u_1 + u_2)) \circ H = g(u_1 + u_2) &= g(u_1) + g(u_2) \\ &= ((h \circ H)(u_1)) \circ H + ((h \circ H)(u_2)) \circ H \\ &= ((h \circ H)(u_1) + (h \circ H)(u_2)) \circ H \\ \implies (h \circ H)(u_1 + u_2) &= (h \circ H)(u_1) + (h \circ H)(u_2) \\ \implies h(H(u_1 + u_2)) &= h(H(u_1) + H(u_2)) \\ \implies H(u_1 + u_2) &= H(u_1) + H(u_2). \end{aligned}$$

∎

PROPOSITION 3.63. *Let $U$ and $V$ be vector spaces, and let $g : U \to U^*$, $h : V \to V^*$ be symmetric bilinear forms. Suppose $H$ is just a function with domain $U$ and target $V$, which is not necessarily linear. If $\frac{1}{2} \in \mathbb{K}$ and $H(0_U) = 0_V$ and $H$ satisfies*

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u)$$

*for all $u, v \in U$, then $H$ also satisfies*

$$((h \circ H)(u)) \circ H = g(u)$$

*for all $u \in U$.*

PROOF. Expanding the RHS of the hypothesis identity using the symmetric property of $g$,

$$\begin{aligned} (g(v - u))(v - u) &= (g(v))(v) - (g(v))(u) - (g(u))(v) + (g(u))(u) \\ &= (g(v))(v) - 2(g(u))(v) + (g(u))(u). \end{aligned}$$

Expanding the LHS, using the symmetric property of $h$ and $H(0_U) = 0_V$,

$$\begin{aligned} & (h(H(v) - H(u)))(H(v) - H(u)) \\ = \quad & (h(H(v)))(H(v)) - (h(H(v)))(H(u)) - (h(H(u)))(H(v)) + (h(H(u)))(H(u)) \\ = \quad & (h(H(v) - H(0_U)))(H(v) - H(0_U)) - 2(h(H(u)))(H(v)) \\ & +(h(H(u) - H(0_U)))(H(u) - H(0_U)) \\ = \quad & (g(v - 0_U))(v - 0_U) - 2(h(H(u)))(H(v)) + (g(u - 0_U))(u - 0_U). \end{aligned}$$

Setting the above quantities equal, cancelling like terms, and using $\frac{1}{2} \in \mathbb{K}$, the conclusion follows. ∎

PROPOSITION 3.64. *Let $U$ and $V$ be vector spaces, and let $g : U \to U^*$ be a metric on $U$. Suppose $h$ is just a function with domain $V$ and target $V^*$, which is not necessarily linear. If $\frac{1}{2} \in \mathbb{K}$ and $H : U \to V$ is a $\mathbb{K}$-linear map satisfying*

$$((h \circ H)(u))(H(u)) = (g(u))(u)$$

*for all $u \in U$, then $H$ satisfies*

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u)$$

*for all $u, v \in U$, and* $\ker(H) = \{0_U\}$.

PROOF. To establish the claimed identity, use the linearity of $H$:

$$LHS = (h(H(v - u)))(H(v - u)) = RHS.$$

Suppose $H(u) = 0_V$. Then, for any $v \in U$,

$$\begin{aligned} ((h \circ H)(v))(H(v)) &= (h(H(v) - H(u)))(H(v) - H(u)) \\ &= (g(v - u))(v - u) \\ &= (g(v))(v) - (g(v))(u) - (g(u))(v) + (g(u))(u) \\ &= ((h \circ H)(v))(H(v)) - 2(g(u))(v) + ((h \circ H)(u))(H(u)), \end{aligned}$$

the last step using the symmetric property of $g$. Using $h(H(u)) \in V^*$ and $H(u) = 0_V$, the last term is 0, so cancelling like terms and using $\frac{1}{2} \in \mathbb{K}$, the conclusion is that $(g(u))(v) = 0$. Since this holds for all $v$, $g(u) = 0_{U^*}$, and $g$ is invertible, so $u = 0_U$. ∎

COROLLARY 3.65. *Given a vector space $V$ and metrics $g$ and $h$ on $V$, if $\frac{1}{2} \in \mathbb{K}$ and $H$ is just a function with domain $V$ and target $V$, which is not necessarily linear, then the following are equivalent.*

(1) *$H : V \rightsquigarrow V$ is right cancellable, and for all $u \in V$,*

$$((h \circ H)(u)) \circ H = g(u).$$

(2) *$H : V \rightsquigarrow V$ is right cancellable, $H(0_V) = 0_V$, and for all $u, v \in V$,*

$$(h(H(v) - H(u)))(H(v) - H(u)) = (g(v - u))(v - u).$$

(3) *$H : V \to V$ is $\mathbb{K}$-linear, and for all $u \in V$,*

$$((h \circ H)(u))(H(u)) = (g(u))(u).$$

(4) *$H : V \to V$ is an isometry with respect to $g$ and $h$.*

Proof. For (1) $\implies$ (3), the linearity is Proposition 3.62 and the identity obviously follows. Since $V$ is finite-dimensional, a $\mathbb{K}$-linear map $V \to V$ with trivial kernel must be invertible by Claim 0.56 (and therefore right cancellable), so Proposition 3.64 gives (3) $\implies$ (2). (2) $\implies$ (1) is Proposition 3.63. It is immediate from Definition 3.23 that (4) $\implies$ (1), and also (4) $\implies$ (3). Finally, the linearity of (3), the identity of (1), and the above mentioned invertibility together imply (4). The implications (1) $\iff$ (4) $\implies$ (3) did not require $\frac{1}{2} \in \mathbb{K}$. ∎

### 3.7.2. More isometries.

Exercise 3.66. Given metrics on $U$ and $V$, the switching map $s : U \otimes V \to V \otimes U : u \otimes v \mapsto v \otimes u$, as in Example 1.29, is an isometry with respect to the induced tensor product metrics. ∎

Lemma 3.67. *Every map $h : \mathbb{K} \to \mathbb{K}^*$ is of the form $h^\nu$, where for $\nu \in \mathbb{K}$, $(h^\nu(\lambda))(\mu) = \nu \cdot \lambda \cdot \mu$. If $\nu \neq 0$, then $h^\nu$ is a metric on $\mathbb{K}$, with inverse map $\frac{1}{\nu} \cdot Tr_{\mathbb{K}}$.*

Proof. For any $h$, let $\nu = (h(1))(1)$; then $(h(\lambda))(\mu) = \lambda \cdot \mu \cdot (h(1))(1)$. For any $\nu$, $h^\nu = \nu \cdot h^1$, and $h^\nu$ is clearly symmetric. If $\nu = 0$ then $h = 0_{\mathrm{Hom}(\mathbb{K},\mathbb{K}^*)}$. If $\nu \neq 0$, then $h^\nu$ is invertible, with inverse $\frac{1}{\nu} \cdot Tr_{\mathbb{K}}$, by Example 2.7:

$$((\frac{1}{\nu} \cdot Tr_{\mathbb{K}}) \circ h^\nu)(\lambda) = \frac{1}{\nu} \cdot Tr_{\mathbb{K}}(h^\nu(\lambda)) = \frac{1}{\nu} \cdot (h^\nu(\lambda))(1) = \frac{1}{\nu} \cdot \nu \cdot \lambda \cdot 1 = \lambda.$$

$$(h^\nu \circ (\frac{1}{\nu} \cdot Tr_{\mathbb{K}}))(A) = h^\nu(\frac{1}{\nu} \cdot A(1)) : \mu \mapsto \nu \cdot \frac{1}{\nu} \cdot A(1) \cdot \mu = A(\mu).$$

∎

A canonical such metric on $\mathbb{K}$ is $h^1 = (Tr_{\mathbb{K}})^{-1}$. $h^1$ is also equal to the map $m : \mathbb{K} \to \mathrm{Hom}(\mathbb{K},\mathbb{K})$ as in Definition 1.20, with inverse $d_{\mathbb{K}}(1) = Tr_{\mathbb{K}}$ as in Lemma 1.22 and Example 2.7. In particular, $h^\nu = \nu \cdot m$.

Exercise 3.68. For three copies of the scalar field, $\mathbb{K}_\alpha$, $\mathbb{K}_\beta$, $\mathbb{K}_\gamma$, with metrics $h^\alpha$, $h^\beta$, $h^\gamma$, if $\beta = \alpha \cdot \gamma \in \mathbb{K}$, then the map $Tr_{\mathbb{K}} : \mathrm{Hom}(\mathbb{K}_\alpha, \mathbb{K}_\beta) \to \mathbb{K}_\gamma$ is an isometry with respect to the induced $b$ metric and $h^\gamma$.

Hint. For $A$, $B \in \mathrm{End}(\mathbb{K})$, the pullback metric is, by Example 2.7,

$$(h^\gamma(Tr_{\mathbb{K}}(A)))(Tr_{\mathbb{K}}(B)) = (h^\gamma(A(1)))(B(1)) = \gamma \cdot A(1) \cdot B(1).$$

The induced metric $b$ on $\mathrm{Hom}(\mathbb{K}_\alpha, \mathbb{K}_\beta)$ gives

$$\begin{aligned} Tr_{\mathbb{K}}(A \circ (h^\alpha)^{-1} \circ B^* \circ h^\beta) &= (A \circ (h^\alpha)^{-1} \circ B^* \circ h^\beta)(1) \\ &= A(\alpha^{-1} \cdot Tr_{\mathbb{K}}(B^*(h^\beta(1)))) \\ &= A(\alpha^{-1} \cdot (h^\beta(1))(B(1))) \\ &= A(\alpha^{-1} \cdot \beta \cdot 1 \cdot B(1)) \\ &= \frac{\beta}{\alpha} \cdot A(1) \cdot B(1). \end{aligned}$$

∎

LEMMA 3.69. *Given a metric $g$ on $U$, the scalar multiplication map from Example 1.28, $l_U : U \otimes \mathbb{K} \to U$, is an isometry, with respect to the tensor product metric $\{g \otimes h^\nu\}$ and $\nu \cdot g$.*

PROOF. Calculating the pullback of $\nu \cdot g$ gives:

$$(\nu \cdot g(l_U(u_1 \otimes \lambda)))(l_U(u_2 \otimes \mu)) = \lambda \cdot \mu \cdot \nu \cdot (g(u_1))(u_2),$$

and the tensor product metric is

$$(\{g \otimes h^\nu\}(u_1 \otimes \lambda))(u_2 \otimes \mu) = \nu \cdot \lambda \cdot \mu \cdot (g(u_1))(u_2).$$

∎

EXERCISE 3.70. Given a metric $g$ on $U$, the canonical map $m : U \to \mathrm{Hom}(\mathbb{K}, U)$ from Definition 1.20, $m(u) : \lambda \mapsto \lambda \cdot u$, is an isometry with respect to $g$ and the metric $b$ induced by $h^1$ and $g$.

HINT. The pullback of the metric $b$ induced by the more general map $h^\nu$ is, using Lemma 3.67,

$$\begin{aligned}
(b(m(u_1)))(m(u_2)) &= Tr_{\mathbb{K}}((h^\nu)^{-1} \circ (m(u_1))^* \circ g \circ (m(u_1))) \\
&= (h^\nu)^{-1}((m(u_1))^*(g((m(u_2))(1)))) \\
&= \nu^{-1} \cdot Tr_{\mathbb{K}}((g((m(u_2))(1))) \circ (m(u_1))) \\
&= \nu^{-1} \cdot (g((m(u_2))(1)))((m(u_1))(1)) \\
&= \nu^{-1} \cdot (g(u_2))(u_1).
\end{aligned}$$

If $U \neq \{0_U\}$, then $\nu = 1$ is necessary for equality. ∎

LEMMA 3.71. *Given a metric $g$ on $V$, and a direct sum $V = \mathbb{K} \oplus U$ with inclusions $Q_i$, if the direct sum is orthogonal with respect to $g$, then the induced metric $Q_1^* \circ g \circ Q_1$ on $\mathbb{K}$ is equal to $h^\nu$, for $\nu = (g(Q_1(1)))(Q_1(1))$.*

PROOF. $Q_1^* \circ g \circ Q_1 = h^\nu$ for some $\nu \neq 0$, by Lemma 3.67 and Theorem 3.58.

$$\nu = (h^\nu(1))(1) = ((Q_1^* \circ g \circ Q_1)(1))(1) = (g(Q_1(1)))(Q_1(1)).$$

∎

EXERCISE 3.72. Given a metric $g$ on $V$ and an orthogonal direct sum $V = \mathbb{K} \oplus U$ as in Lemma 3.71, if $\alpha \in \mathbb{K}$ satisfies

$$(g(Q_1(\alpha)))(Q_1(\alpha)) = 1,$$

then

$$d_{\mathbb{K}U}(\alpha) : \mathrm{Hom}(\mathbb{K}, U) \to U : A \mapsto A(\alpha)$$

is an isometry with respect to the induced metrics.

HINT. Let $h^\nu$ and $g_U$ be the metrics induced by $g$ on $\mathbb{K}$ and $U$ from Lemma 3.71, so $\nu = (g(Q_1(1)))(Q_1(1))$. For $A, B \in \mathrm{Hom}(\mathbb{K}, U)$, the pullback of $g_U$ by $d_{\mathbb{K}U}(\alpha)$ gives:

$$\begin{aligned}
(g_U((d_{\mathbb{K}U}(\alpha))(A)))((d_{\mathbb{K}U}(\alpha))(B)) &= (g_U(A(\alpha)))(B(\alpha)) \\
&= \alpha^2 \cdot (g_U(A(1)))(B(1)).
\end{aligned}$$

The calculation for the induced metric on $\operatorname{Hom}(\mathbb{K}, U)$ is:

$$\begin{aligned}(b(A))(B) &= Tr_{\mathbb{K}}((h^{\nu})^{-1} \circ A^* \circ g_U \circ B) \\ &= (h^{\nu})^{-1}((g_U(B(1))) \circ A) \\ &= \nu^{-1} \cdot (g_U(B(1)))(A(1)).\end{aligned}$$

If $\alpha^2\nu = 1$, then the outputs are equal; the converse holds for $U \neq \{0_U\}$. The above calculation works for any metric on $U$, and is similar to that from Exercise 3.70. ∎

Lemma 3.73. *Given metrics on $U$, $V$, and $W$, the canonical map $n : \operatorname{Hom}(U, V) \otimes W \to \operatorname{Hom}(U, V \otimes W)$ is an isometry with respect to the induced metrics.*

Proof. This follows from the fact that $j$, $l_U$, and $m$ are isometries, Lemma 1.43, where $n = [Id_{\operatorname{Hom}(U,V)} \otimes m^{-1}] \circ j \circ \operatorname{Hom}(l_U, Id_{V \otimes W})$, and Theorems 3.28 and 3.45. It could also be checked directly. ∎

Exercise 3.74. Given a metric $g$ on $U$, the dual metric on $U^*$ and the metric $b$ on $\operatorname{Hom}(U, \mathbb{K})$, induced by $g$ and $h^1$, coincide.

Hint. For $\phi \in U^*$, and the more general metric $h^{\nu}$ on $\mathbb{K}$, the identity

$$((h^{\nu}(1)) \circ \phi)(\lambda) = \nu \cdot 1 \cdot \phi(\lambda) = (\nu \cdot \phi)(\lambda)$$

is used in computing the $b$ metric for $\phi$, $\xi \in U^*$:

$$\begin{aligned}Tr_{\mathbb{K}}(\xi \circ g^{-1} \circ \phi^* \circ h^{\nu}) &= \xi(g^{-1}(\phi^*(h^{\nu}(1)))) = \xi(g^{-1}((h^{\nu}(1)) \circ \phi)) \\ &= \xi(g^{-1}(\nu \cdot \phi)) = \nu \cdot \xi(g^{-1}(\phi)).\end{aligned}$$

The dual metric on $U^*$ results in the quantity $\xi(g^{-1}(\phi))$, so $\nu = 1$ is necessary for equality in the case $U \neq \{0_U\}$, and, in general, if $h^{\nu}$ is the metric on $\mathbb{K}$, then the $b$ metric on $\operatorname{Hom}(U, \mathbb{K})$ is equal to $\nu \cdot d_U \circ g^{-1}$. ∎

Exercise 3.75. Given metrics on $U_1$ and $U_2$, if $A : U_2 \to U_1$ is an isometry, then $A^* : U_1^* \to U_2^*$ is an isometry with respect to the dual metrics. ∎

Exercise 3.76. Given metrics on $U$ and $V$, the map $p : U \otimes V \to V^{**} \otimes U$, as in Notation 1.72, is an isometry with respect to the induced tensor product metrics. ∎

EXERCISE 3.77. Given metrics $g$, $h$, and $y$ on $U$, $V$, and $W$, the map $q$ : $\mathrm{Hom}(V, \mathrm{Hom}(U, W)) \to \mathrm{Hom}(V \otimes U, W)$, as in Definition 1.46, is an isometry with respect to the induced metrics.

HINT. All the maps in the following commutative diagram are isometries.

$$\begin{tikzcd}
V^* \otimes U^* \otimes W \arrow[r, "{[j \otimes Id_W]}"] \arrow[d, "{[Id_{V^*} \otimes k_{UW}]}"] & \mathrm{Hom}(V \otimes U, \mathbb{K} \otimes \mathbb{K}) \otimes W \arrow[d, "{[\mathrm{Hom}(Id_{V \otimes U}, l) \otimes Id_W]}"] \\
V^* \otimes \mathrm{Hom}(U, W) \arrow[d, "k_{V, \mathrm{Hom}(U,W)}"] & (V \otimes U)^* \otimes W \arrow[d, "k_{V \otimes U, W}"] \\
\mathrm{Hom}(V, \mathrm{Hom}(U, W)) \arrow[r, "q"] & \mathrm{Hom}(V \otimes U, W)
\end{tikzcd}$$

∎

LEMMA 3.78. *Given finite-dimensional vector spaces $V$ and $L$, if*

$$Ev_L : L^* \otimes L \to \mathbb{K}$$

*is invertible, then $d_{VL} : V \to \mathrm{Hom}(\mathrm{Hom}(V, L), L)$ is invertible.*

PROOF. Recall $d_{VL}$ from Definition 1.13.

$$\begin{tikzcd}
V \arrow[r, "d_{VL}"] \arrow[d, "d_V"'] & \mathrm{Hom}(\mathrm{Hom}(V, L), L) \arrow[d, "{\mathrm{Hom}(k_{VL}, Id_L)}"] \\
V^{**} \arrow[dr, "{\mathrm{Hom}(Id_{V^*}, Q_1^1)}"] & \mathrm{Hom}(V^* \otimes L, L) \\
\mathrm{Hom}(V^*, L^* \otimes L) \arrow[u, "{\mathrm{Hom}(Id_{V^*}, Ev_L)}"] \arrow[r, "{\mathrm{Hom}(Id_{V^*}, k_{LL})}"'] & \mathrm{Hom}(V^*, \mathrm{End}(L)) \arrow[u, "q"']
\end{tikzcd}$$

The lower triangle is commutative, by the calculation from the Proof of Proposition 2.21, and so is the upper part of the diagram:

$$\begin{aligned}
v &\mapsto (q \circ \mathrm{Hom}(Id_{V^*}, Q_1^1) \circ d_V)(v) \\
&= q(Q_1^1 \circ (d_V(v))) : \\
\phi \otimes u &\mapsto (Q_1^1(\phi(v)))(u) = (\phi(v) \cdot Id_L)(u) = \phi(v) \cdot u, \\
v &\mapsto (\mathrm{Hom}(k_{VL}, Id_L) \circ d_{VL})(v) \\
&= (d_{VL}(v)) \circ k_{VL} : \\
\phi \otimes u &\mapsto (k_{VL}(\phi \otimes u))(v) = \phi(v) \cdot u.
\end{aligned}$$

$d_{VL}$ is invertible because all the other maps in the outer rectangle are invertible. ∎

PROPOSITION 3.79. *For $V$, $L$, and $Ev_L$ as in Lemma 3.78, if there is some metric $y$ on $L$, and $Ev_L$ is an isometry with respect to the induced metric on $L^* \otimes L$ and $h^1$ on $\mathbb{K}$, then $d_{VL}$ is an isometry with respect to any metric $g$ on $V$, and the induced metric on $\mathrm{Hom}(\mathrm{Hom}(V, L), L)$.*

PROOF. By Theorem 3.45 and the hypothesis on $Ev_L$, $\mathrm{Hom}(Id_{V^*}, Ev_L)$ is an isometry with respect to the $b$ metric on $\mathrm{Hom}(V^*, L^* \otimes L)$, and the $b$ metric on $V^{**} = \mathrm{Hom}(V^*, \mathbb{K})$, induced by $d_V \circ g^{-1}$ on $V^*$ and $h^1$ on $\mathbb{K}$. By Theorem 3.27, $d_V$

is an isometry from $V$ to $V^{**}$ with respect to the dual metric, which by Exercise 3.74, is the same as the $b$ metric on $\mathrm{Hom}(V^*, \mathbb{K})$. Referring to the diagram from Lemma 3.78, $d_{VL}$ is equal to a composite of isometries. $\blacksquare$

### 3.7.3. Antisymmetric forms and symplectic forms.

Exercise 3.80. Given a bilinear form $g : V \to V^*$, if $g$ satisfies $(g(v))(v) = 0$ for all $v \in V$, then $g$ is antisymmetric. If $\frac{1}{2} \in \mathbb{K}$, then, conversely, an antisymmetric form $g$ satisfies $(g(v))(v) = 0$. $\blacksquare$

Big Exercise 3.81. Given a bilinear form $g : V \to V^*$, the following are equivalent.

(1) For all $u, v \in V$, if $(g(u))(v) = 0$, then $(g(v))(u) = 0$.
(2) $g \in Sym(V) \cup Alt(V)$.

Hint. (2) $\implies$ (1) is easy; a proof of the well-known converse is given by [J] §6.1. A bilinear form satisfying either equivalent condition is variously described by the literature as "orthosymmetric" or "reflexive." $\blacksquare$

Definition 3.82. A bilinear form $h : U \to U^*$ is a <u>symplectic</u> form means: $h$ is antisymmetric and invertible.

Recall from Theorem 3.15 that the invertibility implies $U$ is finite-dimensional.

Exercise 3.83. Given a symplectic form $h$ on $V$, the bilinear form $d_V \circ h^{-1} : V^* \to V^{**}$ is a symplectic form on $V^*$.

Hint. This is an analogue of Theorem 3.17. The antisymmetric property implies the equality $d_V \circ h^{-1} = -(h^*)^{-1}$. $\blacksquare$

Given a symplectic form $h$ on $V$, the above Exercise suggests there are two opposite ways $h$ could induce a symplectic form on $V^*$:

$$d_V \circ h^{-1} = -(h^*)^{-1}, \tag{3.6}$$

$$-d_V \circ h^{-1} = (h^*)^{-1}. \tag{3.7}$$

Exercise 3.84. The tensor product of symplectic forms is a metric. $\blacksquare$

The following Definition is analogous to Definition 3.23.

Definition 3.85. A map $H : U \to V$ is a <u>symplectic isometry</u>, with respect to symplectic forms $g$ on $U$, and $h$ on $V$, means: $H$ is invertible, and $g = H^* \circ h \circ H$.

Lemma 3.86. *A symplectic form $h : U \to U^*$ is a symplectic isometry with respect to itself and the symplectic form $-d_V \circ h^{-1}$ from (3.7).* $\blacksquare$

Exercise 3.87. Given $V$ with metric $g$ and symplectic form $h$, the following are equivalent.

(1) $g$ is a symplectic isometry with respect to $h$ and the symplectic form $d_V \circ h^{-1}$ from (3.6).
(2) $h^{-1} \circ g \in \mathrm{End}(V)$ is an involution.

$\blacksquare$

EXERCISE 3.88. Given $V$ with metric $g$ and symplectic form $h$, the following are equivalent.

(1) $g$ is a symplectic isometry with respect to $h$ and the symplectic form $-d_V \circ h^{-1}$ from (3.7).
(2) $h$ is an isometry with respect to $g$ and the dual metric $d_V \circ g^{-1}$.
(3) $g^{-1} \circ h \in \mathrm{End}(V)$ is an isometry with respect to $g$.
(4) $g^{-1} \circ h \in \mathrm{End}(V)$ is a symplectic isometry with respect to $h$.

HINT. The equivalence of (2) and (3) follows from Theorem 3.26. ∎

EXERCISE 3.89. Given a symplectic form $h$ on $U$, using either method (3.6) or (3.7) to induce a symplectic form on the dual space, the double dual $U^{**}$ has a canonical symplectic form

$$d_{U^*} \circ (d_U \circ h^{-1})^{-1} = -d_{U^*} \circ (-d_U \circ h^{-1})^{-1} = d_{U^*} \circ h \circ d_U^{-1}.$$

The map $d_U : U \to U^{**}$ is a symplectic isometry with respect to $h$ and the above symplectic form. ∎

BIG EXERCISE 3.90. Several more of the elementary results on metrics can be adapted to symplectic forms. ∎

### 3.7.4. More direct sums.

EXERCISE 3.91. Given linear maps $H : U \to V$ and $h : V \to V^*$, if $H^* \circ h \circ H : U \to U^*$ is invertible, then there is a direct sum $V = U \oplus \ker(H^* \circ h)$. If, in addition, $h$ is symmetric (or antisymmetric), then $h : V \to V^*$ respects the induced direct sums and $H^* \circ h \circ H : U \to U^*$ is a metric (respectively, symplectic form) on $U$. If, further, $h$ is invertible, then $h$ also induces a metric (respectively, symplectic form) on $\ker(H^* \circ h)$.

HINT. $H$ is a linear monomorphism as in Exercise 3.25. Let $Q_1 = H$, and let $P_1 = (H^* \circ h \circ H)^{-1} \circ H^* \circ h$. Then $P_1 \circ Q_1 = Id_U$, and $Q_1 \circ P_1 = H \circ (H^* \circ h \circ H)^{-1} \circ H^* \circ h$ is an idempotent on $V$. The kernel of $Q_1 \circ P_1$ is equal to the kernel of $H^* \circ h$; let $Q_2$ denote the inclusion of this subspace in $V$, and define the projection $P_2$ onto this subspace as in Example 1.113: $P_2 = Id_V - Q_1 \circ P_1 = Q_2 \circ P_2$.

The direct sum $V = U \oplus \ker(H^* \circ h)$ induces a direct sum $V^* = U^* \oplus (\ker(H^* \circ h))^*$ as in Example 1.84. If $h$ is symmetric (or antisymmetric), then $H^* \circ h \circ H$ is also symmetric (respectively, antisymmetric) by Lemma 3.8, and a metric (respectively, symplectic form) on $U$, so $U$ is finite-dimensional and $d_U$ is invertible. Consider the two expressions:

$$\begin{aligned} h \circ Q_1 \circ P_1 &= h \circ H \circ (H^* \circ h \circ H)^{-1} \circ H^* \circ h, \\ P_1^* \circ Q_1^* \circ h &= h^* \circ H^{**} \circ (H^* \circ h^* \circ H^{**})^{-1} \circ H^* \circ h. \end{aligned}$$

If $h = \pm h^* \circ d_V$, then, using Lemma 1.14,

$$\begin{aligned} h \circ Q_1 \circ P_1 &= \pm h^* \circ d_V \circ H \circ (H^* \circ (\pm h^* \circ d_V) \circ H)^{-1} \circ H^* \circ h \\ &= h^* \circ H^{**} \circ d_U \circ (H^* \circ h^* \circ H^{**} \circ d_U)^{-1} \circ H^* \circ h, \end{aligned}$$

so $h \circ Q_1 \circ P_1 = P_1^* \circ Q_1^* \circ h$, and $h$ respects the direct sums.

If, further, $h$ is invertible, then $h$ is a metric (respectively, symplectic form) that respects the direct sums $V \to V^*$, so $V = U \oplus \ker(H^* \circ h)$ is an orthogonal direct sum with respect to $h$, and Theorem 3.58 applies. ∎

EXERCISE 3.92. Given $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, with projection and inclusion maps $P_i$, $Q_i$ on $V$, $P'_i$, $Q'_i$ on $U$, if $A : U_1 \to V_1$ and $B : U_2 \to V_2$ are isometries with respect to metrics $g_i$ on $U_i$, $h_i$ on $V_i$, then

$$A \oplus B = Q_1 \circ A \circ P'_1 + Q_2 \circ B \circ P'_2 : U \to V$$

is an isometry with respect to the induced metrics.

HINT. The invertibility is by Lemma 1.86. The rest of the claim is that

$$g_1 \oplus g_2 = (A \oplus B)^* \circ (h_1 \oplus h_2) \circ (A \oplus B).$$

The RHS can be expanded:

$$\begin{aligned} RHS &= (P_1'^* \circ A^* \circ Q_1^* + P_2'^* \circ B^* \circ Q_2^*) \\ &\quad \circ (P_1^* \circ h_1 \circ P_1 + P_2^* \circ h_2 \circ P_2) \\ &\quad \circ (Q_1 \circ A \circ P'_1 + Q_2 \circ B \circ P'_2) \\ &= P_1'^* \circ A^* \circ h_1 \circ A \circ P'_1 + P_2'^* \circ B^* \circ h_2 \circ B \circ P'_2 \\ &= P_1'^* \circ g_1 \circ P'_1 + P_2'^* \circ g_2 \circ P'_2 = LHS. \end{aligned}$$

The last step uses $g_1 = A^* \circ h_1 \circ A$, $g_2 = B^* \circ h_2 \circ B$. ∎

EXERCISE 3.93. Given metrics $g_1$ and $g_2$ on $V_1$ and $V_2$, if $V = V_1 \oplus V_2$ and $W = V_1 \oplus V_2$ are direct sums with data $P'_i$, $Q'_i$ and $P_i$, $Q_i$, respectively, then the map $Q'_1 \circ P_1 + Q'_2 \circ P_2 : W \to V$ is an isometry with respect to the direct sum metrics from Corollary 3.18.

HINT. This is a special case of Exercise 3.92. The construction of the invertible map $Q'_1 \circ P_1 + Q'_2 \circ P_2 : W \to V$ is a special case of the map from Lemma 1.86. ∎

EXERCISE 3.94. Given metrics $g_1$ and $g_2$ on $V_1$ and $V_2$, if $V = V_1 \oplus V_2$, then the dual of the metric $g_1 \oplus g_2$ from Corollary 3.18 is $d_V \circ (g_1 \oplus g_2)^{-1} : V^* \to V^{**}$, as in Theorem 3.17. For the direct sum $V^* = V_1^* \oplus V_2^*$ from Example 1.84, the direct sum of the dual metrics is $(d_{V_1} \circ g_1^{-1}) \oplus (d_{V_2} \circ g_2^{-1})$. These two metrics on $V^*$ are equal.

HINT. Lemma 1.14 applies to the direct sum formula (3.3) and the inverse (3.4). ∎

EXAMPLE 3.95. Given $\frac{1}{2} \in \mathbb{K}$, and given $V$ with metric $g$ and an involution $K_1 : V \to V$, producing a direct sum $V_1 \oplus V_2$ as in Lemma 1.119, suppose the bilinear forms $Q_i^* \circ g \circ Q_i$ are metrics for $i = 1, 2$ (this is the case, for example, when $K_1$ is an isometry, by Lemma 3.55 and Theorem 3.58). If $K_2$ is another involution on $V$ that is an isometry and anticommutes with $K_1$, then $K_2$ respects the direct sums $V_1 \oplus V_2 \to V_2 \oplus V_1$ as in Lemma 1.127, and the induced maps $P_2 \circ K_2 \circ Q_1 : V_1 \to V_2$ and $P_1 \circ K_2 \circ Q_2 : V_2 \to V_1$, as in Theorem 1.136, are isometries by Lemma 3.31.

LEMMA 3.96. *Given $\frac{1}{2} \in \mathbb{K}$, and given $V$ with metric $g$ and an involution $K : V \to V$, producing a direct sum $V = V_1 \oplus V_2$ with data $P_i$, $Q_i$ as in Lemma 1.119, suppose the direct sum is orthogonal with respect to $g$ (this is the case, for example, when $K$ is an isometry, by Lemma 3.55). Let $K'$ be another involution on $V$ that is an isometry and anticommutes with $K$, and which produces a direct sum $V = V'_1 \oplus V'_2$, with data $P'_i$, $Q'_i$. If $\beta \in \mathbb{K}$ satisfies $\beta^2 = 2$, then for $i = 1, 2$, $I = 1, 2$, the map $\beta \cdot P'_I \circ Q_i : V_i \to V'_I$ is an isometry.*

PROOF. The map $\beta \cdot P'_I \circ Q_i : V_i \to V'_I$ is invertible by Theorem 1.137. The induced metric on $V_i$ is $Q_i^* \circ g \circ Q_i$ and on $V'_I$ is $(Q'_I)^* \circ g \circ Q'_I$, by Lemma 3.55 and Theorem 3.58. From the Proof of Lemma 3.55, $g \circ Q'_I \circ P'_I = (P'_I)^* \circ (Q'_I)^* \circ g$.

$$\begin{aligned} & (\beta \cdot P'_I \circ Q_i)^* \circ ((Q'_I)^* \circ g \circ Q'_I) \circ (\beta \cdot P'_I \circ Q_i) \\ = \;& \beta^2 \cdot Q_i^* \circ (P'_I)^* \circ (Q'_I)^* \circ g \circ Q'_I \circ P'_I \circ Q_i \\ = \;& \beta^2 \cdot Q_i^* \circ g \circ Q'_I \circ P'_I \circ Q_i \\ = \;& \beta^2 \cdot Q_i^* \circ g \circ \frac{1}{2} \cdot (Id_V \pm K') \circ Q_i. \end{aligned}$$

By hypothesis, $g$ respects the direct sum $V_1 \oplus V_2$, but $K'$ reverses the direct sum as in Lemma 1.127. So, $Q_i^* \circ g \circ K' \circ Q_i = 0_{\mathrm{Hom}(V_i, V_i^*)}$ and the second term in the last line drops out. ∎

EXERCISE 3.97. Given metrics $g$ and $h$ on $U$ and $V$, let $U = U_1 \oplus U_2$ and $V = V_1 \oplus V_2$ be orthogonal direct sums with respective data $P_i$, $Q_i$, $P'_i$, $Q'_i$. If $H : U \to V$ is an isometry such that $P'_2 \circ H \circ Q_1 = 0_{\mathrm{Hom}(U_1, V_2)}$, and $P'_1 \circ H \circ Q_1$ is a linear epimorphism, then $P'_1 \circ H \circ Q_2 = 0_{\mathrm{Hom}(U_2, V_1)}$, so $H$ respects the direct sums.

HINT.

$$\begin{aligned} 0_{\mathrm{Hom}(U_1, U_2^*)} &= Q_2^* \circ g \circ Q_1 \\ &= Q_2^* \circ H^* \circ h \circ H \circ Q_1 \\ &= Q_2^* \circ H^* \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2)^* \circ h \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ H \circ Q_1 \\ &= (P'_1 \circ H \circ Q_2)^* \circ Q_1'^* \circ h \circ Q'_1 \circ P'_1 \circ H \circ Q_1. \end{aligned}$$

$Q_1'^* \circ h \circ Q'_1$ is invertible by Theorem 3.58, so $P'_1 \circ H \circ Q_2 = 0_{\mathrm{Hom}(U_2, V_1)}$ by the linear epimorphism property (Definition 0.52). ∎

EXERCISE 3.98. If $U_1 = V_1$ in Exercise 3.97, then the epimorphism property is not needed in the hypothesis.

HINT.

$$\begin{aligned} (P_1 \circ H^{-1} \circ Q'_1) \circ (P'_1 \circ H \circ Q_1) &= P_1 \circ H^{-1} \circ (Q'_1 \circ P'_1 + Q'_2 \circ P'_2) \circ H \circ Q_1 \\ &= P_1 \circ Q_1 = Id_{V_1}. \end{aligned}$$

Claim 0.56 applies. ∎

EXERCISE 3.99. Given any vector space $V$, if $U = U_1 \oplus U_2$ is a direct sum with projections $P_i$ and inclusions $Q_i$, then as in Example 1.83,

$$\mathrm{Hom}(U, V) = \mathrm{Hom}(U_1, V) \oplus \mathrm{Hom}(U_2, V),$$

with projections $\mathrm{Hom}(Q_i, Id_V)$ and inclusions $\mathrm{Hom}(P_i, Id_V)$. Given metrics $g$ and $h$ on $U$ and $V$, if $U_1 \oplus U_2$ is an orthogonal direct sum, then $\mathrm{Hom}(U_1, V) \oplus \mathrm{Hom}(U_2, V)$ is an orthogonal direct sum with respect to the induced $b$ metric.

HINT. Consider $A : U_i \to V$, $B : U_I \to V$.

$$\begin{aligned} ((\mathrm{Hom}(P_I, Id_V)^* \circ b \circ \mathrm{Hom}(P_i, Id_V))(A))(B) &= (b(A \circ P_i))(B \circ P_I) \\ &= Tr_V(B \circ P_I \circ g^{-1} \circ (A \circ P_i)^* \circ h) \\ &= Tr_V(B \circ P_I \circ g^{-1} \circ P_i^* \circ A^* \circ h). \end{aligned}$$

By Lemma 1.89, since $g : U \to U^*$ respects the direct sums, so does $g^{-1} : U^* \to U$, so for $i \neq I$, $P_I \circ g^{-1} \circ P_i^* = 0_{\mathrm{Hom}(U_i^*, U_I)}$. This makes $(b(A \circ P_i))(B \circ P_I)$ equal to zero, proving orthogonality. ∎

EXERCISE 3.100. Given any vector space $U$, if $V = V_1 \oplus V_2$ is a direct sum with projections $P_i$ and inclusions $Q_i$, then as in Example 1.82, $\mathrm{Hom}(U, V) = \mathrm{Hom}(U, V_1) \oplus \mathrm{Hom}(U, V_2)$, with projections $\mathrm{Hom}(Id_U, P_i)$ and inclusions $\mathrm{Hom}(Id_U, Q_i)$. Given metrics $g$ and $h$ on $U$ and $V$, if $V_1 \oplus V_2$ is an orthogonal direct sum, then $\mathrm{Hom}(U, V_1) \oplus \mathrm{Hom}(U, V_2)$ is an orthogonal direct sum with respect to the induced $b$ metric.

HINT. Consider $A : U \to V_i$, $B : U \to V_I$.

$$\begin{aligned}((\mathrm{Hom}(Id_U, Q_I))^* \circ b \circ \mathrm{Hom}(Id_U, Q_i))(A))(B) &= (b(Q_i \circ A))(Q_I \circ B) \\ &= Tr_V(Q_I \circ B \circ g^{-1} \circ (Q_i \circ A)^* \circ h) \\ &= Tr_{V_I}(B \circ g^{-1} \circ A^* \circ Q_i^* \circ h \circ Q_I).\end{aligned}$$

For $i \neq I$, this quantity is zero. ∎

EXERCISE 3.101. Given a metric $g$ on $U$, if $U = U_1 \oplus U_2$ is an orthogonal direct sum with data $Q_i$, $P_i$, and $g_1$, $g_2$ are the metrics induced on $U_1$, $U_2$ (from Theorem 3.58), and $K : U \to U^*$, then

$$Tr_g(K) = Tr_{g_1}(Q_1^* \circ K \circ Q_1) + Tr_{g_2}(Q_2^* \circ K \circ Q_2).$$

HINT. By Theorem 3.58, $g_i^{-1} = P_i \circ g^{-1} \circ P_i^*$, and from the hint for Exercise 3.99, $P_I \circ g^{-1} \circ P_i^* = 0_{\mathrm{Hom}(U_i^*, U_I)}$ for $i \neq I$. Using Lemma 2.6,

$$\begin{aligned}Tr_g(K) &= Tr_V(g^{-1} \circ (Q_1 \circ P_1 + Q_2 \circ P_2)^* \circ K \circ (Q_1 \circ P_1 + Q_2 \circ P_2)) \\ &= Tr_{V_1}(P_1 \circ g^{-1} \circ (P_1^* \circ Q_1^* + P_2^* \circ Q_2^*) \circ K \circ Q_1) \\ &\quad + Tr_{V_2}(P_2 \circ g^{-1} \circ (P_1^* \circ Q_1^* + P_2^* \circ Q_2^*) \circ K \circ Q_2) \\ &= Tr_{V_1}(P_1 \circ g^{-1} \circ P_1^* \circ Q_1^* \circ K \circ Q_1) \\ &\quad + Tr_{V_2}(P_2 \circ g^{-1} \circ P_2^* \circ Q_2^* \circ K \circ Q_2) \\ &= Tr_{V_1}(g_1^{-1} \circ Q_1^* \circ K \circ Q_1) + Tr_{V_2}(g_2^{-1} \circ Q_2^* \circ K \circ Q_2).\end{aligned}$$

∎

LEMMA 3.102. *Let $V = U \oplus U^*$, with projections and inclusions $P_i$, $Q_i$. The direct sum induces a symmetric form on $V$,*

$$P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1. \tag{3.8}$$

*If $U$ is finite-dimensional, then this symmetric form is a metric.*

PROOF.

$$\begin{aligned}(P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1)^* \circ d_V &= P_2^* \circ P_1^{**} \circ d_V + P_1^* \circ d_U^* \circ P_2^{**} \circ d_V \\ &= P_2^* \circ d_U \circ P_1 + P_1^* \circ d_U^* \circ d_{U^*} \circ P_2 \\ &= P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1.\end{aligned}$$

$$\begin{aligned}(P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1) \circ (Q_1 \circ d_U^{-1} \circ Q_2^* + Q_2 \circ Q_1^*) &= P_2^* \circ Q_2^* + P_1^* \circ Q_1^* \\ &= Id_{V^*}.\end{aligned}$$

$$\begin{aligned}(Q_1\circ d_U^{-1}\circ Q_2^*+Q_2\circ Q_1^*)\circ(P_1^*\circ P_2+P_2^*\circ d_U\circ P_1) &= Q_1\circ P_1+Q_2\circ P_2\\ &= Id_V.\end{aligned}$$

∎

EXAMPLE 3.103. Given a metric $g_U$ on $U$, if $V=U\oplus U^*$, with direct sum data $P_i$, $Q_i$, then the direct sum of the metric $g_U$ and its dual $d_U\circ g_U^{-1}$ is a metric on $V$:

$$g_U\oplus g_{U^*}=P_1^*\circ g_U\circ P_1+P_2^*\circ d_U\circ g_U^{-1}\circ P_2,$$

as in Theorem 3.17 and Corollary 3.18.

The map $K=Q_2\circ g_U\circ P_1+Q_1\circ g_U^{-1}\circ P_2$ is an involution on $V$, as in Equation (1.19) from Theorem 1.136, and it is an isometry with respect to both the above induced metric $g_U\oplus g_{U^*}$, and the canonical metric $g_V$ from (3.8) in Lemma 3.102. In particular, if $\frac{1}{2}\in\mathbb{K}$, then Lemma 3.55 applies, so that the direct sum $V=V_1\oplus V_2$, where

$$\begin{aligned}P_1' &= \frac{1}{2}(Id_V+K)=\frac{1}{2}(Id_V+Q_2\circ g_U\circ P_1+Q_1\circ g_U^{-1}\circ P_2)\\ P_2' &= \frac{1}{2}(Id_V-K)=\frac{1}{2}(Id_V-Q_2\circ g_U\circ P_1-Q_1\circ g_U^{-1}\circ P_2),\end{aligned}$$

is orthogonal with respect to both metrics on $V$. Each of the two metrics on $V$ induces a metric on $V_1$ and on $V_2$.

EXERCISE 3.104. For $V=U\oplus U^*$ and $V=V_1\oplus V_2$ as in the above Example, the two induced metrics on $V_1$ are identical, while those on $V_2$ are opposite.

HINT. It is more convenient to check the equality of the inverses of the induced metrics on $V_1$, using (3.4) from Corollary 3.18 and the formulas from Theorem 3.58:

$$\begin{aligned}&P_1'\circ g_V^{-1}\circ(P_1')^*\\ =\;&\frac{1}{2}(Id_V+K)\circ(Q_1\circ d_U^{-1}\circ Q_2^*+Q_2\circ Q_1^*)\circ\frac{1}{2}(Id_V+K)^*\\ =\;&P_1'\circ(g_U\oplus g_{U^*})^{-1}\circ(P_1')^*\\ =\;&\frac{1}{2}(Id_V+K)\circ(Q_1\circ g_U^{-1}\circ Q_1^*+Q_2\circ g_U\circ d_U^{-1}\circ Q_2^*)\circ\frac{1}{2}(Id_V+K)^*\\ =\;&\frac{1}{2}(Q_1\circ g_U^{-1}\circ Q_1^*+Q_2\circ g_U^*\circ Q_2^*+Q_2\circ Q_1^*+Q_1\circ d_U^{-1}\circ Q_2^*)\\ =\;&\frac{1}{2}(g_V^{-1}+(g_U\oplus g_{U^*})^{-1}).\end{aligned}$$

The calculations for the metrics induced on $V_2$ are similar. ∎

EXAMPLE 3.105. Let $V=U\oplus U^*$, with direct sum data $P_i$, $Q_i$. The direct sum induces an antisymmetric form on $V$,

$$P_2^*\circ d_U\circ P_1-P_1^*\circ P_2. \tag{3.9}$$

If $U$ is finite-dimensional, then this antisymmetric form is symplectic (Definition 3.82). The construction is similar to the induced symmetric form (3.8) from Lemma 3.102, and canonical up to sign (as in (3.6), (3.7)). The inverse of the symplectic form is $Q_1\circ d_U^{-1}\circ Q_2^*-Q_2\circ Q_1^*$.

### 3.7.5. Isotropic maps and graphs.

DEFINITION 3.106. Given a bilinear form $g : V \to V^*$, a linear map $A : U \to V$ is isotropic with respect to $g$ means that the pullback of $g$ by $A$ is zero:

$$A^* \circ g \circ A = 0_{\mathrm{Hom}(U,U^*)}.$$

EXERCISE 3.107. Given $V = V_1 \oplus V_2$ with projections and inclusions $(P_1, P_2)$, $(Q_1, Q_2)$, and a bilinear form $h : V \to V^*$, the following are equivalent.

(1) $Q_1$ and $Q_2$ are both isotropic with respect to $h$.
(2) The involution $K = Q_1 \circ P_1 - Q_2 \circ P_2$ satisfies $h = -K^* \circ h \circ K$.

If, further, $\frac{1}{2} \in \mathbb{K}$ and $K \in \mathrm{End}(V)$ is any involution satisfying $h = -K^* \circ h \circ K$, then the direct sum produced by $K$ has both of the above equivalent properties.

HINT. The expression $Q_1 \circ P_1 - Q_2 \circ P_2$ is as in Example 1.122. ∎

EXERCISE 3.108. Given $V = V_1 \oplus V_2$ with inclusions $Q_i$, bilinear forms $g_1 : V_1 \to V_1^*$, $g_2 : V_2 \to V_2^*$, and a map $A : V_1 \to V_2$, the following are equivalent.

(1) $g_1 = A^* \circ g_2 \circ A$.
(2) The map $Q_1 + Q_2 \circ A : V_1 \to V$ is isotropic with respect to the bilinear form $g_1 \oplus (-g_2)$.

HINT. The first property is that $g_1$ is the pullback of $g_2$ by $A$ as in Definition 3.7; special cases include $A$ being an isometry (Definition 3.23) or a symplectic isometry (Definition 3.85).

The second property refers to the direct sum of bilinear forms as in (3.3) from Notation 3.9. The expression $Q_1 + Q_2 \circ A$ is from the notion that a "graph" of a linear map can be defined in terms of a direct sum, as in Exercise 1.107. ∎

EXERCISE 3.109. ([LP]) Let $V = U \oplus U^*$. Given maps $E : W \to U$ and $h : U \to W^*$, the following are equivalent.

(1) The bilinear form $h \circ E : W \to W^*$ is antisymmetric.
(2) The map $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \to V$ is isotropic with respect to the symmetric form (3.8) on $V$ from Lemma 3.102.

Further, if $E$ is a linear monomorphism, then so is $Q_1 \circ E + Q_2 \circ h^* \circ d_W$.

HINT. By Definition 3.106, the second property is that the pullback of the symmetric form (3.8) on $V = U \oplus U^*$ from Lemma 3.102 by the map $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \to V$ is $0_{\mathrm{Hom}(W,W^*)}$. The transpose $T_W(h \circ E)$ is $E^* \circ h^* \circ d_W$.

$$\begin{aligned} & (Q_1 \circ E + Q_2 \circ h^* \circ d_W)^* \circ (P_1^* \circ P_2 + P_2^* \circ d_U \circ P_1) \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \\ = \;& E^* \circ h^* \circ d_W + d_W^* \circ h^{**} \circ d_U \circ E \\ = \;& E^* \circ h^* \circ d_W + h \circ E. \end{aligned}$$

For any maps $F$, $G$, if

$$(Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ F = (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ G,$$

then

$$\begin{aligned} P_1 \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ F &= P_1 \circ (Q_1 \circ E + Q_2 \circ h^* \circ d_W) \circ G \\ = E \circ F &= E \circ G, \end{aligned}$$

so if $E$ is a linear monomorphism (Definition 0.47), then $F = G$, proving the second claim. ∎

If $W = U$ and $E = Id_U$, then this construction is exactly the graph of $h^* \circ d_U$, as in Exercise 1.107. A generalization of the construction appears in Section 4.2.

EXERCISE 3.110. ([**LP**]) Let $V = U \oplus U^*$. Given maps $E : W \to U$ and $h : U \to W^*$, the following are equivalent.

(1) The bilinear form $h \circ E : W \to W^*$ is symmetric.
(2) The map $Q_1 \circ E + Q_2 \circ h^* \circ d_W : W \to V$ is isotropic with respect to the antisymmetric form (3.9) from Example 3.105.

■

### 3.7.6. The adjoint.

DEFINITION 3.111. Metrics $g$, $h$, on $U$, $V$ induce an adjoint map,

$$\mathrm{Hom}(h, g^{-1}) \circ t_{UV} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(V, U) : A \mapsto g^{-1} \circ A^* \circ h. \tag{3.10}$$

EXERCISE 3.112. Given metrics $g$ and $h$ on $U$ and $V$, the map (3.10) is an isometry with respect to the induced $b$ metrics. Also, if $A : U \to V$ is an isometry, then its adjoint is an isometry $V \to U$.

HINT. The first assertion follows from the fact that $g$, $h$, and $t_{UV}$ are isometries. The second claim follows from the following equation, which uses the symmetry of $g$ and $h$, and Lemma 1.14:

$$(g^{-1} \circ A^* \circ h)^* = h^* \circ A^{**} \circ (g^{-1})^* = h \circ d_V^{-1} \circ A^{**} \circ d_U \circ g^{-1} = h \circ A \circ g^{-1}, \tag{3.11}$$

and the hypothesis $g = A^* \circ h \circ A$:

$$\begin{aligned}(g^{-1} \circ A^* \circ h)^* \circ g \circ (g^{-1} \circ A^* \circ h) &= (h \circ A \circ g^{-1}) \circ A^* \circ h \\ &= h \circ A \circ A^{-1} = h.\end{aligned}$$

■

LEMMA 3.113. *Given metrics $g$ and $h$ on $U$ and $V$, the composite of adjoint maps,*

$$(\mathrm{Hom}(g, h^{-1}) \circ t_{VU}) \circ (\mathrm{Hom}(h, g^{-1}) \circ t_{UV}) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, V)$$

*is the identity. In particular, the adjoint map $\mathrm{Hom}(g, g^{-1}) \circ t_{UU} : \mathrm{End}(U) \to \mathrm{End}(U)$ is an involution.*

PROOF. Using (3.11),

$$h^{-1} \circ (g^{-1} \circ A^* \circ h)^* \circ g = h^{-1} \circ (h \circ A \circ g^{-1}) \circ g = A.$$

■

EXERCISE 3.114. Given a metric $g$ on $U$, if $Tr_U(Id_U) \neq 0$, then the adjoint map $\mathrm{Hom}(g, g^{-1}) \circ t_{UU}$ respects any direct sum $\mathrm{End}(U) = \mathbb{K} \oplus \mathrm{End}_0(U)$ as in Example 2.9. The restriction of the adjoint map to $\mathrm{End}_0(U)$ is an involution and an isometry.

HINT. The direct sum refers to the construction of Example 2.9, and it is easily checked that $P_I \circ \mathrm{Hom}(g, g^{-1}) \circ t_{UU} \circ Q_i$ is zero for $i \neq I$. The direct sum is orthogonal as in Theorem 3.52, and Theorem 3.61 applies to the map induced by the adjoint on $\mathrm{End}_0(U)$. ■

THEOREM 3.115. *Given a metric $g$ on $U$, the following diagram is commutative, where $s$ and $s'$ are switching involutions.*

$$\begin{tikzcd}
U \otimes U \arrow[rr, "s"] \arrow[d, "{[g\otimes g]}"] & & U \otimes U \arrow[d, "{[g\otimes g]}"] \\
U^* \otimes U^* \arrow[rr, "s'"] \arrow[d, "k_{UU^*}"] & & U^* \otimes U^* \arrow[d, "k_{UU^*}"] \\
\mathrm{Hom}(U, U^*) \arrow[r, "t_{UU^*}"] & \mathrm{Hom}(U^{**}, U^*) \arrow[r, "\mathrm{Hom}(d_U, Id_{U^*})"] & \mathrm{Hom}(U, U^*) \\
\mathrm{End}(U) \arrow[r, "t"] \arrow[u, "\mathrm{Hom}(Id_U, g)"] & \mathrm{End}(U^*) \arrow[r, "\mathrm{Hom}(g, g^{-1})"] \arrow[u, "\mathrm{Hom}(g^*, Id_{U^*})"] & \mathrm{End}(U) \arrow[u, "\mathrm{Hom}(Id_U, g)"] \\
U^* \otimes U \arrow[r, "p"] \arrow[u, "k"] & U^{**} \otimes U^* \arrow[r, "{[g^*\otimes g^{-1}]}"] \arrow[u, "k'"] & U^* \otimes U \arrow[u, "k"] \\
U \otimes U \arrow[rr, "s"] \arrow[u, "{[g\otimes Id_U]}"] & & U \otimes U \arrow[u, "{[g\otimes Id_U]}"]
\end{tikzcd}$$

*All the horizontal compositions of arrows define involutions, and if $\frac{1}{2} \in \mathbb{K}$, then they produce orthogonal direct sums on the spaces in the left column.*

PROOF. The composite in the third row is $T_U$, and the second square from the top does not involve the metric $g$ — it was considered in Lemma 3.6.

The composite in the fifth row, $[g^*\otimes g^{-1}]\circ p$, is the only involution not considered earlier. The commutativity of all the squares is easy to check.

The direct sums are produced by the involutions as in Lemma 1.119. The orthogonality of the direct sum for $\mathrm{Hom}(U, U^*)$ was checked in Theorem 3.56, and the orthogonality of the other direct sums similarly follows from Lemma 3.55 since all the horizontal arrows are isometries and involutions, or from Theorem 3.61 since all the vertical arrows are isometries which respect the direct sums, by Lemma 1.126. In particular, the direct sum $U^* \otimes U^* = S^2(U^*) \oplus \Lambda^2(U^*)$ from Lemma 3.6 is orthogonal. ∎

DEFINITION 3.116. Given a metric $g$ on $U$, if $\frac{1}{2} \in \mathbb{K}$, then the orthogonal direct sum on $\mathrm{End}(U)$, produced by the involution $\mathrm{Hom}(g, g^{-1}) \circ t$ as in Theorem 3.115, defines subspaces of self-adjoint ($A = g^{-1} \circ A^* \circ g$) and skew-adjoint ($A = -g^{-1} \circ A^* \circ g$) endomorphisms.

EXAMPLE 3.117. Given a metric $g$ on $U$, if $\frac{1}{2} \in \mathbb{K}$, then the bilinear form $h : U \to U^*$ is a symmetric (or, antisymmetric) form if and only if $g^{-1} \circ h \in \mathrm{End}(U)$ is self-adjoint (respectively, skew-adjoint). This is the action of the middle left vertical arrow, and its inverse, from Theorem 3.115, respecting the direct sums $\mathrm{Hom}(U, U^*) \to \mathrm{End}(U)$.

EXERCISE 3.118. Given metrics $g$, $h$ on $U$, $V$, if $\frac{1}{2} \in \mathbb{K}$ then for any map $A : U \to V$,

$$\mathrm{Hom}(g^{-1} \circ A^* \circ h, A) : \mathrm{End}(U) \to \mathrm{End}(V)$$

respects the direct sum from Definition 3.116.

HINT. The following diagram is commutative, using Lemma 1.8 and the symmetric property of $g$ and $h$, so Lemma 1.126 applies.

$$\begin{tikzcd}
\mathrm{End}(U) \arrow[r, "t_{UU}"] \arrow[d, "\mathrm{Hom}(g^{-1}\circ A^*\circ h, A)"] & \mathrm{End}(U^*) \arrow[r, "\mathrm{Hom}(g,g^{-1})"] & \mathrm{End}(U) \arrow[d, "\mathrm{Hom}(g^{-1}\circ A^*\circ h, A)"] \\
\mathrm{End}(V) \arrow[r, "t_{VV}"] & \mathrm{End}(V^*) \arrow[r, "\mathrm{Hom}(h,h^{-1})"] & \mathrm{End}(V)
\end{tikzcd}$$

∎

EXERCISE 3.119. Given a metric $g$ on $V$, if $\frac{1}{2} \in \mathbb{K}$, then a skew-adjoint $A \in \mathrm{End}(V)$ satisfies $Tr_V(A) = 0$. If, further, $Tr_V(Id_V) \neq 0$, then any $A \in \mathrm{End}(V)$ can be written as a sum of three terms,

$$A = \frac{Tr_V(A)}{Tr_V(Id_V)} \cdot Id_V + A_1 + A_2,$$

where $A_1$ and $A_2$ have trace 0, $A_1$ is self-adjoint, and $A_2$ is skew-adjoint.

HINT. The first claim follows from Lemma 2.5 and Lemma 2.6. The second claim is an analogue of Corollary 3.36. Apply Theorem 1.125 to $Tr_V$ and the involution $\mathrm{Hom}(g, g^{-1}) \circ t$ on $\mathrm{End}(V)$ to get a direct sum decomposition. ∎

EXERCISE 3.120. Given a metric $g$ on $U$, any scalar $\alpha \in \mathbb{K}$, and any vector $u \in U$, the endomorphism

$$\alpha \cdot k_{UU}((g(u)) \otimes u) \in \mathrm{End}(U)$$

is self-adjoint. If, further, $\alpha\cdot(g(u))(u) = 1$, then $\alpha\cdot k_{UU}((g(u))\otimes u)$ is an idempotent.

HINT. From the commutativity of the diagram in Theorem 3.115,

$$\begin{aligned}
& (\mathrm{Hom}(g, g^{-1}) \circ t_{UU})(k_{UU}((g(u)) \otimes u)) \\
= \; & (\mathrm{Hom}(g, g^{-1}) \circ t_{UU} \circ k_{UU} \circ [g \otimes Id_U])(u \otimes u) \\
= \; & (k_{UU} \circ [g \otimes Id_U] \circ s)(u \otimes u) \\
= \; & k_{UU}((g(u)) \otimes u).
\end{aligned}$$

The easily checked idempotent property is related to Exercise 2.15. ∎

EXERCISE 3.121. Given metrics $g$, $h$ on $U$, $V$, any vector $u \in U$, and any map $A : U \to V$, the two self-adjoint endomorphisms from Exercise 3.120 are related by the map from Exercise 3.118:

$$\mathrm{Hom}(g^{-1} \circ A^* \circ h, A)(k_{UU}((g(u)) \otimes u)) = k_{VV}((h(A(u))) \otimes (A(u))).$$

HINT. The left square is commutative by Lemma 1.36, and the right square is commutative by Lemma 1.62 and Equation (3.11).

$$\begin{tikzcd}
U \otimes U \arrow[r, "{[g\otimes Id_U]}"] \arrow[d, "{[A\otimes B]}"] & U^* \otimes U \arrow[r, "k_{UU}"] \arrow[d, "{[(h\circ A\circ g^{-1})\otimes B]}"] & \mathrm{End}(U) \arrow[d, "{\mathrm{Hom}(g^{-1}\circ A^*\circ h, B)}"] \\
V \otimes V \arrow[r, "{[h\otimes Id_V]}"] & V^* \otimes V \arrow[r, "k_{VV}"] & \mathrm{End}(V)
\end{tikzcd}$$

The equality follows from the case where $B = A$, and starting with $u\otimes u \in U\otimes U$. ∎

EXERCISE 3.122. Given a metric $g$ on $U$, and an endomorphism $A \in \mathrm{End}(U)$, any pair of two of the following three statements implies the remaining third statement.

(1) $A$ is an involution.
(2) $A$ is self-adjoint.
(3) $A$ is an isometry.

$\blacksquare$

### 3.7.7. Some formulas from applied mathematics.

REMARK 3.123. The following few statements are related to the Householder reflection $R$.

EXERCISE 3.124. Given a metric $g$ on $U$, and an element $u \in U$, if $(g(u))(u) \neq 0$, then the endomorphism

$$R = Id_U - \frac{2}{(g(u))(u)} \cdot k_{UU}((g(u)) \otimes u)$$

is self-adjoint, an involution, and an isometry.

HINT. The second term is from Exercise 3.120. Lemma 1.123 and Exercise 3.122 apply. $\blacksquare$

PROPOSITION 3.125. *Given a metric $g$ on $V$ and $\frac{1}{2} \in \mathbb{K}$, if $u, v \in V$ satisfy $(g(u))(u) = (g(v))(v) \neq 0$, then there exists an isometry $H \in \mathrm{End}(V)$ such that $H(u) = v$.*

PROOF. Such an isometry may not be unique; the following construction is not canonical, it depends on two cases.

Case 1. If $(g(u+v))(u+v) \neq 0$, then consider the isometry from Exercise 3.124, applied to the vector $u+v$:

$$\begin{aligned}
R &= Id_V - \frac{2}{(g(u+v))(u+v)} \cdot k_{VV}((g(u+v)) \otimes (u+v)) : \\
u &\mapsto u - \frac{2}{(g(u+v))(u+v)} \cdot (g(u+v))(u) \cdot (u+v) \\
&= u - \frac{2}{2 \cdot (g(u))(u) + 2 \cdot (g(u))(v)} \cdot ((g(u))(u) + (g(v))(u)) \cdot (u+v) \\
&= -v.
\end{aligned}$$

Let $H = -R$.

Case 2. If $(g(u+v))(u+v) = 0$, the calculation

$$(g(u+v))(u+v) + (g(u-v))(u-v) = 4 \cdot (g(u))(u),$$

and the assumption $\frac{1}{2} \in \mathbb{K}$, imply that $(g(u-v))(u-v) \neq 0$, so we can use the isometry from Exercise 3.124, applied to the vector $u-v$:

$$\begin{aligned}
H = R &= Id_V - \frac{2}{(g(u-v))(u-v)} \cdot k_{VV}((g(u-v)) \otimes (u-v)) : \\
u &\mapsto u - \frac{2}{(g(u-v))(u-v)} \cdot (g(u-v))(u) \cdot (u-v) \\
&= u - \frac{2}{2 \cdot (g(u))(u) - 2 \cdot (g(u))(v)} \cdot ((g(u))(u) - (g(v))(u)) \cdot (u-v) \\
&= v. \quad \blacksquare
\end{aligned}$$

EXERCISE 3.126. Given a metric $g$ on $V$ and $v \in V$, if $(g(v))(v) \neq 0$, then there exists a direct sum $V = \mathbb{K} \oplus \ker(g(v))$ such that any direct sum equivalent to it has the properties that it is orthogonal and for any $A \in \mathrm{End}(V)$,

$$Tr_V(Q_1 \circ P_1 \circ A) = \frac{(g(v))(A(v))}{(g(v))(v)}.$$

If, further, $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by the involution $-R$, for $R$ as in Exercise 3.124, is such an equivalent direct sum.

HINT. Since $g(v) \neq 0_{V^*}$, Lemmas 1.100 and 1.101 give a direct sum $V = \mathbb{K} \oplus \ker(g(v))$, which is canonical up to equivalence, as follows. Let $Q'_2$ be the inclusion of the subspace $\ker(g(v))$ in $V$. For any $\alpha, \beta \in \mathbb{K}$ with $\alpha \cdot \beta \cdot (g(v))(v) = 1$, define

$$\begin{aligned} Q_1^\beta : \mathbb{K} &\to V : \gamma \mapsto \beta \cdot \gamma \cdot v, \\ P_1^\alpha = \alpha \cdot g(v) : V &\to \mathbb{K}, \\ P'_2 = Id_V - Q_1^\beta \circ P_1^\alpha : V &\to \ker(g(v)). \end{aligned}$$

For the orthogonality of the direct sum, it is straightforward to check, using the symmetric property of $g$, that $(P_1^\alpha)^* \circ (Q_1^\beta)^* \circ g = g \circ Q_1^\beta \circ P_1^\alpha$, or that $(Q_1^\beta)^* \circ g \circ Q'_2$ and $(Q'_2)^* \circ g \circ Q_1^\beta$ are both zero. This is also a special case of Exercise 3.91 with $h = g$ and $H = Q_1^\beta$.

It is also easy to check that

$$Q_1^\beta \circ P_1^\alpha \circ A = k_{VV}(\beta \cdot \alpha \cdot ((g(v)) \circ A) \otimes v) \in \mathrm{End}(V), \tag{3.12}$$

so by the definition of trace,

$$Tr_V(Q_1^\beta \circ P_1^\alpha \circ A) = Ev_V(\beta \cdot \alpha \cdot ((g(v)) \circ A) \otimes v) \tag{3.13}$$

$$= \beta \cdot \alpha \cdot (g(v))(A(v)) = \frac{(g(v))(A(v))}{(g(v))(v)}, \tag{3.14}$$

where the RHS of (3.14) does not depend on the choice of $\alpha$, $\beta$. Further, if maps $P_i$, $Q_i$ define any direct sum equivalent to the above orthogonal direct sum, then that direct sum is also orthogonal by Lemma 3.50, and $Q_1^\beta \circ P_1^\alpha = Q_1 \circ P_1$ as in Lemma 1.95, so the LHS of (3.13) is invariant under equivalent direct sums.

Finally, setting $A = Id_V$ in (3.12) gives $R = Id_V - 2 \cdot Q_1^\beta \circ P_1^\alpha$. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum produced by $-R$ as in Lemma 1.119 has $Q_1 \circ P_1 = \frac{1}{2} \cdot (Id_V + (-R)) = Q_1^\beta \circ P_1^\alpha$ and the direct sums are equivalent. The $A = Id_V$ case of (3.14) also gives the formula from Example 2.8.

The above steps did not use the invertibility of $g$, although the notion of orthogonal direct sum was defined only with respect to an invertible metric $g$. ∎

EXERCISE 3.127. Given a metric $g$ on $V$ and a direct sum of the form $V = \mathbb{K} \oplus U$ with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$, let $v = Q_1(1)$. If the direct sum is orthogonal with respect to $g$, then it is equivalent to a direct sum $V = \mathbb{K} \oplus \ker(g(v))$ from Exercise 3.126. The involution from Lemma 3.54,

$$-K = -Q_1 \circ P_1 + Q_2 \circ P_2,$$

coincides with the involution from Exercise 3.124,

$$\begin{aligned} R &= Id_V - \frac{2}{(g(v))(v)} \cdot k_{VV}((g(v)) \otimes v) \\ &= Id_V - \frac{2}{((g \circ Q_1)(1))(Q_1(1))} \cdot k_{VV}(((g \circ Q_1)(1)) \otimes (Q_1(1))) \in \operatorname{End}(V). \end{aligned}$$

HINT. First, Lemma 3.71 applies to the orthogonal direct sum: $(g(v))(v) = (g(Q_1(1)))(Q_1(1)) = \nu \neq 0$. Using orthogonality again, $g \circ Q_1 \circ P_1 = P_1^* \circ Q_1^* \circ g$, so for any $w \in V$,

$$\begin{aligned} g(Q_1(P_1(w))) &= P_1^*(Q_1^*(g(w))) \\ = P_1(w) \cdot g(Q_1(1)) = P_1(w) \cdot g(v) &= g(w) \circ Q_1 \circ P_1 : v \mapsto \\ P_1(w) \cdot (g(v))(v) &= (g(w))(Q_1(P_1(v))) = (g(w))(P_1(v) \cdot Q_1(1)) \\ &= P_1(Q_1(1)) \cdot (g(w))(v) = (g(w))(v) \\ \implies P_1(w) &= \frac{(g(w))(v)}{(g(v))(v)}. \end{aligned}$$

The equivalence of the direct sums follows, using the symmetric property of $g$:

$$\begin{aligned} (Q_1 \circ P_1)(w) &= P_1(w) \cdot Q_1(1) = \frac{(g(w))(v)}{(g(v))(v)} \cdot v, \\ (Q_1^\beta \circ P_1^\alpha)(w) &= \alpha \cdot \beta \cdot (g(v))(w) \cdot v = \frac{(g(v))(w)}{(g(v))(v)} \cdot v. \end{aligned}$$

The claimed equality also follows, as in (3.12):

$$-K = Id_V - 2 \cdot Q_1 \circ P_1 = Id_V - 2 \cdot Q_1^\beta \circ P_1^\alpha = R.$$

∎

EXERCISE 3.128. Given a metric $g$ on $V$, if $Tr_V(Id_V) \neq 0$, then for an orthogonal direct sum $\operatorname{End}(V) = \mathbb{K} \oplus \operatorname{End}_0(V)$ with inclusion $Q_1^\beta$ as in Example 2.9 and Theorem 3.52, the induced metric on $\mathbb{K}$ is $h^\nu$, where $\nu = \beta^2 \cdot Tr_V(Id_V)$ does not depend on $g$. The involution $R$ on $\operatorname{End}(V)$ from Exercise 3.127 does not depend on $g$ or $\beta$, and reverses the trace: for any $A \in \operatorname{End}(V)$, $Tr_V(R(A)) = -Tr_V(A)$.

HINT. Using Lemma 3.71,

$$\nu = (b(Q_1^\beta(1)))(Q_1^\beta(1)) = Tr_V(\beta \cdot Id_V \circ g^{-1} \circ (\beta \cdot Id_V)^* \circ g) = \beta^2 \cdot Tr_V(Id_V).$$

For $A \in \operatorname{End}(V)$,

$$\begin{aligned} R(A) &= A - \frac{2}{\nu} \cdot (b(Q_1^\beta(1)))(A) \cdot Q_1^\beta(1) \\ &= A - \frac{2}{\nu} \cdot Tr_V(A \circ g^{-1} \circ (\beta \cdot Id_V)^* \circ g) \cdot \beta \cdot Id_V \\ &= A - \frac{2 \cdot Tr_V(A) \cdot \beta^2}{\beta^2 \cdot Tr_V(Id_V)} \cdot Id_V. \end{aligned}$$

∎

REMARK 3.129. The following few exercises are related to the block vec operation from [O].

In the following diagram,

$$\begin{tikzcd}
 & \mathrm{Hom}(U\otimes V, W^*\otimes X^*) \arrow[dr, "\mathrm{Hom}(Id_{U\otimes V},\tilde{j})"] & \\
\mathrm{Hom}(U,W^*)\otimes\mathrm{Hom}(V,X^*) \arrow[ur, "j"] & & \mathrm{Hom}(U\otimes V,(W\otimes X)^*) \arrow[ddd, "\Box"] \\
 & (U\otimes V)^*\otimes(W\otimes X)^* \arrow[ur, "k_{U\otimes V,(W\otimes X)^*}"] & \\
U^*\otimes W^*\otimes V^*\otimes X^* \arrow[uu, "{[k_{UW^*}\otimes k_{VX^*}]}"] \arrow[r, "s_1"] \arrow[dr, "s_3"'] & U^*\otimes V^*\otimes W^*\otimes X^* \arrow[u, "{[\tilde{j}\otimes\tilde{j}]}"'] \arrow[d, "s_2"'] & \\
 & V^*\otimes X^*\otimes U^*\otimes W^* \arrow[d, "{[\tilde{j}\otimes\tilde{j}]}"'] & \mathrm{Hom}(V\otimes X,(U\otimes W)^*) \\
 & (V\otimes X)^*\otimes(U\otimes W)^* \arrow[ur, "k_{V\otimes X,(U\otimes W)^*}"'] &
\end{tikzcd}$$

$s_1$, $s_2$, and $s_3 = s_2 \circ s_1$ are switching maps, and the various $\tilde{j}$ maps are as in Notation 2.43. The top block is commutative, it is similar to the diagram from Lemma 2.32.

NOTATION 3.130. If three of the four spaces $U$, $V$, $W$, $X$ are finite-dimensional, then all of the arrows in the above diagram are invertible. Define the map

$$\Box : \mathrm{Hom}(U \otimes V, (W \otimes X)^*) \to \mathrm{Hom}(V \otimes X, (U \otimes W)^*)$$

to equal the composite going counter-clockwise around the lower right square in the diagram.

EXERCISE 3.131. ([O] Theorem 1) If three of the four spaces $U$, $V$, $W$, $X$ are finite-dimensional, then for any $A \in \mathrm{Hom}(U \otimes V, (W \otimes X)^*)$, the following are equivalent.

(1) There exist $h_1 \in \mathrm{Hom}(U, W^*)$, $h_2 \in \mathrm{Hom}(V, X^*)$ such that

$$A = \tilde{j} \circ [h_1 \otimes h_2].$$

(2) There exist $\phi_1 \in (V \otimes X)^*$ and $\phi_2 \in (U \otimes W)^*$ such that

$$\Box(A) = k_{V\otimes X,(U\otimes W)^*}(\phi_1 \otimes \phi_2).$$

In the special case $W = U$, $X = V$, (1) can be re-written using Notation 3.13:

(1′) There exist $h_1 \in \mathrm{Hom}(U, U^*)$, $h_2 \in \mathrm{Hom}(V, V^*)$ such that $A = \{h_1 \otimes h_2\}$.

∎

EXERCISE 3.132. ([O] Corollary 1) If $V$ is finite-dimensional, then for any $A \in \mathrm{Hom}(V \otimes V, (V \otimes V)^*)$, the following are equivalent.

(1) There exists $h \in \mathrm{Hom}(V, V^*)$ such that $A = \{h \otimes h\}$.
(2) There exists $\phi \in (V \otimes V)^*$ such that $\Box(A) = k_{V\otimes V,(V\otimes V)^*}(\phi \otimes \phi)$.

Either (1) or (2) implies that the bilinear form $\Box(A)$ is symmetric. ∎

REMARK 3.133. The following two Propositions relating the $b$ metric to a trace on a tensor product space are analogous to a formula involving the "commutation matrix" $K$ (from Remark 1.76), which appears in [HJ] §4.3, and [Magnus] (exercise 3.9: $trK(A'\otimes B)=trA'B$).

PROPOSITION 3.134. *Given metrics $g$ and $h$ on $U$ and $V$, for $A$, $B\in\mathrm{Hom}(U,V)$,*

$$Tr_{V^*\otimes U}([(h\circ d_V^{-1})\otimes g^{-1}]\circ p\circ[B^*\otimes A])=Tr_U(g^{-1}\circ B^*\circ h\circ A).$$

PROOF. In the following diagram,

$$\begin{tikzcd}
U^*\otimes V\otimes U^*\otimes V \arrow[d, "{[[(d_U\circ g^{-1})\otimes h]\otimes Id_{U^*\otimes V}]}"] & U^*\otimes V\otimes U^*\otimes V \arrow[l, "s"'] \arrow[r, "a_1"] \arrow[d, "{[p\otimes Id_{U^*\otimes V}]}"] & \mathrm{Hom}(U,V)\otimes\mathrm{Hom}(U,V) \arrow[d, "{[t_{UV}\otimes Id_{\mathrm{Hom}(U,V)}]}"'] \\
U^{**}\otimes V^*\otimes U^*\otimes V \arrow[d, "{[j\otimes Id_{U^*\otimes V}]}"] & V^{**}\otimes U^*\otimes U^*\otimes V \arrow[r, "a_2"] \arrow[d, "{[[Id_{V^{**}}\otimes g^{-1}]\otimes[Id_{U^*}\otimes h]]}"] & \mathrm{Hom}(V^*,U^*)\otimes\mathrm{Hom}(U,V) \arrow[dd, "j"'] \\
\mathrm{Hom}(U^*\otimes V,\mathbb{K}\otimes\mathbb{K})\otimes U^*\otimes V \arrow[d, "{[\mathrm{Hom}(Id_{U^*\otimes V},l)\otimes Id_{U^*\otimes V}]}"] & V^{**}\otimes U\otimes U^*\otimes V^* \arrow[d, "s"] & \\
(U^*\otimes V)^*\otimes U^*\otimes V \arrow[dd, "Ev_{U^*\otimes V}"'] & V^{**}\otimes V^*\otimes U^*\otimes U \arrow[d, "{[k_{V^*V^*}\otimes k_{UU}]}"] \arrow[ddl, "{l\circ[Ev_{V^*}\otimes Ev_U]}"'] & \mathrm{Hom}(V^*\otimes U,U^*\otimes V) \arrow[dd, "{\mathrm{Hom}(Id_{V^*\otimes U},p)}"'] \\
 & \mathrm{End}(V^*)\otimes\mathrm{End}(U) \arrow[d, "j_2"] \arrow[dl, "{l\circ[Tr_{V^*}\otimes Tr_U]}"] & \\
\mathbb{K} & \mathrm{End}(V^*\otimes U) \arrow[l, "Tr_{V^*\otimes U}"] & \mathrm{Hom}(V^*\otimes U,V^{**}\otimes U^*) \arrow[l, "a_3"]
\end{tikzcd}$$

the arrow $s$ in the top row switches the two $V$ factors, and the abbreviated arrow labels are

$$\begin{aligned}
a_1 &= [k_{UV}\otimes k_{UV}]\\
a_2 &= [k_{U^*V^*}\otimes k_{UV}]\\
a_3 &= \mathrm{Hom}(Id_{V^*\otimes U},[(h\circ d_V^{-1})\otimes g^{-1}]).
\end{aligned}$$

The top right square is commutative by Lemmas 1.36 and 1.75. The lower left triangle is commutative by Corollary 2.36, and the triangle above that by the definition of trace. Starting with $\Phi\otimes\phi\otimes\xi\otimes v\in V^{**}\otimes U^*\otimes U^*\otimes V$, the lower right square is commutative:

$$\begin{aligned}
\Phi\otimes\phi\otimes\xi\otimes v &\mapsto (a_3\circ\mathrm{Hom}(Id_{V^*\otimes U},p)\circ j\circ a_2)(\Phi\otimes\phi\otimes\xi\otimes v)\\
&= [(h\circ d_V^{-1})\otimes g^{-1}]\circ p\circ[(k_{U^*V^*}(\Phi\otimes\phi))\otimes(k_{UV}(\xi\otimes v))]:\\
\psi\otimes u &\mapsto (h((\xi(u))\cdot v))\otimes(g^{-1}((\Phi(\psi))\cdot\phi)),\\
\Phi\otimes\phi\otimes\xi\otimes v &\mapsto (j_2\circ[k_{V^*V^*}\otimes k_{UU}]\circ s\circ[[Id_{V^{**}}\otimes g^{-1}]\otimes[Id_{U^*}\otimes h]])(\Phi\otimes\phi\otimes\xi\otimes v)\\
&= [(k_{V^*V^*}(\Phi\otimes(h(v))))\otimes(k_{UU}(\xi\otimes(g^{-1}(\phi))))]:\\
\psi\otimes u &\mapsto (\Phi(\psi)\cdot h(v))\otimes(\xi(u)\cdot g^{-1}(\phi)).
\end{aligned}$$

Starting with $\phi\otimes w\otimes\xi\otimes v\in U^*\otimes V\otimes U^*\otimes V$, the upper left square is commutative:

$$\begin{aligned}
\phi\otimes w\otimes\xi\otimes v &\mapsto (l\circ[Ev_{V^*}\otimes Ev_U]\circ s\circ[[Id_{V^{**}}\otimes g^{-1}]\otimes[Id_{U^*}\otimes h]]\circ[p\otimes Id_{U^*\otimes V}])(\phi\otimes w\otimes\xi\otimes v)\\
&= (l\circ[Ev_{V^*}\otimes Ev_U]\circ s)((d_V(w))\otimes(g^{-1}(\phi))\otimes\xi\otimes(h(v)))\\
&= ((h(v))(w))\cdot(\xi(g^{-1}(\phi))),\\
\phi\otimes w\otimes\xi\otimes v &\mapsto (Ev_{U^*\otimes V}\circ[(\mathrm{Hom}(Id_{U^*\otimes V},l)\circ j\circ[(d_U\circ g^{-1})\otimes h])\otimes Id_{U^*\otimes V}]\circ s)(\phi\otimes w\otimes\xi\otimes v)\\
&= Ev_{U^*\otimes V}((l\circ[((d_U\circ g^{-1})(\phi))\otimes(h(v))])\otimes\xi\otimes w)\\
&= (\xi(g^{-1}(\phi)))\cdot((h(v))(w)).
\end{aligned}$$

This last quantity is also the result of the tensor product metric:

$$(\{(d_U\circ g^{-1})\otimes h\}(\phi\otimes w))(\xi\otimes v) = (\xi(g^{-1}(\phi)))\cdot((h(v))(w)),$$

from Corollary 3.19. So the claimed equality follows from the commutativity of the diagram, and the fact that $k_{UV}^{-1}$ is an isometry (Theorem 3.41). Starting with $B\otimes A\in\mathrm{Hom}(U,V)\otimes\mathrm{Hom}(U,V)$:

$$\begin{aligned}
LHS &= (Tr_{V^*\otimes U}\circ a_3\circ\mathrm{Hom}(Id_{V^*\otimes U},p)\circ j\circ[t_{UV}\otimes Id_{\mathrm{Hom}(U,V)}])(B\otimes A)\\
&= (Ev_{U^*\otimes V}\circ[(\mathrm{Hom}(Id_{U^*\otimes V},l)\circ j\circ[(d_U\circ g^{-1})\otimes h])\otimes Id_{U^*\otimes V}]\circ s\circ a_1^{-1})(B\otimes A)\\
&= (\{(d_U\circ g^{-1})\otimes h\}(k_{UV}^{-1}(B)))(k_{UV}^{-1}(A))\\
&= (b(B))(A) = RHS.
\end{aligned}$$

∎

PROPOSITION 3.135. *Given metrics $g$ and $h$ on $U$ and $V$, for $A$, $B\in\mathrm{Hom}(U,V)$,*

$$Tr_{\{(d_V\circ h^{-1})\otimes g\}}(f_{UV}\circ[B^*\otimes A]) = Tr_g(B^*\circ h\circ A).$$

PROOF. By Lemma 1.74 and the previous Proposition,

$$\begin{aligned}
LHS &= Tr_{V^*\otimes U}([(d_V\circ h^{-1})\otimes g]^{-1}\circ j^{-1}\circ\mathrm{Hom}(Id_{V^*\otimes U},l)^{-1}\circ f_{UV}\circ[B^*\otimes A])\\
&= Tr_{V^*\otimes U}([(h\circ d_V^{-1})\otimes g^{-1}]\circ p\circ[B^*\otimes A])\\
&= Tr_U(g^{-1}\circ B^*\circ h\circ A) = RHS.
\end{aligned}$$

∎

### 3.7.8. Eigenvalues.

EXERCISE 3.136. Suppose $h$ and $g$ are bilinear forms on $V$, and $g$ is symmetric. If $h(v_1) = \lambda_1 \cdot g(v_1)$, and $(T_V(h))(v_2) = \lambda_2 \cdot g(v_2)$, then either $\lambda_1 = \lambda_2$, or $(g(v_1))(v_2) = 0$.

HINT.

$$\begin{aligned}(\lambda_1 - \lambda_2) \cdot (g(v_1))(v_2) &= (\lambda_1 \cdot g(v_1))(v_2) - (\lambda_2 \cdot g(v_2))(v_1) \\ &= (h(v_1))(v_2) - ((T_V(h))(v_2))(v_1) = 0.\end{aligned}$$

$\blacksquare$

EXERCISE 3.137. Suppose $h$ and $g$ are bilinear forms on $V$, and $g$ is antisymmetric. If $h(v_1) = \lambda_1 \cdot g(v_1)$, and $(T_V(h))(v_2) = \lambda_2 \cdot g(v_2)$, then either $\lambda_1 = -\lambda_2$, or $(g(v_1))(v_2) = 0$. $\blacksquare$

EXERCISE 3.138. If $h$ and $g$ are both symmetric forms (or both antisymmetric), and $h(v_1) = \lambda_1 \cdot g(v_1)$, and $h(v_2) = \lambda_2 \cdot g(v_2)$, then either $\lambda_1 = \lambda_2$, or $(g(v_1))(v_2) = 0$. $\blacksquare$

EXERCISE 3.139. If $g$ is a bilinear form on $V$, and $E$ is an endomorphism of $V$ such that $g \circ E = E^* \circ g : V \to V^*$, and $E(v_1) = \lambda_1 \cdot v_1$, and $E(v_2) = \lambda_2 \cdot v_2$, then either $\lambda_1 = \lambda_2$, or $(g(v_1))(v_2) = 0$.

HINT. When $g$ is a metric, the hypothesis is that $E$ is self-adjoint.

$$\begin{aligned}(\lambda_1 - \lambda_2) \cdot (g(v_1))(v_2) &= (\lambda_1 \cdot g(v_1))(v_2) - (\lambda_2 \cdot g(v_1))(v_2) \\ &= (g(E(v_1)))(v_2) - (g(v_1))(E(v_2)) \\ &= ((g \circ E)(v_1))(v_2) - ((E^* \circ g)(v_1))(v_2) = 0.\end{aligned}$$

$\blacksquare$

EXERCISE 3.140. If $g$ is a bilinear form on $V$, and $E$ is an endomorphism of $V$ such that $g \circ E = -E^* \circ g : V \to V^*$, and $E(v_1) = \lambda_1 \cdot v_1$, and $E(v_2) = \lambda_2 \cdot v_2$, then either $\lambda_1 = -\lambda_2$, or $(g(v_1))(v_2) = 0$. In particular, if $\frac{1}{2} \in \mathbb{K}$, then either $\lambda_1 = 0$, or $(g(v_1))(v_1) = 0$.

HINT. This is a skew-adjoint version of the previous Exercise. $\blacksquare$

EXERCISE 3.141. Given a metric $g$ on $U$, a self-adjoint endomorphism $H : U \to U$, and a nonzero element $v \in U$, there exists $\lambda \in \mathbb{K}$ such that $H(v) = \lambda \cdot v$ if and only if $H$ commutes with the endomorphism $k((g(v)) \otimes v)$ from Exercise 3.120.

HINT. The diagram from Exercise 3.121 gives these two equalities:

$$\begin{aligned}H \circ (k((g(v)) \otimes v)) &= k((g(v)) \otimes (H(v))), \\ (k((g(v)) \otimes v)) \circ H &= k((g(H(v))) \otimes v).\end{aligned}$$

If $H(v) = \lambda \cdot v$, then the two quantities are equal. Conversely, if they are equal, then for any $u \in U$,

$$\begin{aligned}(k((g(v)) \otimes (H(v))))(u) &= (k((g(H(v))) \otimes v))(u) \\ (g(v))(u) \cdot (H(v)) &= (g(H(v)))(u) \cdot v.\end{aligned}$$

Since $v \neq 0_U$, the non-degeneracy of $g$ implies there is some $u$ so that $(g(v))(u) \neq 0$. Let $\lambda = \frac{(g(H(v)))(u)}{(g(v))(u)}$. $\blacksquare$

EXERCISE 3.142. If $g$ is a bilinear form on $V$, and $E$ is an endomorphism of $V$ such that $E^* \circ g \circ E = g : V \to V^*$, and $E(v_1) = \lambda_1 \cdot v_1$, and $E(v_2) = \lambda_2 \cdot v_2$, then either $\lambda_1 \cdot \lambda_2 = 1$, or $(g(v_1))(v_2) = 0$. In particular, either $\lambda_1^2 = 1$, or $(g(v_1))(v_1) = 0$.

HINT.

$$\begin{aligned}(g(v_1))(v_2) &= ((E^* \circ g \circ E)(v_1))(v_2) = (g(E(v_1)))(E(v_2)) \\ &= \lambda_1 \cdot \lambda_2 \cdot (g(v_1))(v_2).\end{aligned}$$

∎

### 3.7.9. Canonical metrics.

EXAMPLE 3.143. Given $V$ finite-dimensional, the canonical invertible map

$$(k^*)^{-1} \circ e : \mathrm{End}(V) \to \mathrm{End}(V)^*$$

from Lemma 2.1 is a metric on $\mathrm{End}(V)$. It is symmetric by Lemma 1.14 and Lemma 2.1:

$$((k^*)^{-1} \circ e)^* \circ d_{\mathrm{End}(V)} = e^* \circ (k^{-1})^{**} \circ d_{\mathrm{End}(V)} = e^* \circ d \circ k^{-1} = (k^*)^{-1} \circ e.$$

This metric on $\mathrm{End}(V)$ should be called the <u>canonical metric</u>, to distinguish it from the $b$ metric, induced by a choice of metric on $V$. The non-degeneracy of the metric was considered in Proposition 2.16, where it was also shown that for $A$, $B \in \mathrm{End}(V)$,

$$(((k^*)^{-1} \circ e)(A))(B) = Tr_V(A \circ B). \tag{3.15}$$

EXAMPLE 3.144. Given $V$ finite-dimensional, the canonical map $f : V^* \otimes V \to (V^* \otimes V)^*$ is invertible, and is symmetric by Lemma 1.71, so it is a metric on $V^* \otimes V$. The dual metric on $(V^* \otimes V)^*$ is $d \circ f^{-1} = (f^*)^{-1}$.

This metric on $V^* \otimes V$ is also canonical, and, in general, different from the tensor product metric induced by a choice of metric on $V$. By Lemma 1.74, the metric $f$ is equal to the composite $\mathrm{Hom}(Id_{V^* \otimes V}, l) \circ j \circ p : V^* \otimes V \to (V^* \otimes V)^*$.

EXERCISE 3.145. The dual metric $d_{\mathbb{K}} \circ (h^1)^{-1}$ on $\mathbb{K}^*$ coincides with the $(k^*)^{-1} \circ e$ metric on $\mathrm{End}(\mathbb{K})$.

HINT. The metric $h^1$ is as in Lemma 3.67. For $\phi$, $\xi \in \mathbb{K}^*$,

$$\begin{aligned}((d_{\mathbb{K}} \circ (h^1)^{-1})(\phi))(\xi) &= \xi(Tr_{\mathbb{K}}(\phi)) = \xi(\phi(1)) = \phi(1) \cdot \xi(1) \\ (((k^*)^{-1} \circ e)(\phi))(\xi) &= Tr_{\mathbb{K}}(\phi \circ \xi) = \phi(\xi(1)) = \xi(1) \cdot \phi(1).\end{aligned}$$

∎

EXERCISE 3.146. With respect to the canonical metrics $(k^*)^{-1} \circ e$ on $\mathrm{End}(\mathbb{K})$ and $h^1$ on $\mathbb{K}$, $Tr_{\mathbb{K}}$ is an isometry.

HINT. The canonical metric, applied to $A$, $B \in \mathrm{End}(\mathbb{K})$, is:

$$(((k^*)^{-1} \circ e)(A))(B) = Tr_{\mathbb{K}}(A \circ B) = (A \circ B)(1) = A(B(1)) = B(1) \cdot A(1).$$

This coincides with the pullback:

$$(h^1(Tr_{\mathbb{K}}(A)))(Tr_{\mathbb{K}}(B)) = (h^1(A(1)))(B(1)) = A(1) \cdot B(1).$$

∎

EXERCISE 3.147. Given finite-dimensional $V$, the canonical map $k : V^* \otimes V \to \operatorname{End}(V)$ is an isometry with respect to the canonical metrics $f$ and $(k^*)^{-1} \circ e$.

HINT. The pullback of $(k^*)^{-1} \circ e$ by $k$ agrees with $f$:

$$k^* \circ (k^*)^{-1} \circ e \circ k = f.$$

∎

EXERCISE 3.148. Given finite-dimensional $V$, the canonical map $e : \operatorname{End}(V) \to (V^* \otimes V)^*$ is an isometry with respect to the canonical metrics $(k^*)^{-1} \circ e$ and $(f^*)^{-1}$.

HINT. The pullback of $(f^*)^{-1}$ by $e$ is:

$$e^* \circ (f^*)^{-1} \circ e = e^* \circ (e^*)^{-1} \circ (k^*)^{-1} \circ e = (k^*)^{-1} \circ e.$$

∎

It follows that $f_{VV}$ is an isometry, but this also follows from Theorem 3.26.

EXERCISE 3.149. ([$\mathbf{G}_2$] §I.8) Given finite-dimensional $U$, $V$, if $A : U \to V$ is invertible, then $\operatorname{Hom}(A^{-1}, A) : \operatorname{End}(U) \to \operatorname{End}(V)$ is an isometry with respect to the $(k^*)^{-1} \circ e$ metrics.

HINT. From the Proof of Lemma 2.6:

$$\begin{aligned} (k'^*)^{-1} \circ e' &= (k'^*)^{-1} \circ e' \circ \operatorname{Hom}(A, A^{-1}) \circ \operatorname{Hom}(A^{-1}, A) \\ &= \operatorname{Hom}(A^{-1}, A)^* \circ (k^*)^{-1} \circ e \circ \operatorname{Hom}(A^{-1}, A). \end{aligned}$$

∎

EXERCISE 3.150. For $U$, $V$, and invertible $A$ as in the previous Exercise, $[(A^{-1})^* \otimes A] : U^* \otimes U \to V^* \otimes V$ is an isometry with respect to $f_{UU}$ and $f_{VV}$.

HINT. This follows from Lemma 1.62:

$$[(A^{-1})^* \otimes A] = k_{VV}^{-1} \circ \operatorname{Hom}(A^{-1}, A) \circ k_{UU},$$

and could also be checked directly. ∎

EXERCISE 3.151. Given finite-dimensional $V$, the transpose $t : \operatorname{End}(V) \to \operatorname{End}(V^*) : A \mapsto A^*$ is an isometry with respect to the canonical $(k^*)^{-1} \circ e$ metrics.

HINT. From the Proof of Lemma 2.5:

$$t^* \circ (k'^*)^{-1} \circ e' \circ t = (k^*)^{-1} \circ e.$$

∎

EXERCISE 3.152. Given finite-dimensional $U, V$, the map $j : \operatorname{End}(U) \otimes \operatorname{End}(V) \to \operatorname{End}(U \otimes V)$ is an isometry with respect to the tensor product of canonical metrics, and the canonical metric on $\operatorname{End}(U \otimes V)$.

HINT. By Corollary 2.36,

$$\begin{aligned} Tr_{U \otimes V}((j(A_1 \otimes B_1)) \circ (j(A_2 \otimes B_2))) &= Tr_{U \otimes V}(j((A_1 \circ A_2) \otimes (B_1 \circ B_2))) \\ &= Tr_U(A_1 \circ A_2) \cdot Tr_V(B_1 \circ B_2). \end{aligned}$$

∎

EXERCISE 3.153. Given finite-dimensional $U$, if $Tr_U(Id_U) \neq 0$, then a direct sum $\mathrm{End}(U) = \mathbb{K} \oplus \mathrm{End}_0(U)$ from Example 2.9 is orthogonal with respect to the canonical metric $(k^*)^{-1} \circ e$ on $\mathrm{End}(U)$, and this induces a canonical metric on $\mathrm{End}_0(U)$. The involution from Exercise 3.124, defined in terms of the canonical metric and the canonical element $Id_U$, is given for $A \in \mathrm{End}(U)$ by:

$$R : A \mapsto A - 2 \cdot \frac{Tr_U(A)}{Tr_U(Id_U)} \cdot Id_V,$$

which is the same as the involution $-K$ from Lemma 3.54 and the involution $R$ from Exercise 3.128.

HINT. The orthogonality is easy to check; this is also a special case of Exercises 3.126 and 3.127. ∎

EXERCISE 3.154. Given a metric $g$ on $U$, the adjoint involution $\mathrm{Hom}(g, g^{-1}) \circ t_{UU}$ on $\mathrm{End}(U)$ is an isometry with respect to the canonical metric. If $\frac{1}{2} \in \mathbb{K}$, then the direct sum decomposition into self-adjoint and skew-adjoint endomorphisms, from Definition 3.116, is orthogonal with respect to the canonical metric. On the space of self-adjoint endomorphisms, the metric induced by the canonical metric coincides with the metric induced by the induced $b$ metric. On the space of skew-adjoint endomorphisms, the two induced metrics are opposite. ∎

EXERCISE 3.155. For any bilinear form $g : \mathrm{End}(V) \to \mathrm{End}(V)^*$ (and in particular, any metric $g$ on $\mathrm{End}(V)$), if $V$ is finite-dimensional then there exists $F \in \mathrm{End}(\mathrm{End}(V))$ so that for all $A, B \in \mathrm{End}(V)$,

$$(g(A))(B) = Tr_V((F(A)) \circ B).$$

HINT. Define $F = e^{-1} \circ k^* \circ g$. Then by Proposition 2.17,

$$(g(A))(B) = Tr_V((e^{-1}(k^*(g(A)))) \circ B).$$

The canonical metric on $\mathrm{End}(V)$ from Example 3.143 is the case $F = Id_{\mathrm{End}(V)}$. The $b$ metric from Definition 3.40 induced by a metric $h$ on $V$,

$$(b(A))(B) = Tr_V(h^{-1} \circ A^* \circ h \circ B),$$

is the case where $F$ is the adjoint involution from Definition 3.111 and Lemma 3.113. ∎

EXAMPLE 3.156. For the generalized transpose from Definition 1.7 and Example 1.53,

$$t_{UV}^W \in \mathrm{Hom}(\mathrm{Hom}(U,V), \mathrm{Hom}(\mathrm{Hom}(V,W), \mathrm{Hom}(U,W))),$$

and any bilinear form $g : \mathrm{Hom}(U,W) \to \mathrm{Hom}(U,W)^*$, the map

$$\mathrm{Hom}(Id_{\mathrm{Hom}(U,V)}, \mathrm{Hom}(Id_{\mathrm{Hom}(V,W)}, g))$$

transforms $t_{UV}^W$ to the scalar valued trilinear form

$$\mathrm{Hom}(Id_{\mathrm{Hom}(V,W)}, g) \circ t_{UV}^W \in \mathrm{Hom}(\mathrm{Hom}(U,V), \mathrm{Hom}(\mathrm{Hom}(V,W), (\mathrm{Hom}(U,W))^*)).$$

For $A \in \mathrm{Hom}(U,V)$, $B \in \mathrm{Hom}(V,W)$, and $C \in \mathrm{Hom}(U,W)$,

$$\begin{array}{rcl} \mathrm{Hom}(Id_{\mathrm{Hom}(V,W)}, g) \circ t_{UV}^W : A & \mapsto & g \circ (t_{UV}^W(A)) = g \circ \mathrm{Hom}(A, Id_U) : \\ B & \mapsto & g(B \circ A) : \\ C & \mapsto & (g(B \circ A))(C). \end{array}$$

In the special case where $g$ is the metric $b$ from Definition 3.40 induced by metrics $g_1$ on $U$ and $g_2$ on $W$,

$$\begin{aligned}(g(B\circ A))(C) &= Tr_U(g_1^{-1}\circ(B\circ A)^*\circ g_2\circ C)\\ &= Tr_{U^*}(A^*\circ B^*\circ g_2\circ C\circ g_1^{-1}).\end{aligned}$$

In a different special case where $W=U$ and $g$ is the canonical metric $(k^*)^{-1}\circ e$ on $\mathrm{End}(U)$ from Example 3.143,

$$(g(B\circ A))(C) = Tr_U(B\circ A\circ C).$$

Example 3.157. For any metrics $g$ and $h$ on $\mathrm{End}(V)$, consider the $b$ metric from Definition 3.40 induced by $g$ and $h$ on $\mathrm{End}(\mathrm{End}(V))$,

$$(b(E))(F) = Tr_{\mathrm{End}(V)}(F\circ g^{-1}\circ E^*\circ h),$$

for $E,F\in\mathrm{End}(\mathrm{End}(V))$. The canonical metric on $\mathrm{End}(\mathrm{End}(V))$,

$$(k^*_{\mathrm{End}(V),\mathrm{End}(V)})^{-1}\circ e_{\mathrm{End}(V),\mathrm{End}(V)}, \tag{3.16}$$

is not necessarily the same as the $b$ metric. The metrics can be shown to be different by example, if there exist $A,B\in\mathrm{End}(V)$, $\Psi,\Phi\in\mathrm{End}(V)^*$ such that $(h(A))(B)=0$ and $\Psi(A)\neq 0$ and $\Phi(B)\neq 0$. From Equation (3.5) and Equation (3.15),

$$\begin{aligned}&(b(k_{\mathrm{End}(V),\mathrm{End}(V)}(\Phi\otimes A)))(k_{\mathrm{End}(V),\mathrm{End}(V)}(\Psi\otimes B))\\ =\ &\Psi(g^{-1}(\Phi)))\cdot(h(A))(B)=0.\end{aligned}$$

The canonical metric applied to the same inputs has output

$$\begin{aligned}&Tr_{\mathrm{End}(V)}((k_{\mathrm{End}(V),\mathrm{End}(V)}(\Phi\otimes A))\circ(k_{\mathrm{End}(V),\mathrm{End}(V)}(\Psi\otimes B)))\\ =\ &Tr_{\mathrm{End}(V)}(\Phi(B)\cdot k_{\mathrm{End}(V),\mathrm{End}(V)}(\Psi\otimes A))\\ =\ &\Phi(B)\cdot\Psi(A)\neq 0.\end{aligned}$$

So even for $g$ and $h$ as in Example 3.143, the canonical metric is not the same as the metric canonically induced by canonical metrics!

CHAPTER 4

# Vector Valued Bilinear Forms

The notion of a bilinear form $h : V \to V^*$ can be generalized from the "scalar valued" case to a "vector valued" (or "$W$-valued," or "twisted") form $h : V \to \mathrm{Hom}(V, W)$, so that for inputs $v_1$, $v_2 \in V$, the output $(h(v_1))(v_2)$ is an element of $W$. In the same way as Example 1.55, vector valued bilinear functions $\mathbf{B} : V \times V \rightsquigarrow W$ correspond to $W$-valued bilinear forms on $V$, elements of the space $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$. Most of the properties of the scalar valued case generalize, but some of the canonical maps are different.

## 4.1. Transpose for vector valued forms

There would appear to be multiple ways to use the already considered canonical maps to define a transpose operation that switches the inputs for a $W$-valued form. One way would be to transform $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ into $\mathrm{Hom}(V, V^*) \otimes W$, and then apply $[T_V \otimes Id_W]$, where $T_V$ is the transpose for scalar valued forms from Definition 3.2. Another way would be to start from scratch with canonical maps from Chapter 1, which is the approach taken with Lemma 4.1 and Definition 4.2. Of course, these two ways end up with the same result, as shown in Lemma 4.5.

The following Lemma considers a more general domain $\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W))$, where $V_1$ and $V_2$ are not necessarily the same. The $d$ map in the diagram is a generalized double duality from Definition 1.13, the $t$ map is a generalized transpose from Definition 1.7, the canonical $q$ maps are as in Definition 1.46, and $s$ is a switching map.

LEMMA 4.1. *For any $V_1$, $V_2$, $W$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \arrow[r, "q_1"] \arrow[d, "t^W_{V_1, \mathrm{Hom}(V_2, W)}"] & \mathrm{Hom}(V_1 \otimes V_2, W) \arrow[dd, "\mathrm{Hom}(s, Id_W)"] \\
\mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2, W), W), \mathrm{Hom}(V_1, W)) \arrow[d, "\mathrm{Hom}(d_{V_2 W}, Id_{\mathrm{Hom}(V_1, W)})"] & \\
\mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W)) \arrow[r, "q_2"] & \mathrm{Hom}(V_2 \otimes V_1, W)
\end{tikzcd}$$

PROOF. For $u \in V_1$, $v \in V_2$, and $A \in \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W))$,

$$\begin{aligned}
& (\mathrm{Hom}(s, Id_W) \circ q_1)(A) : \\
v \otimes u \quad \mapsto \quad & ((q_1(A)) \circ s)(v \otimes u) = (q_1(A))(u \otimes v) = (A(u))(v), \\
& (q_2 \circ \mathrm{Hom}(d_{V_2 W}, Id_{\mathrm{Hom}(V_1, W)}) \circ t^W_{V_1, \mathrm{Hom}(V_2, W)})(A) : \\
v \otimes u \quad \mapsto \quad & (q_2((t^W_{V_1, \mathrm{Hom}(V_2, W)}(A)) \circ d_{V_2 W}))(v \otimes u) \\
= \quad & ((t^W_{V_1, \mathrm{Hom}(V_2, W)}(A))(d_{V_2 W}(v)))(u) = ((d_{V_2 W}(v)) \circ A)(u) = (A(u))(v).
\end{aligned}$$ ∎

Note that the composite $\mathrm{Hom}(s, Id_W)\circ q_1$ could be abbreviated as a re-ordered $q$ map, as in (1.6) from Notation 1.49. However, the relationship between the switching maps and the transpose maps in the following Definition will be more clear without this abbreviation, so here in Section 4.1 and in Section 4.2, all the $q$ maps will be the version from Definition 1.46 and not the variant (1.5) from Notation 1.49.

DEFINITION 4.2. Corresponding to the left column in the above diagram, let

$$T_{V_1,V_2;W} = \mathrm{Hom}(d_{V_2W}, Id_{\mathrm{Hom}(V_1,W)})\circ t^W_{V_1,\mathrm{Hom}(V_2,W)}. \tag{4.1}$$

NOTATION 4.3. In the special case $V = V_1 = V_2$, abbreviate $T_{V,V;W} = T_{V;W}$. In the case $W = \mathbb{K}$, $T_{V;\mathbb{K}}$ is exactly $T_V$ from Definition 3.2.

The above expression (4.1) for $T_{V_1,V_2;W}$ (and $T_{V;W}$) uses only Hom spaces and maps from Section 1.1, without referring to tensor products, the scalar field $\mathbb{K}$, scalar multiplication, or any dual space like $V^*$. The spaces and $s$ and $q$ maps in Lemma 4.1 use tensor products but no scalars.

LEMMA 4.4. *For any vector spaces $V_1$, $V_2$, $W$, $T_{V_1,V_2;W}$ is invertible. In particular, for $V_1 = V_2$, $T_{V_1;W}$ is an involution on* $\mathrm{Hom}(V_1, \mathrm{Hom}(V_1,W))$.

PROOF. The first claim follows from Lemma 4.1 and the invertibility of the $q$ maps (Lemma 1.47), and the diagram also shows that

$$T_{V_2,V_1;W} = T^{-1}_{V_1,V_2;W}. \tag{4.2}$$

The second claim is a special case of (4.2). For $h : V_1 \to \mathrm{Hom}(V_2,W)$, and $v \in V_1$, $u \in V_2$, it follows from Definition 4.2 that

$$((T_{V_1,V_2;W}(h))(u))(v) = (h(v))(u), \tag{4.3}$$

as in (3.1) from Lemma 3.3. Instead of using Lemma 4.1 or (4.3), Equation (4.2) can be checked directly from Definition 4.2, using Lemma 1.6, Lemma 1.14, and Lemma 1.17:

$$\begin{aligned}
T_{V_2,V_1;W}(T_{V_1,V_2;W}(h)) &= \mathrm{Hom}((t^W_{V_1,\mathrm{Hom}(V_2,W)}(h))\circ d_{V_2W}, Id_W)\circ d_{V_1W} \\
&= \mathrm{Hom}(d_{V_2W}, Id_W)\circ \mathrm{Hom}(\mathrm{Hom}(h, Id_W), Id_W)\circ d_{V_1W} \\
&= \mathrm{Hom}(d_{V_2W}, Id_W)\circ d_{\mathrm{Hom}(V_2,W),W}\circ h \\
&= h.
\end{aligned}$$

Relabeling the subscripts then gives the composite in the other order. $\blacksquare$

The following Lemma uses a canonical $n$ map from Definition 1.40, so that for $g\otimes w \in \mathrm{Hom}(V,V^*)\otimes W$, $(n(g\otimes w))(v) = (g(v))\otimes w$.

LEMMA 4.5. *For any $V$, $W$, the following diagram is commutative. If $V$ or $W$ is finite-dimensional, then the $k_{VW}$ and $n$ maps in the diagram are invertible.*

$$\begin{tikzcd}
\mathrm{Hom}(V,\mathrm{Hom}(V,W)) \arrow[d, "T_{V;W}"'] \arrow[dr, "t^W_{V,\mathrm{Hom}(V,W)}"] & \mathrm{Hom}(V,V^*\otimes W) \arrow[l, "\mathrm{Hom}(Id_V,k_{VW})"'] & \mathrm{Hom}(V,V^*)\otimes W \arrow[l, "n"'] \arrow[d, "{[T_V\otimes Id_W]}"'] \\
 & \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V,W),W),\mathrm{Hom}(V,W)) \arrow[dl, "\mathrm{Hom}(d_{VW},Id_{\mathrm{Hom}(V,W)})"] & \\
\mathrm{Hom}(V,\mathrm{Hom}(V,W)) & \mathrm{Hom}(V,V^*\otimes W) \arrow[l, "\mathrm{Hom}(Id_V,k_{VW})"'] & \mathrm{Hom}(V,V^*)\otimes W \arrow[l, "n"']
\end{tikzcd}$$

PROOF. The left triangle is Definition 4.2 for $T_{V;W}$. For the right part of the diagram, starting with $g\otimes w \in \mathrm{Hom}(V, V^*) \otimes W$,

$$\begin{array}{rcl}
g \otimes w & \mapsto & (\mathrm{Hom}(d_{VW}, Id_{\mathrm{Hom}(V,W)}) \circ t^{W}_{V,\mathrm{Hom}(V,W)} \circ \mathrm{Hom}(Id_V, k_{VW}) \circ n)(g\otimes w) : \\
v & \mapsto & ((t^{W}_{V,\mathrm{Hom}(V,W)}(k_{VW} \circ (n(g \otimes w)))) \circ d_{VW})(v) \\
& = & (d_{VW}(v)) \circ k_{VW} \circ (n(g \otimes w)) : \\
u & \mapsto & (d_{VW}(v))(k_{VW}((g(u)) \otimes w)) \\
& = & (k_{VW}((g(u)) \otimes w))(v) = (g(u))(v) \cdot w, \\
g\otimes w & \mapsto & (\mathrm{Hom}(Id_V, k_{VW}) \circ n \circ [T_V \otimes Id_W])(g \otimes w) : \\
v & \mapsto & k_{VW}((n((T_V(g)) \otimes w))(v)) \\
& = & k_{VW}(((T_V(g))(v)) \otimes w) : \\
u & \mapsto & ((T_V(g))(v))(u) \cdot w = (g(u))(v) \cdot w.
\end{array}$$

The invertibility of the canonical maps was stated in Lemma 1.64 and Lemma 1.44. ∎

LEMMA 4.6. *For any vector spaces $U_1$, $U_2$, $V_1$, $V_2$, $W_1$, $W_2$, and any maps $E : U_1 \to V_1$, $F : U_2 \to V_2$, $G : W_1 \to W_2$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W_1)) \arrow[r, "T_{V_1,V_2;W_1}"] \arrow[d, "\mathrm{Hom}(E,\mathrm{Hom}(F,G))"'] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W_1)) \arrow[d, "\mathrm{Hom}(F,\mathrm{Hom}(E,G))"] \\
\mathrm{Hom}(U_1, \mathrm{Hom}(U_2, W_2)) \arrow[r, "T_{U_1,U_2,W_2}"] & \mathrm{Hom}(U_2, \mathrm{Hom}(U_1, W_2))
\end{tikzcd}$$

PROOF. The claim could be checked by calculating how the composites act on pairs of input vectors, as in Equation (4.3) from the Proof of Lemma 4.4. The following proof instead shows how the claim follows from only the elementary properties of the $t$ and $d$ maps.

The diagram can be expanded using Definition 4.2 and Lemma 1.6:

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W_1)) \arrow[rr, "T_{V_1,V_2;W_1}"] \arrow[d, "\mathrm{Hom}(Id_{V_1},\mathrm{Hom}(Id_{V_2},G))"'] & & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W_1)) \arrow[d, "\mathrm{Hom}(Id_{V_2},\mathrm{Hom}(Id_{V_1},G))"'] \\
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W_2)) \arrow[r, "t^{W_2}_{V_1,\mathrm{Hom}(V_2,W_2)}"'] \arrow[d, "\mathrm{Hom}(E,\mathrm{Hom}(F,Id_{W_2}))"'] & M_1 \arrow[r, "\mathrm{Hom}(d_{V_2W_2},Id_{\mathrm{Hom}(V_1,W_2)})"'] \arrow[d, "a_1"'] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W_2)) \arrow[d, "\mathrm{Hom}(F,\mathrm{Hom}(E,Id_{W_2}))"'] \\
\mathrm{Hom}(U_1, \mathrm{Hom}(U_2, W_2)) \arrow[r, "t^{W_2}_{U_1,\mathrm{Hom}(U_2,W_2)}"'] & M_2 \arrow[r, "\mathrm{Hom}(d_{U_2W_2},Id_{\mathrm{Hom}(U_1,W_2)})"'] & \mathrm{Hom}(U_2, \mathrm{Hom}(U_1, W_2))
\end{tikzcd}$$

where

$$\begin{array}{rcl}
M_1 & = & \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2, W_2), W_2), \mathrm{Hom}(V_1, W_2)) \\
M_2 & = & \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(U_2, W_2), W_2), \mathrm{Hom}(U_1, W_2)) \\
a_1 & = & \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(F, Id_{W_2}), Id_{W_2}), \mathrm{Hom}(E, Id_{W_2})).
\end{array}$$

The lower left square is commutative by Lemma 1.8. The lower right square is commutative by Lemma 1.6 and Lemma 1.14. These steps are analogous to the steps in the Proof of Lemma 3.8.

The commutativity of the upper block states that for $h : V_1 \to \mathrm{Hom}(V_2, W_1)$,

$$T_{V_1,V_2;W_2}(\mathrm{Hom}(Id_{V_2}, G) \circ h) = \mathrm{Hom}(Id_{V_1}, G) \circ (T_{V_1,V_2;W_1}(h)).$$

The following diagram expands the upper block of the previous diagram, so that the compositions down the left and right sides are $T_{V_1,V_2;W_1}$ and $T_{V_1,V_2;W_2}$ from Definition 4.2, and the claim of the Lemma follows from the commutativity of the diagram.

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W_1)) \arrow[rr, "\mathrm{Hom}(Id_{V_1},\mathrm{Hom}(Id_{V_2},G))"] \arrow[ddr, "t^{W_2}_{V_1,\mathrm{Hom}(V_2,W_1)}"] \arrow[dddd, "t^{W_1}_{V_1,\mathrm{Hom}(V_2,W_1)}"] & & \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W_2)) \arrow[d, "t^{W_2}_{V_1,\mathrm{Hom}(V_2,W_2)}"'] \\
& & \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2, W_2), W_2), \mathrm{Hom}(V_1, W_2)) \arrow[dddd, "\mathrm{Hom}(d_{V_2W_2},Id_{\mathrm{Hom}(V_1,W_2)})"'] \\
& \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2,W_1),W_2), \mathrm{Hom}(V_1,W_2)) \arrow[ur, "a_2"] \arrow[d, "a_3"'] & \\
& \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2,W_1),W_1), \mathrm{Hom}(V_1,W_2)) \arrow[ddr, "\mathrm{Hom}(d_{V_2W_1},Id_{\mathrm{Hom}(V_1,W_2)})"] & \\
\mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(V_2, W_1), W_1), \mathrm{Hom}(V_1, W_1)) \arrow[ur, "a_4"] \arrow[d, "\mathrm{Hom}(d_{V_2W_1},Id_{\mathrm{Hom}(V_1,W_1)})"] & & \\
\mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W_1)) \arrow[rr, "\mathrm{Hom}(Id_{V_2},\mathrm{Hom}(Id_{V_1},G))"'] & & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W_2))
\end{tikzcd}$$

The inside arrows are:

$$\begin{aligned}
a_2 &= \mathrm{Hom}(\mathrm{Hom}(\mathrm{Hom}(Id_{V_2}, G), Id_{W_2}), Id_{\mathrm{Hom}(V_1,W_2)}) \\
a_3 &= \mathrm{Hom}(\mathrm{Hom}(Id_{\mathrm{Hom}(V_2,W_1)}, G), Id_{\mathrm{Hom}(V_1,W_2)}) \\
a_4 &= \mathrm{Hom}(Id_{\mathrm{Hom}(\mathrm{Hom}(V_2,W_1),W_1)}, \mathrm{Hom}(Id_{V_1}, G)).
\end{aligned}$$

The upper square and the left square are both commutative by Lemma 1.8. The lower square is commutative by Lemma 1.6. The commutativity of the right block follows from Lemma 1.6 and this special case of Lemma 1.14:

$$\mathrm{Hom}(Id_{\mathrm{Hom}(V_2,W_1)}, G) \circ d_{V_2W_1} = \mathrm{Hom}(\mathrm{Hom}(Id_{V_2}, G), Id_{W_2}) \circ d_{V_2W_2}.$$

∎

Similarly to Lemma 4.5, the following few Lemmas use canonical $n$ maps — all labeled $n$, even when some spaces appear in a different order, as in Notation 1.41, so their domain, target, and formula are as indicated by their position in the diagram.

LEMMA 4.7. *For any $U$, $V_1$, $V_2$, $W$, the following diagram is commutative. If $U$ is finite-dimensional, or $V_1$ and $V_2$ are both finite-dimensional, then all the maps in the diagram are invertible.*

$$\begin{tikzcd}
\mathrm{Hom}(V_1, U\otimes\mathrm{Hom}(V_2,W)) \arrow[r, "\mathrm{Hom}(Id_{V_1},n_1)"] & \mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W\otimes U)) \arrow[ddd, "T_{V_1,V_2;W\otimes U}"] \\
\mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W))\otimes U \arrow[u, "n_2"] \arrow[d, "{[T_{V_1,V_2,W}\otimes Id_U]}"] & \\
\mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W))\otimes U \arrow[d, "n_3"] & \\
\mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W)\otimes U) \arrow[r, "\mathrm{Hom}(Id_{V_2},n_4)"] & \mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W\otimes U))
\end{tikzcd}$$

PROOF. Replacing the $T_{V_1,V_2;W}$ and $T_{V_1,V_2;W\otimes U}$ downward arrows in the above diagram by the composites with the $q$ maps from Lemma 4.1 gives this diagram.

$$\begin{tikzcd}
\mathrm{Hom}(V_1, U\otimes\mathrm{Hom}(V_2,W)) \arrow[r, "\mathrm{Hom}(Id_{V_1},n_1)"] & \mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W\otimes U)) \arrow[dd, "q_3"] \\
\mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W))\otimes U \arrow[u, "n_2"] \arrow[d, "{[q_1\otimes Id_U]}"] & \\
\mathrm{Hom}(V_1\otimes V_2,W)\otimes U \arrow[r, "n_5"'] \arrow[d, "{[\mathrm{Hom}(s,Id_W)\otimes Id_U]}"] & \mathrm{Hom}(V_1\otimes V_2,W\otimes U) \arrow[d, "{\mathrm{Hom}(s,Id_{W\otimes U})}"] \\
\mathrm{Hom}(V_2\otimes V_1,W)\otimes U \arrow[r, "n_6"'] & \mathrm{Hom}(V_2\otimes V_1,W\otimes U) \\
\mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W))\otimes U \arrow[u, "{[q_2\otimes Id_U]}"] \arrow[d, "n_3"] & \\
\mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W)\otimes U) \arrow[r, "\mathrm{Hom}(Id_{V_2},n_4)"] & \mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W\otimes U)) \arrow[uu, "q_4"]
\end{tikzcd}$$

The middle block is commutative by Lemma 1.42. To check the top square, for $h\otimes u\in\mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W))\otimes U$, $v\otimes x\in V_1\otimes V_2$,

$$\begin{aligned}
(q_3\circ\mathrm{Hom}(Id_{V_1},n_1)\circ n_2)(h\otimes u): v\otimes x &\mapsto (q_3(n_1\circ(n_2(h\otimes u))))(v\otimes x)\\
&= ((n_1\circ(n_2(h\otimes u)))(v))(x)\\
&= (n_1(u\otimes(h(v))))(x)\\
&= ((h(v))(x))\otimes u,\\
(n_5\circ[q_1\otimes Id_U])(h\otimes u): v\otimes x &\mapsto (n_5((q_1(h))\otimes u))(v\otimes x)\\
&= ((q_1(h))(v\otimes x))\otimes u\\
&= ((h(v))(x))\otimes u.
\end{aligned}$$

The lowest square is analogous, with some re-ordering. ∎

The composite $\mathrm{Hom}(Id_{V_1}, n_1) \circ n_2$ appearing in the upper left corner of the diagram from Lemma 4.7 is equal to a composite of the following form, using different variants of the $n$ maps, as in Notation 1.41:

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W) \otimes U) \arrow[r] & \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W \otimes U)) \\
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \otimes U \arrow[u] &
\end{tikzcd} \tag{4.4}$$

The spaces can be re-ordered in various ways to state results analogous to Lemma 4.7, with other versions of the $n$ maps but essentially the same Proof. The following Lemma 4.8 is an analogue of Lemma 4.7 but with a longer composite of $n$ maps.

LEMMA 4.8. *For any $U_1$, $U_2$, $V_1$, $V_2$, $W$, the following diagram is commutative.*

$$\begin{tikzcd}
U_1 \otimes \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \otimes U_2 \arrow[r, "{[Id_{U_1} \otimes [T_{V_1,V_2;W} \otimes Id_{U_2}]]}"] \arrow[d, "{[Id_{U_1} \otimes n_2]}"] & U_1 \otimes \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W)) \otimes U_2 \arrow[d, "{[Id_{U_1} \otimes n_3]}"'] \\
U_1 \otimes \mathrm{Hom}(V_1, U_2 \otimes \mathrm{Hom}(V_2, W)) \arrow[d, "n_7"] & U_1 \otimes \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W) \otimes U_2) \arrow[d, "n_8"'] \\
\mathrm{Hom}(V_1, U_1 \otimes U_2 \otimes \mathrm{Hom}(V_2, W)) \arrow[d, "{\mathrm{Hom}(Id_{V_1}, [Id_{U_1} \otimes n_1])}"] & \mathrm{Hom}(V_2, U_1 \otimes \mathrm{Hom}(V_1, W) \otimes U_2) \arrow[d, "{\mathrm{Hom}(Id_{V_2}, [Id_{U_1} \otimes n_4])}"'] \\
\mathrm{Hom}(V_1, U_1 \otimes \mathrm{Hom}(V_2, W \otimes U_2)) \arrow[d, "{\mathrm{Hom}(Id_{V_1}, n_9)}"] & \mathrm{Hom}(V_2, U_1 \otimes \mathrm{Hom}(V_1, W \otimes U_2)) \arrow[d, "{\mathrm{Hom}(Id_{V_2}, n_{10})}"'] \\
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, U_1 \otimes W \otimes U_2)) \arrow[r, "T_{V_1,V_2;U_1 \otimes W \otimes U_2}"'] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, U_1 \otimes W \otimes U_2))
\end{tikzcd}$$

PROOF. As remarked after the Proof of Lemma 4.7, the vertical composites of $n$ maps could be rearranged into composites of different $n$ maps with the spaces in different order, as in (4.4), or using variations on Lemma 1.45. Such rearrangements could be used in a proof of this Lemma as stated or to state and prove analogous results.

The following Proof is more direct, and analogous to that of Lemma 4.7, replacing the $T$ maps with composites involving the $q$ maps from Lemma 4.1. In the statement of the Lemma and the following diagram, the maps labeled $n_1, \ldots, n_4, q_1, q_2$

are the same that appear in Lemma 4.7.

$$\begin{tikzcd}
U_1 \otimes \mathrm{Hom}(V_1 \otimes V_2, W) \otimes U_2 \arrow[r, "{[Id_{U_1} \otimes [\mathrm{Hom}(s, Id_W) \otimes Id_{U_2}]]}"] & U_1 \otimes \mathrm{Hom}(V_2 \otimes V_1, W) \otimes U_2 \\
U_1 \otimes \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \otimes U_2 \arrow[u, "{[Id_{U_1} \otimes [q_1 \otimes Id_{U_2}]]}"'] \arrow[r, "{[Id_{U_1} \otimes [T_{V_1,V_2;W} \otimes Id_{U_2}]]}"] \arrow[d, "{n_7 \circ [Id_{U_1} \otimes n_2]}"] & U_1 \otimes \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W)) \otimes U_2 \arrow[u, "{[Id_{U_1} \otimes [q_2 \otimes Id_{U_2}]]}"] \arrow[d, "{n_8 \circ [Id_{U_1} \otimes n_3]}"'] \\
\mathrm{Hom}(V_1, U_1 \otimes U_2 \otimes \mathrm{Hom}(V_2, W)) \arrow[d, "{\mathrm{Hom}(Id_{V_1}, n_9 \circ [Id_{U_1} \otimes n_1])}"] & \mathrm{Hom}(V_2, U_1 \otimes \mathrm{Hom}(V_1, W) \otimes U_2) \arrow[d, "{\mathrm{Hom}(Id_{V_2}, n_{10} \circ [Id_{U_1} \otimes n_4])}"'] \\
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, U_1 \otimes W \otimes U_2)) \arrow[r, "{T_{V_1,V_2;U_1 \otimes W \otimes U_2}}"'] \arrow[d, "q_5"] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, U_1 \otimes W \otimes U_2)) \arrow[d, "q_6"] \\
\mathrm{Hom}(V_1 \otimes V_2, U_1 \otimes W \otimes U_2) \arrow[r, "{\mathrm{Hom}(s, Id_{U_1 \otimes W \otimes U_2})}"] & \mathrm{Hom}(V_2 \otimes V_1, U_1 \otimes W \otimes U_2)
\end{tikzcd}$$

The $q$ and $s$ maps are invertible, and the commutativity around the outside of the diagram can be checked directly, using the formula for $q^{-1}$ from Lemma 1.47. Starting in the upper left corner with $U_1 \otimes \mathrm{Hom}(V_1 \otimes V_2, W) \otimes U_2$, for $A \in \mathrm{Hom}(V_1 \otimes V_2, W)$, $u \in U_1$, $v \in U_2$, $y \in V_1$, $z \in V_2$,

$$\begin{aligned}
u \otimes A \otimes v &\mapsto q_6(n_{10} \circ [Id_{U_1} \otimes n_4] \circ (n_8([Id_{U_1} \otimes n_3](u \otimes (q_2^{-1}(A \circ s)) \otimes v)))) : \\
z \otimes y &\mapsto ((n_{10} \circ [Id_{U_1} \otimes n_4] \circ (n_8([Id_{U_1} \otimes n_3](u \otimes (q_2^{-1}(A \circ s)) \otimes v))))(z))(y) \\
&= (n_{10}([Id_{U_1} \otimes n_4](u \otimes ((q_2^{-1}(A \circ s))(z)) \otimes v)))(y) \\
&= u \otimes (((q_2^{-1}(A \circ s))(z))(y)) \otimes v \\
&= u \otimes ((A \circ s)(z \otimes y)) \otimes v = u \otimes (A(y \otimes z)) \otimes v, \\
u \otimes A \otimes v &\mapsto (q_5(n_9 \circ [Id_{U_1} \otimes n_1] \circ (n_7([Id_{U_1} \otimes n_2](u \otimes (q_1^{-1}(A)) \otimes v))))) \circ s : \\
z \otimes y &\mapsto ((n_9 \circ [Id_{U_1} \otimes n_1] \circ (n_7([Id_{U_1} \otimes n_2](u \otimes (q_1^{-1}(A)) \otimes v))))(y))(z) \\
&= (n_9([Id_{U_1} \otimes n_1](u \otimes v \otimes ((q_1^{-1}(A))(y)))))(z) \\
&= u \otimes (((q_1^{-1}(A))(y))(z)) \otimes v = u \otimes (A(y \otimes z)) \otimes v.
\end{aligned}$$

∎

The following Lemma 4.9 shows how $T_{V_1,V_2;W}$ is related to some switching maps, which are involutions in the case $V_1 = V_2$; an analogue for the scalar case $T_V$ is Theorem 3.115.

LEMMA 4.9. *For any $V_1$, $V_2$, $W$, the following diagram is commutative. If $V_1$ and $V_2$ are both finite-dimensional, then all the arrows are invertible.*

$$\begin{tikzcd}
V_1^*\otimes V_2^*\otimes W \arrow[rr, "{[s_2\otimes Id_W]}"] \arrow[d, "{[Id_{V_1^*}\otimes k_{V_2W}]}"'] \arrow[ddr, "k_{V_1,V_2^*\otimes W}"] & & V_2^*\otimes V_1^*\otimes W \arrow[dl, "{[Id_{V_2^*}\otimes k_{V_1W}]}"'] \arrow[d, "k_{V_2,V_1^*\otimes W}"] \\
V_1^*\otimes \mathrm{Hom}(V_2,W) \arrow[dd, "k_{V_1,\mathrm{Hom}(V_2,W)}"] & V_2^*\otimes \mathrm{Hom}(V_1,W) \arrow[d, "n"'] \arrow[ddr, "k_{V_2,\mathrm{Hom}(V_1,W)}"'] & \mathrm{Hom}(V_2,V_1^*\otimes W) \arrow[dd, "{\mathrm{Hom}(Id_{V_2},k_{V_1W})}"'] \\
 & \mathrm{Hom}(V_1,V_2^*\otimes W) \arrow[dl, "{\mathrm{Hom}(Id_{V_1},k_{V_2W})}"] & \\
\mathrm{Hom}(V_1,\mathrm{Hom}(V_2,W)) \arrow[rr, "T_{V_1,V_2;W}"'] \arrow[d, "q_1"'] & & \mathrm{Hom}(V_2,\mathrm{Hom}(V_1,W)) \arrow[d, "q_2"] \\
\mathrm{Hom}(V_1\otimes V_2,W) \arrow[rr, "{\mathrm{Hom}(s_1,Id_W)}"] & & \mathrm{Hom}(V_2\otimes V_1,W) \\
(V_1\otimes V_2)^*\otimes W \arrow[u, "k_{V_1\otimes V_2,W}"] \arrow[rr, "{[s_1^*\otimes Id_W]}"] & & (V_2\otimes V_1)^*\otimes W \arrow[u, "k_{V_2\otimes V_1,W}"'] \\
\mathrm{Hom}(V_1\otimes V_2,\mathbb{K}\otimes\mathbb{K})\otimes W \arrow[u, "{[\mathrm{Hom}(Id_{V_1\otimes V_2},l)\otimes Id_W]}"'] & & \mathrm{Hom}(V_2\otimes V_1,\mathbb{K}\otimes\mathbb{K})\otimes W \arrow[u, "{[\mathrm{Hom}(Id_{V_2\otimes V_1},l)\otimes Id_W]}"] \\
V_1^*\otimes V_2^*\otimes W \arrow[u, "{[j_1\otimes Id_W]}"] \arrow[rr, "{[s_2\otimes Id_W]}"] & & V_2^*\otimes V_1^*\otimes W \arrow[u, "{[j_2\otimes Id_W]}"']
\end{tikzcd}$$

PROOF. The block with the $q$ maps is exactly Lemma 4.1, and the next lower block is commutative by Lemma 1.62. The lowest block is easy to check, where the abbreviation $\tilde{j}_1 = \mathrm{Hom}(Id_{V_1\otimes V_2}, l)\circ j_1$ as in Notation 2.43 could be used, so that the vertical composite is $[\tilde{j}_1\otimes Id_W]$. Inside the upper rectangle, the blocks on the left and right are commutative by Lemma 1.62 again, and its upper block is commutative by a variation on Lemma 1.65. The lower block in the upper rectangle is commutative: using Definition 4.2 and $\phi\otimes A\in V_2^*\otimes\mathrm{Hom}(V_1,W)$, $u\in V_1$, $v\in V_2$,

$$\begin{array}{rcl}
 & & T_{V_1,V_2;W}\circ\mathrm{Hom}(Id_{V_1},k_{V_2,W})\circ n : \\
\phi\otimes A & \mapsto & (t^W_{V_1,\mathrm{Hom}(V_2,W)}(k_{V_2,W}\circ(n(\phi\otimes A))))\circ d_{V_2,W} : \\
v & \mapsto & (d_{V_2,W}(v))\circ k_{V_2,W}\circ(n(\phi\otimes A)) : \\
u & \mapsto & (k_{V_2,W}(\phi\otimes(A(u))))(v) \\
 & = & \phi(v)\cdot(A(u)) = ((k_{V_2,\mathrm{Hom}(V_1,W)}(\phi\otimes A))(v))(u).
\end{array}$$

So the top rectangle is commutative. ∎

LEMMA 4.10. *For any $V_1$, $V_2$, $V_3$, $W$, if $V_1$ and $V_2$ are finite-dimensional, then all the arrows in the following diagram are invertible and the diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \otimes V_3 \arrow[r, "{[T_{V_1,V_2;W} \otimes Id_{V_3}]}"] \arrow[d, "n_1"'] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W)) \otimes V_3 \arrow[d, "n_2"] \\
\mathrm{Hom}(V_1, V_3 \otimes \mathrm{Hom}(V_2, W)) & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W) \otimes V_3) \\
\mathrm{Hom}(V_1, V_3 \otimes V_2^* \otimes W) \arrow[u, "{\mathrm{Hom}(Id_{V_1}, [Id_{V_3} \otimes k_{V_2 W}])}"'] & \mathrm{Hom}(V_2, V_1^* \otimes W \otimes V_3) \arrow[u, "{\mathrm{Hom}(Id_{V_2}, [k_{V_1 W} \otimes Id_{V_3}])}"'] \\
V_1^* \otimes V_3 \otimes V_2^* \otimes W \arrow[r, "s_1"] \arrow[u, "k_{V_1, V_3 \otimes V_2^* \otimes W}"'] & V_2^* \otimes V_1^* \otimes W \otimes V_3 \arrow[u, "k_{V_2, V_1^* \otimes W \otimes V_3}"']
\end{tikzcd}$$

PROOF. The following diagram is commutative, where the column on the left matches the left column in the above diagram, and the column on the right uses two maps from the diagram in Lemma 4.9.

$$\begin{tikzcd}
\mathrm{Hom}(V_1, V_3 \otimes \mathrm{Hom}(V_2, W)) & \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \otimes V_3 \arrow[l, "n_1"] \\
\mathrm{Hom}(V_1, V_3 \otimes V_2^* \otimes W) \arrow[u, "{\mathrm{Hom}(Id_{V_1}, [Id_{V_3} \otimes k_{V_2 W}])}"] & \mathrm{Hom}(V_1, V_2^* \otimes W) \otimes V_3 \arrow[l, "n_3"] \arrow[u, "{[\mathrm{Hom}(Id_{V_1}, k_{V_2 W}) \otimes Id_{V_3}]}"'] \\
V_1^* \otimes V_3 \otimes V_2^* \otimes W \arrow[u, "k_{V_1, V_3 \otimes V_2^* \otimes W}"] & V_1^* \otimes V_2^* \otimes W \otimes V_3 \arrow[l, "{[Id_{V_1^*} \otimes s]}"] \arrow[u, "{[k_{V_1, V_2^* \otimes W} \otimes Id_{V_3}]}"']
\end{tikzcd}$$

The upper square is commutative by Lemma 1.42, and the lower square by a variation on Lemma 1.65. The above diagram is abbreviated to appear in the left block of the following diagram, by labeling its four corners ($M_{11}$, $M_{12}$, $M_{21}$, $M_{22}$), and its upward vertical composites as the vertical arrows $a_1$, $[a_2 \otimes Id_{V_3}]$.

$$\begin{tikzcd}
M_{11} & M_{12} \arrow[l, "n_1"'] \arrow[r, "{[T_{V_1,V_2;W} \otimes Id_{V_3}]}"] & M_{13} \arrow[r, "n_2"] & \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W) \otimes V_3) \\
 & & & \mathrm{Hom}(V_2, V_1^* \otimes W \otimes V_3) \arrow[u, "{\mathrm{Hom}(Id_{V_2}, [k_{V_1 W} \otimes Id_{V_3}])}"] \\
M_{21} \arrow[uu, "a_1"] & M_{22} \arrow[l, "{[Id_{V_1^*} \otimes s]}"] \arrow[r, "{[[s_2 \otimes Id_W] \otimes Id_{V_3}]}"] \arrow[uu, "{[a_2 \otimes Id_{V_3}]}"] & M_{23} \arrow[uu, "{[a_3 \otimes Id_{V_3}]}"] \arrow[ur, "k_{V_2, V_1^* \otimes W \otimes V_3}"']
\end{tikzcd}$$

The other two spaces are similarly related to the right column in the top square from Lemma 4.9:

$$\begin{aligned}
M_{13} &= \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W)) \otimes V_3 \\
M_{23} &= V_2^* \otimes V_1^* \otimes W \otimes V_3, \\
a_3 &= \mathrm{Hom}(Id_{V_2}, k_{V_1 W}) \circ k_{V_2, V_1^* \otimes W},
\end{aligned}$$

so that the commutativity of the middle block follows from the commutativity of the top block from Lemma 4.9, together with Lemma 1.36. The right block is a mirror image analogue of the left block, but without the switching map, so the verification that it is commutative again uses Lemma 1.42 and Lemma 1.65. Letting $s_1 = [[s_2 \otimes Id_W] \otimes Id_{V_3}] \circ [Id_{V_1^*} \otimes s^{-1}]$, the commutativity around the outside of the diagram gives the claim of the Lemma. ∎

EXERCISE 4.11. If $V_1$, $V_2$, $W$ have metrics $g_1$, $g_2$, $h$, then

$$T_{V_1,V_2;W} : \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, W)) \to \mathrm{Hom}(V_2, \mathrm{Hom}(V_1, W))$$

is an isometry with respect to the induced metrics.

HINT. From Lemma 4.1, $T_{V_1,V_2;W} = q_2^{-1} \circ \mathrm{Hom}(s, Id_W) \circ q_1$ is a composite of isometries, using Theorem 3.45, Exercise 3.66, and Exercise 3.77. In the special case $V_1 = V_2$, $g_1 = g_2$, Exercise 3.122 applies to the involution $T_{V_1;W}$. A further special case is $W = \mathbb{K}$, so Corollary 3.48 on the involution $T_V$ follows from this claim and Exercise 3.74. ∎

The transpose for bilinear forms can be applied to vector valued trilinear forms: elements of $\mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, U)))$, to switch the first and second, or second and third, inputs in expressions such as $((h(x))(y))(z) \in U$. A map switching the first and third inputs can be expressed in terms of $T$ maps, or in terms of $q$ and $s$ maps in analogy with Lemma 4.1, but in more than one way, as shown by the following Lemma 4.12 and Equation (4.5).

LEMMA 4.12. *For any vector spaces $X$, $Y$, $Z$, $U$, the following diagram is commutative.*

$$\begin{tikzcd}
 & \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, U))) \arrow[dl, "\mathrm{Hom}(Id_X, T_{Y,Z;U})"'] \arrow[dr, "T_{X,Y;\mathrm{Hom}(Z,U)}"] & \\
\mathrm{Hom}(X, \mathrm{Hom}(Z, \mathrm{Hom}(Y, U))) \arrow[d, "T_{X,Z;\mathrm{Hom}(Y,U)}"] & & \mathrm{Hom}(Y, \mathrm{Hom}(X, \mathrm{Hom}(Z, U))) \arrow[d, "\mathrm{Hom}(Id_Y, T_{X,Z;U})"'] \\
\mathrm{Hom}(Z, \mathrm{Hom}(X, \mathrm{Hom}(Y, U))) \arrow[dr, "\mathrm{Hom}(Id_Z, T_{X,Y;U})"'] & & \mathrm{Hom}(Y, \mathrm{Hom}(Z, \mathrm{Hom}(X, U))) \arrow[dl, "T_{Y,Z;\mathrm{Hom}(X,U)}"] \\
 & \mathrm{Hom}(Z, \mathrm{Hom}(Y, \mathrm{Hom}(X, U))) &
\end{tikzcd}$$

PROOF. The claim follows from finding a map

$$a_1 : \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathrm{Hom}(Z, U))) \to \mathrm{Hom}(Z, \mathrm{Hom}(Y, \mathrm{Hom}(X, U))),$$

equal to both downward composites on the left and right sides of the above diagram. This first diagram corresponds to the right side.

$$\begin{tikzcd}
\mathrm{Hom}(X,\mathrm{Hom}(Y,\mathrm{Hom}(Z,U))) \arrow[drr, "T_{X,Y;\mathrm{Hom}(Z,U)}"] \arrow[ddr, "q_1"] & & \\
& & \mathrm{Hom}(Y,\mathrm{Hom}(X,\mathrm{Hom}(Z,U))) \arrow[ddl, "q_2"] \arrow[dddl, "\mathrm{Hom}(Id_Y,q_5)"] \arrow[ddddddd, "\mathrm{Hom}(Id_Y,T_{X,Z;U})"] \\
& \mathrm{Hom}(X\otimes Y,\mathrm{Hom}(Z,U)) \arrow[dl, "q_3"'] \arrow[d, "\mathrm{Hom}(s_1,Id_{\mathrm{Hom}(Z,U)})"] & \\
\mathrm{Hom}(X\otimes Y\otimes Z,U) \arrow[d, "\mathrm{Hom}({[s_1\otimes Id_Z]},Id_U)"] & \mathrm{Hom}(Y\otimes X,\mathrm{Hom}(Z,U)) \arrow[dl, "q_3'"] & \\
\mathrm{Hom}(Y\otimes X\otimes Z,U) \arrow[d, "\mathrm{Hom}({[Id_Y\otimes s_2]},Id_U)"] & \mathrm{Hom}(Y,\mathrm{Hom}(X\otimes Z,U)) \arrow[l, "q_4"'] \arrow[d, "\mathrm{Hom}(Id_Y,\mathrm{Hom}(s_2,Id_U))"] & \\
\mathrm{Hom}(Y\otimes Z\otimes X,U) \arrow[d, "\mathrm{Hom}({[s_3\otimes Id_X]},Id_U)"] & \mathrm{Hom}(Y,\mathrm{Hom}(Z\otimes X),U) \arrow[l, "q_4'"] & \\
\mathrm{Hom}(Z\otimes Y\otimes X,U) & \mathrm{Hom}(Y\otimes Z,\mathrm{Hom}(X,U)) \arrow[ul, "q_3''"'] \arrow[d, "\mathrm{Hom}(s_3,Id_{\mathrm{Hom}(X,U)})"] & \\
& \mathrm{Hom}(Z\otimes Y,\mathrm{Hom}(X,U)) \arrow[ul, "q_3'''"] & \\
& & \mathrm{Hom}(Y,\mathrm{Hom}(Z,\mathrm{Hom}(X,U))) \arrow[uuul, "\mathrm{Hom}(Id_Y,q_6)"'] \arrow[uul, "q_2'"'] \arrow[dll, "T_{Y,Z;\mathrm{Hom}(X,U)}"] \\
\mathrm{Hom}(Z,\mathrm{Hom}(Y,\mathrm{Hom}(X,U))) \arrow[uur, "q_1'"] & &
\end{tikzcd}$$

All the arrows are invertible — the $q$ maps convert the transpose maps to switching maps (recall the convention from Lemma 4.1 that all the $q$ maps here are ordered as in Definition 1.46). The right blocks are commutative by Lemma 4.1 and Definition 4.2, with the right center block also using Lemma 1.6. The three left blocks are commutative by Lemma 1.50 and the two center blocks by Lemma 1.51.

This second diagram, corresponding to the left side of the claim, is not exactly a mirror image of the first but is commutative in the same way, using Lemma 4.1, Lemma 1.6, Lemma 1.50, and Lemma 1.51.

$$\begin{tikzcd}
& & \mathrm{Hom}(X,\mathrm{Hom}(Y,\mathrm{Hom}(Z,U))) \arrow[dll, "\mathrm{Hom}(Id_X,T_{Y,Z;U})"'] \arrow[ddl, "\mathrm{Hom}(Id_X,q_7)"] \\
\mathrm{Hom}(X,\mathrm{Hom}(Z,\mathrm{Hom}(Y,U))) \arrow[ddr, "\mathrm{Hom}(Id_X,q_8)"] \arrow[dddr, "q_{11}"'] \arrow[ddddddd, "T_{X,Z;\mathrm{Hom}(Y,U)}"] & & \\
& \mathrm{Hom}(X,\mathrm{Hom}(Y\otimes Z,U)) \arrow[d, "\mathrm{Hom}(Id_X,\mathrm{Hom}(s_3,Id_U))"] \arrow[dr, "q_9"] & \\
& \mathrm{Hom}(X,\mathrm{Hom}(Z\otimes Y,U)) \arrow[dr, "q_9'"'] & \mathrm{Hom}(X\otimes Y\otimes Z,U) \arrow[d, "\mathrm{Hom}([Id_X\otimes s_3],Id_U)"'] \\
& \mathrm{Hom}(X\otimes Z,\mathrm{Hom}(Y,U)) \arrow[r, "q_{10}"] \arrow[d, "\mathrm{Hom}(s_2,Id_{\mathrm{Hom}(Y,U)})"'] & \mathrm{Hom}(X\otimes Z\otimes Y,U) \arrow[d, "\mathrm{Hom}([s_2\otimes Id_Y],Id_U)"'] \\
& \mathrm{Hom}(Z\otimes X,\mathrm{Hom}(Y,U)) \arrow[r, "q_{10}'"'] & \mathrm{Hom}(Z\otimes X\otimes Y,U) \arrow[d, "\mathrm{Hom}([Id_Z\otimes s_1],Id_U)"'] \\
& \mathrm{Hom}(Z,\mathrm{Hom}(X\otimes Y,U)) \arrow[ur, "q_9''"] \arrow[d, "\mathrm{Hom}(Id_Z,\mathrm{Hom}(s_1,Id_U))"] & \mathrm{Hom}(Z\otimes Y\otimes X,U) \\
& \mathrm{Hom}(Z,\mathrm{Hom}(Y\otimes X,U)) \arrow[ur, "q_9'''"'] & \\
\mathrm{Hom}(Z,\mathrm{Hom}(X,\mathrm{Hom}(Y,U))) \arrow[uuur, "q_{12}"] \arrow[uur, "\mathrm{Hom}(Id_Z,q_8')"'] \arrow[drr, "\mathrm{Hom}(Id_Z,T_{X,Y;U})"'] & & \\
& & \mathrm{Hom}(Z,\mathrm{Hom}(Y,\mathrm{Hom}(X,U))) \arrow[uul, "\mathrm{Hom}(Id_Z,q_7')"']
\end{tikzcd}$$

The composites of permutations in the diagrams are equal to the same switching map $s_4$:

$$\begin{aligned}
[s_3 \otimes Id_X] \circ [Id_Y \otimes s_2] \circ [s_1 \otimes Id_Z] &= [Id_Z \otimes s_1] \circ [s_2 \otimes Id_Y] \circ [Id_X \otimes s_3] \\
&= s_4 : X \otimes Y \otimes Z \to Z \otimes Y \otimes X.
\end{aligned}$$

By Lemma 1.51, $q_3 \circ q_1 = q_9 \circ \mathrm{Hom}(Id_X, q_7)$ and $q_3''' \circ q_1' = q_9''' \circ \mathrm{Hom}(Id_Z \circ q_7')$. So, the two diagrams fit together as claimed, with the downward composite in the left column of the first diagram being equal to the composite in the right column of the second, giving the required map

$$\begin{aligned}(4.5)\, a_1 &= (q_3'''\circ q_1')^{-1}\circ \mathrm{Hom}([s_3\otimes Id_X]\circ[Id_Y\otimes s_2]\circ[s_1\otimes Id_Z], Id_U)\circ q_3\circ q_1\\ &= \mathrm{Hom}(Id_Z,(q_7')^{-1})\circ(q_9''')^{-1}\circ \mathrm{Hom}(s_4, Id_U)\circ q_9\circ \mathrm{Hom}(Id_X, q_7).\end{aligned}$$

∎

## 4.2. Symmetric bilinear forms

Definition 4.13. A $W$-valued form $h\in \mathrm{Hom}(V,\mathrm{Hom}(V,W))$ is symmetric means: $h = T_{V;W}(h)$. $h$ is antisymmetric means: $h = -T_{V;W}(h)$. Let $Sym(V;W)$ denote the subspace of symmetric forms, and $Alt(V;W)$ the subspace of antisymmetric forms.

It follows from Lemma 1.119 and Lemma 4.4 that if $\frac{1}{2}\in\mathbb{K}$, then $T_{V;W}$ produces a direct sum

$$\mathrm{Hom}(V,\mathrm{Hom}(V,W)) = Sym(V;W)\oplus Alt(V;W). \tag{4.6}$$

Remark 4.14. The direct sum (4.6) is canonical, and so is the decomposition of any form $h$ into its symmetric and antisymmetric parts

$$\frac{1}{2}(h+T_{V;W}(h)) + \frac{1}{2}(h - T_{V;W}(h)). \tag{4.7}$$

However, there are several other involutions appearing in the $V = V_1 = V_2$ case of Lemma 4.9, and some of the other spaces admit distinct but equivalent direct sums as in Example 1.144. Recalling the direct sum $V\otimes V = S^2V\oplus\Lambda^2V$ produced by the involution $s$ as in Example 1.124, Example 1.145 applies to the involutions $T_{V;W}$ and $\mathrm{Hom}(s, Id_W)$ from Lemma 4.1 and Lemma 4.9, so the map

$$q : \mathrm{Hom}(V,\mathrm{Hom}(V,W))\to \mathrm{Hom}(V\otimes V, W)$$

respects both of the direct sums on the target:

$$\begin{aligned}\mathrm{Hom}(V\otimes V,W) &= \{A : A\circ s = A\}\oplus\{A : A\circ s = -A\},\\ \mathrm{Hom}(V\otimes V,W) &= \mathrm{Hom}(S^2V,W)\oplus\mathrm{Hom}(\Lambda^2V,W).\end{aligned}$$

Example 4.15. It follows from Lemma 4.5 that for a map of the form

$$h = (\mathrm{Hom}(Id_V, k_{VW})\circ n_1)(g\otimes w),$$

with $g : V\to V^*$, $w\in W$, if $g$ is symmetric, or antisymmetric, then so is $h$.

Definition 4.16. For any $U$, $V$, $W$, the pullback of a $W$-valued form $h : V\to \mathrm{Hom}(V,W)$ by a map $H : U\to V$ is another $W$-valued form $\mathrm{Hom}(H, Id_W)\circ h\circ H : U\to\mathrm{Hom}(U,W)$.

In the case $W=\mathbb{K}$, this coincides with the previously defined pullback (Definition 3.7).

Lemma 4.17. *For maps $H : U\to V$, $G : W_1\to W_2$, and a form $h : V\to \mathrm{Hom}(V,W_1)$,*

$$T_{U;W_2}(\mathrm{Hom}(H,G)\circ h\circ H) = \mathrm{Hom}(H,G)\circ(T_{V;W_1}(h))\circ H.$$

*The map*

$$\mathrm{Hom}(H,\mathrm{Hom}(H,G)) : \mathrm{Hom}(V,\mathrm{Hom}(V,W_1))\to\mathrm{Hom}(U,\mathrm{Hom}(U,W_2))$$

*respects the direct sums $Sym(V;W_1)\oplus Alt(V;W_1)\to Sym(U;W_2)\oplus Alt(U;W_2)$.*

PROOF. The first claim is a special case of Lemma 4.6. The claim about the direct sums follows from Lemma 1.126 and Lemma 4.4. ∎

The $G = Id_W$ case of Lemma 4.17 shows that the pullback by $H : U \to V$ of a symmetric form $h : V \to \mathrm{Hom}(V, W)$ is a symmetric form $U \to \mathrm{Hom}(U, W)$, and similarly, the pullback of an antisymmetric form is antisymmetric.

NOTATION 4.18. For $h_1 : V_1 \to \mathrm{Hom}(V_1, W)$, and $h_2 : V_2 \to \mathrm{Hom}(V_2, W)$, and a direct sum $V = V_1 \oplus V_2$, let $h_1 \oplus h_2 : V \to \mathrm{Hom}(V, W)$ denote the form

$$\mathrm{Hom}(P_1, Id_W) \circ h_1 \circ P_1 + \mathrm{Hom}(P_2, Id_W) \circ h_2 \circ P_2.$$

In the $W = \mathbb{K}$ case, this is exactly the construction of Notation 3.9.

LEMMA 4.19. $T_{V;W}(h_1 \oplus h_2) = (T_{V_1;W}(h_1)) \oplus (T_{V_2;W}(h_2))$.

PROOF. The proof proceeds exactly as in Theorem 3.10, using Lemma 1.14 and Lemma 1.6. ∎

It follows that the direct sum of symmetric $W$-valued forms is symmetric, and similarly, the direct sum of antisymmetric forms is antisymmetric.

Working with the tensor product of vector valued forms is simpler than the scalar case (Notation 3.13), since the scalar multiplication is omitted. If $h_1 : V_1 \to \mathrm{Hom}(V_1, W_1)$ and $h_2 : V_2 \to \mathrm{Hom}(V_2, W_2)$ are two vector valued forms, then the map

$$j \circ [h_1 \otimes h_2] : V_1 \otimes V_2 \to \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2)$$

has output

$$((j \circ [h_1 \otimes h_2])(u_1 \otimes u_2))(v_1 \otimes v_2) = ((h_1(u_1))(v_1)) \otimes ((h_2(u_2))(v_2)) \in W_1 \otimes W_2,$$

so it is a $W_1 \otimes W_2$-valued form.

THEOREM 4.20.

$$T_{V_1 \otimes V_2; W_1 \otimes W_2}(j \circ [h_1 \otimes h_2]) = j \circ [(T_{V_1;W_1}(h_1)) \otimes (T_{V_2;W_2}(h_2))].$$

PROOF. In analogy with the proof of Theorem 3.12, the following diagram is commutative:

$$\begin{tikzcd}
V_1 \otimes V_2 \arrow[r, "{[d_{V_1W_1} \otimes d_{V_2W_2}]}"] \arrow[d, "d_{V_1 \otimes V_2, W_1 \otimes W_2}"] & M_1 \arrow[d, "j'"] \\
\mathrm{Hom}(\mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2), W_1 \otimes W_2) \arrow[r, "\mathrm{Hom}(j, Id_{W_1 \otimes W_2})"'] & M_2
\end{tikzcd}$$

where

$$\begin{aligned}
M_1 &= \mathrm{Hom}(\mathrm{Hom}(V_1, W_1), W_1) \otimes \mathrm{Hom}(\mathrm{Hom}(V_2, W_2), W_2) \\
M_2 &= \mathrm{Hom}(\mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2), W_1 \otimes W_2);
\end{aligned}$$

$$\begin{aligned}
v_1 \otimes v_2 &\mapsto (\mathrm{Hom}(j, Id_{W_1 \otimes W_2}) \circ d_{V_1 \otimes V_2, W_1 \otimes W_2})(v_1 \otimes v_2) \\
&= (d_{V_1 \otimes V_2, W_1 \otimes W_2}(v_1 \otimes v_2)) \circ j : \\
A \otimes B &\mapsto [A \otimes B](v_1 \otimes v_2) = (A(v_1)) \otimes (B(v_2)), \\
v_1 \otimes v_2 &\mapsto (j' \circ [d_{V_1W_1} \otimes d_{V_2W_2}])(v_1 \otimes v_2) \\
&= j'((d_{V_1W_1}(v_1)) \otimes (d_{V_2W_2}(v_2))) : \\
A \otimes B &\mapsto (A(v_1)) \otimes (B(v_2)).
\end{aligned}$$

The statement of the Theorem follows from Lemma 1.6, the above diagram, Lemma 1.37, and Lemma 1.36:

$$\begin{aligned}
LHS &= (t^{W_1\otimes W_2}_{V_1\otimes V_2,\mathrm{Hom}(V_1\otimes V_2,W_1\otimes W_2)}(j\circ[h_1\otimes h_2]))\circ d_{V_1\otimes V_2,W_1\otimes W_2}\\
&= \mathrm{Hom}([h_1\otimes h_2],Id_{W_1\otimes W_2})\circ\mathrm{Hom}(j,Id_{W_1\otimes W_2})\circ d_{V_1\otimes V_2,W_1\otimes W_2}\\
&= \mathrm{Hom}([h_1\otimes h_2],Id_{W_1\otimes W_2})\circ j'\circ[d_{V_1W_1}\otimes d_{V_2W_2}]\\
&= j\circ[\mathrm{Hom}(h_1,Id_{W_1})\otimes\mathrm{Hom}(h_2,Id_{W_2})]\circ[d_{V_1W_1}\otimes d_{V_2W_2}]\\
&= j\circ[(T_{V_1;W_1}(h_1))\otimes(T_{V_2;W_2}(h_2))].
\end{aligned}$$

∎

It follows that the tensor product of symmetric forms is symmetric, as is the tensor product of antisymmetric forms.

In the $W_1=\mathbb{K}$ case, the tensor product of a scalar valued form $h_1:V_1\to V_1^*$ and a vector valued form $h_2:V_2\to\mathrm{Hom}(V_2,W)$ is a form $j\circ[h_1\otimes h_2]$ with values in $\mathbb{K}\otimes W$. The map $\mathrm{Hom}(Id_{V_1\otimes V_2},l_W)\circ j\circ[h_1\otimes h_2]$ is a $W$-valued form.

COROLLARY 4.21. *For $h_1:V_1\to V_1^*$ and $h_2:V_2\to\mathrm{Hom}(V_2,W)$, the following $W$-valued forms are equal.*

$$\begin{aligned}
&T_{V_1\otimes V_2;W}(\mathrm{Hom}(Id_{V_1\otimes V_2},l_W)\circ j\circ[h_1\otimes h_2])\\
=\ &\mathrm{Hom}(Id_{V_1\otimes V_2},l_W)\circ j\circ[(T_{V_1}(h_1))\otimes(T_{V_2;W}(h_2))].
\end{aligned}$$

PROOF. The equality follows immediately from Lemma 4.6, the previous Theorem, and the equality $T_{V;\mathbb{K}}=T_V$. ∎

EXERCISE 4.22. Let $V=U\oplus\mathrm{Hom}(U,L)$ be a direct sum with data $P_i$, $Q_i$. Then,

$$\mathrm{Hom}(P_1,Id_L)\circ P_2+\mathrm{Hom}(P_2,Id_L)\circ d_{UL}\circ P_1 \tag{4.8}$$

is a symmetric $L$-valued form on $V$. If $d_{UL}$ is invertible, then this form is also invertible.

HINT. The proof that the form (4.8) is symmetric is the same as the calculation from Lemma 3.102, but using Lemma 1.14 and Lemma 1.17 in their full generality. If $d_{UL}$ is invertible (for example, as in Proposition 3.78), then the inverse of the form is

$$Q_1\circ d_{UL}^{-1}\circ\mathrm{Hom}(Q_2,Id_L)+Q_2\circ\mathrm{Hom}(Q_1,Id_L).$$

∎

EXERCISE 4.23. Let $V = U \oplus \mathrm{Hom}(U, L)$ as in Exercise 4.22. Given maps $E : W \to U$ and $h : U \to \mathrm{Hom}(W, L)$, the following are equivalent.

(1) The $L$-valued form $h \circ E : W \to \mathrm{Hom}(W, L)$ is antisymmetric.
(2) The pullback of the symmetric $L$-valued form (4.8) by the map

$$Q_1 \circ E + Q_2 \circ \mathrm{Hom}(h, Id_L) \circ d_{WL} : W \to V$$

is $0_{\mathrm{Hom}(W,\mathrm{Hom}(W,L))}$.

HINT. The statement is analogous to Exercise 3.109, but uses the generalized notion of pullback from Definition 4.16. ∎

EXERCISE 4.24. Suppose $V = V_1 \oplus V_2$, and there is an invertible map $g : V \to \mathrm{Hom}(V, L)$ so that these pullbacks are zero:

$$\begin{aligned}\mathrm{Hom}(Q_1, Id_L) \circ g \circ Q_1 &= 0_{\mathrm{Hom}(V_1,\mathrm{Hom}(V_1,L))},\\ \mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_2 &= 0_{\mathrm{Hom}(V_2,\mathrm{Hom}(V_2,L))}.\end{aligned}$$

Then these maps are invertible:

$$\mathrm{Hom}(Q_1, Id_L) \circ g \circ Q_2 : V_2 \to \mathrm{Hom}(V_1, L),$$

$$\mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_1 : V_1 \to \mathrm{Hom}(V_2, L).$$

HINT. The inverses are $P_2 \circ g^{-1} \circ \mathrm{Hom}(P_1, Id_L)$, $P_1 \circ g^{-1} \circ \mathrm{Hom}(P_2, Id_L)$. ∎

EXERCISE 4.25. ([**EPW**]) Let $V = V_1 \oplus V_2$ and $g : V \to \mathrm{Hom}(V, L)$ be as in the previous Exercise. Then there is another direct sum $V = V_1 \oplus \mathrm{Hom}(V_1, L)$, defined by data $P_1$, $Q_1$, and

$$\begin{aligned}P_2' &= \mathrm{Hom}(Q_1, Id_L) \circ g : V \to \mathrm{Hom}(V_1, L),\\ Q_2' &= g^{-1} \circ \mathrm{Hom}(P_1, Id_L) : \mathrm{Hom}(V_1, L) \to V.\end{aligned}$$

If, also, $g$ is symmetric, then $g$ is equal to the $L$-valued form induced by this direct sum, as in Exercise 4.22:

$$g = \mathrm{Hom}(P_1, Id_L) \circ P_2' + \mathrm{Hom}(P_2', Id_L) \circ d_{V_1,L} \circ P_1.$$

HINT. It is easy to check that $P_2' \circ Q_1$ is zero, and $P_2' \circ Q_2'$ is the identity.

$$\begin{aligned}&\mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_1 \circ P_1 \circ Q_2'\\ =\;& \mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_1 \circ P_1 \circ g^{-1} \circ \mathrm{Hom}(P_1 \circ Id_L)\\ =\;& \mathrm{Hom}(Q_2, Id_L) \circ g \circ (Id_V - Q_2 \circ P_2) \circ g^{-1} \circ \mathrm{Hom}(P_1, Id_L)\\ =\;& 0_{\mathrm{Hom}(\mathrm{Hom}(V_1,L),\mathrm{Hom}(V_2,L))}.\end{aligned}$$

By the previous Exercise, $\mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_1$ is invertible, so

$$P_1 \circ Q_2' = 0_{\mathrm{Hom}(\mathrm{Hom}(V_1,L),V_1)}.$$

$$\begin{aligned}&Q_1 \circ P_1 + Q_2' \circ P_2'\\ =\;& Q_1 \circ P_1 + (Q_1 \circ P_1 + Q_2 \circ P_2) \circ Q_2' \circ P_2' \circ (Q_1 \circ P_1 + Q_2 \circ P_2)\\ =\;& Q_1 \circ P_1 + Q_2 \circ P_2 \circ g^{-1} \circ \mathrm{Hom}(P_1, Id_L) \circ \mathrm{Hom}(Q_1, Id_L) \circ g \circ Q_2 \circ P_2\\ =\;& Q_1 \circ P_1 + Q_2 \circ P_2 = Id_V,\end{aligned}$$

using the inverse formula from the previous Exercise. As for the claimed equality,

$$\begin{aligned}
RHS &= \mathrm{Hom}(P_1, Id_L) \circ \mathrm{Hom}(Q_1, Id_L) \circ g \\
&\quad + \mathrm{Hom}(\mathrm{Hom}(Q_1, Id_L) \circ g, Id_L) \circ d_{V_1,L} \circ P_1 \\
&= \mathrm{Hom}(P_1, Id_L) \circ \mathrm{Hom}(Q_1, Id_L) \circ g \circ (Q_1 \circ P_1 + Q_2 \circ P_2) + \\
&\quad \mathrm{Hom}(g, Id_L) \circ \mathrm{Hom}(\mathrm{Hom}(Q_1, Id_L), Id_L) \circ d_{V_1,L} \circ P_1 \\
&= \mathrm{Hom}(P_1, Id_L) \circ \mathrm{Hom}(Q_1, Id_L) \circ g \circ Q_2 \circ P_2 \\
&\quad + \mathrm{Hom}(P_2, Id_L) \circ \mathrm{Hom}(Q_2, Id_L) \circ g \circ Q_2 \circ P_2 \\
&\quad + \mathrm{Hom}(g, Id_L) \circ d_{VL} \circ Q_1 \circ P_1 \\
&= \mathrm{Hom}(Q_1 \circ P_1 + Q_2 \circ P_2, Id_L) \circ g \circ Q_2 \circ P_2 + (T_{VL}(g)) \circ Q_1 \circ P_1 \\
&= g \circ (Q_2 \circ P_2 + Q_1 \circ P_1) = g.
\end{aligned}$$

∎

## 4.3. Vector valued trace with respect to a metric

A metric $g$ on $V$ suggests that the scalar trace $Tr_g$ (Definition 3.32) can be generalized to a vector valued trace map on $W$-valued forms, but at first there would appear to be two constructions of a map $Tr_{g;W} : \mathrm{Hom}(V, \mathrm{Hom}(V, W)) \to W$. One way would be to combine the previously constructed vector valued trace $Tr_{V^*;W}$ (Definition 2.50) and composition with $g^{-1}$, and another would be to start with the scalar trace $Tr_g$, and tensor with $Id_W$. Of course, the two approaches have the same result.

Lemma 4.26. *Given a metric $g$ on $V$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V,\mathrm{Hom}(V,W)) \arrow[r, "\mathrm{Hom}(Id_V, k_{VW}^{-1})"] \arrow[d, "\mathrm{Hom}(g^{-1}, k_{VW}^{-1})"'] & \mathrm{Hom}(V, V^* \otimes W) \arrow[r, shift left, "n_1^{-1}"] \arrow[dl, "\mathrm{Hom}(g^{-1}, Id_{V^*\otimes W})"'] & \mathrm{Hom}(V, V^*) \otimes W \arrow[l, shift left, "n_1"] \arrow[dl, "{[\mathrm{Hom}(g^{-1}, Id_{V^*}) \otimes Id_W]}"'] \arrow[ddl, "{[Tr_g \otimes Id_W]}"] \\
\mathrm{Hom}(V^*, V^* \otimes W) \arrow[d, "Tr_{V^*;W}"] & \mathrm{Hom}(V^*, V^*) \otimes W \arrow[l, "n"] \arrow[d, "{[Tr_{V^*} \otimes Id_W]}"'] & \\
W & \mathbb{K} \otimes W \arrow[l, "l_W"] &
\end{tikzcd}$$

Proof. The upper left triangle is commutative by Lemma 1.6, and the upper right block is commutative by Lemma 1.42, with invertible $n$ maps by the finite-dimensionality of $V$. The lower square is the definition of $Tr_{V^*;W}$, and the right triangle uses the definition of $Tr_g$. ∎

Definition 4.27. Given a metric $g$ on $V$, an arbitrary vector space $W$, and a $W$-valued form $h : V \to \mathrm{Hom}(V, W)$, the <u>$W$-valued trace</u> with respect to $g$ is the following element of $W$:

$$Tr_{g;W}(h) = Tr_{V^*;W}(k_{VW}^{-1} \circ h \circ g^{-1}).$$

Corollary 2.58 also gives the equality

$$Tr_{V^*;W}(k_{VW}^{-1}\circ h\circ g^{-1}) = Tr_{V;W}([g^{-1}\otimes Id_W]\circ k_{VW}^{-1}\circ h).$$

By the previous Lemma,

$$Tr_{g;W} = Tr_{V^*;W}\circ \mathrm{Hom}(g^{-1},k_{VW}^{-1}) = l_W\circ[Tr_g\otimes Id_W]\circ n_1^{-1}\circ \mathrm{Hom}(Id_V,k_{VW}^{-1}).$$

EXAMPLE 4.28. Given a metric $g$ on $V$, if $h$ is of the form $h = (\mathrm{Hom}(Id_V,k_{VW})\circ n_1)(E\otimes w)$, for $E : V\to V^*$ and $w\in W$, then $Tr_{g;W}(h) = Tr_g(E)\cdot w$, and if $Tr_g(E) = 0$, then $Tr_{g;W}(h) = 0_W$.

The previously defined scalar valued trace with respect to $g$ (Definition 3.32) is exactly the $W = \mathbb{K}$ case of the vector valued case:

THEOREM 4.29. *Given a metric $g$ on $V$, for $h : V\to V^*$, $Tr_{g;\mathbb{K}}(h) = Tr_g(h)$.*

PROOF. By Lemma 1.63, $k_{V\mathbb{K}} : V^*\otimes\mathbb{K}\to V^*$ is exactly the scalar multiplication appearing in Theorem 2.54, so that

$$Tr_{g;\mathbb{K}}(h) = Tr_{V^*;\mathbb{K}}(k_{V\mathbb{K}}^{-1}\circ h\circ g^{-1}) = Tr_{V^*}(h\circ g^{-1}) = Tr_g(h).$$

∎

THEOREM 4.30. *For any metric $h^\nu$ on $\mathbb{K}$, as in Lemma 3.67, and a form $h : \mathbb{K}\to\mathrm{Hom}(\mathbb{K},W)$,*

$$Tr_{h^\nu;W}(h) = \frac{1}{\nu}\cdot(h(1))(1).$$

PROOF.

$$\begin{aligned}
Tr_{h^\nu;W}(h) &= Tr_{\mathbb{K};W}([(h^\nu)^{-1}\otimes Id_W]\circ k_{\mathbb{K}W}^{-1}\circ h)\\
&= (l_W\circ[(h^\nu)^{-1}\otimes Id_W]\circ k_{\mathbb{K}W}^{-1}\circ h)(1)\\
&= (\frac{1}{\nu}\cdot m^{-1})(h(1))\\
&= \frac{1}{\nu}(h(1))(1),
\end{aligned}$$

where the first step uses Corollary 2.57, and the last step uses the formula $m^{-1} = d_{\mathbb{K}W}(1)$, from Definition 1.20. The intermediate step uses the commutativity of the diagram

$$\begin{tikzcd}
W & \mathrm{Hom}(\mathbb{K},W) \arrow[l, "m^{-1}"'] \\
\mathbb{K}\otimes W \arrow[u, "\nu\cdot l_W"] \arrow[r, "{[h^\nu\otimes Id_W]}"] & \mathbb{K}^*\otimes W \arrow[u, "k_{\mathbb{K}W}"']
\end{tikzcd}$$

$$\begin{aligned}
\lambda\otimes w \quad &\mapsto \quad (m^{-1}\circ k_{\mathbb{K}W}\circ[h^\nu\otimes Id_W])(\lambda\otimes w)\\
&= \quad (k_{\mathbb{K}W}((h^\nu(\lambda))\otimes w))(1)\\
&= \quad \nu\cdot\lambda\cdot 1\cdot w\\
&= \quad (\nu\cdot l_W)(\lambda\otimes w).
\end{aligned}$$

∎

THEOREM 4.31. *Given a metric $g$ on $V$, $Tr_{g;W}(T_{V;W}(h)) = Tr_{g;W}(h)$.*

PROOF. Since $V$ must be finite-dimensional, Lemma 4.5 and Lemma 4.26 apply.

$$\begin{aligned}
Tr_{g;W} \circ T_{V;W} &= l_W \circ [Tr_g \otimes Id_W] \circ n_1^{-1} \circ \mathrm{Hom}(Id_V, k_{VW}^{-1}) \\
&\quad \circ \mathrm{Hom}(Id_V, k_{VW}) \circ n_1 \circ [T_V \otimes Id_W] \circ n_1^{-1} \circ \mathrm{Hom}(Id_V, k_{VW}^{-1}) \\
&= l_W \circ [Tr_g \otimes Id_W] \circ [T_V \otimes Id_W] \circ n_1^{-1} \circ \mathrm{Hom}(Id_V, k_{VW}^{-1}) \\
&= l_W \circ [Tr_g \otimes Id_W] \circ n_1^{-1} \circ \mathrm{Hom}(Id_V, k_{VW}^{-1}) \\
&= Tr_{g;W},
\end{aligned}$$

by Lemma 1.36 and Theorem 3.33, which stated that $Tr_g \circ T_V = Tr_g$. ∎

COROLLARY 4.32. *Given a metric $g$ on $V$, if $h : V \to \mathrm{Hom}(V, W)$ is antisymmetric and $\frac{1}{2} \in \mathbb{K}$, then $Tr_{g;W}(h) = 0_W$.* ∎

PROPOSITION 4.33. *Given a metric $g$ on $V$, the $W$-valued trace is invariant under pullback, that is, if $H : U \to V$ is invertible, then*

$$Tr_{H^* \circ g \circ H;W}(\mathrm{Hom}(H, Id_W) \circ h \circ H) = Tr_{g;W}(h).$$

PROOF. Using Corollary 2.58 and Lemma 1.62,

$$\begin{aligned}
LHS &= Tr_{U^*;W}(k_{UW}^{-1} \circ \mathrm{Hom}(H, Id_W) \circ h \circ H \circ H^{-1} \circ g^{-1} \circ (H^*)^{-1}) \\
&= Tr_{V^*;W}([(H^*)^{-1} \otimes Id_W] \circ k_{UW}^{-1} \circ \mathrm{Hom}(H, Id_W) \circ h \circ g^{-1}) \\
&= Tr_{V^*;W}(k_{VW}^{-1} \circ h \circ g^{-1}) = RHS.
\end{aligned}$$

This statement and proof are analogous to Proposition 3.37. ∎

THEOREM 4.34. *Given a metric $g$ on $V$, for any map $B : W \to W'$,*

$$Tr_{g;W'}(\mathrm{Hom}(Id_V, B) \circ h) = B(Tr_{g;W}(h)).$$

PROOF. Using Corollary 2.59 and Lemma 1.62,

$$\begin{aligned}
LHS &= Tr_{V^*;W'}(k_{VW'}^{-1} \circ \mathrm{Hom}(Id_V, B) \circ h \circ g^{-1}) \\
&= Tr_{V^*,W'}([Id_{V^*} \otimes B] \circ k_{VW}^{-1} \circ h \circ g^{-1}) \\
&= B(Tr_{V^*;W}(k_{VW}^{-1} \circ h \circ g^{-1})) = RHS.
\end{aligned}$$

∎

COROLLARY 4.35. *Given a metric $g$ on $V$, and maps $H : U \to V$, $B : W \to W'$, if $H$ is invertible then the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V, \mathrm{Hom}(V, W)) \arrow[r, "Tr_{g,W}"] \arrow[d, "\mathrm{Hom}(H, \mathrm{Hom}(H, B))"'] & W \arrow[d, "B"] \\
\mathrm{Hom}(U, \mathrm{Hom}(U, W')) \arrow[r, "Tr_{H^* \circ g \circ H;W'}"] & W'
\end{tikzcd}$$

PROOF. This follows from Proposition 4.33 and Theorem 4.34. ∎

PROPOSITION 4.36. *Given metrics $g_1$, $g_2$ on $V_1$, $V_2$, and a direct sum $V = V_1 \oplus V_2$, for $W$-valued forms $h_1 : V_1 \to \mathrm{Hom}(V_1, W)$, $h_2 : V_2 \to \mathrm{Hom}(V_2, W)$,*

$$Tr_{g_1 \oplus g_2;W}(h_1 \oplus h_2) = Tr_{g_1;W}(h_1) + Tr_{g_2;W}(h_2) \in W.$$

PROOF. First, Lemma 1.6 and Lemma 1.62 apply to simplify the following map from $\mathrm{Hom}(V_I, W)$ to $V_i^* \otimes W$:

$$\begin{aligned} & [Q_i^* \otimes Id_W] \circ k_{VW}^{-1} \circ \mathrm{Hom}(P_I, Id_W) \\ = \; & k_{V_iW}^{-1} \circ \mathrm{Hom}(Q_i, Id_W) \circ \mathrm{Hom}(P_I, Id_W) = k_{V_iW}^{-1} \circ \mathrm{Hom}(P_I \circ Q_i, Id_W) \\ = \; & k_{V_iW}^{-1}, \text{ if } i = I, \text{ or } 0_{\mathrm{Hom}(\mathrm{Hom}(V_I,W), V_i^* \otimes W)} \text{ if } i \neq I. \end{aligned}$$

Then the formula (3.4) for $(g_1 \oplus g_2)^{-1}$ from Corollary 3.18 applies:

$$\begin{aligned} LHS &= Tr_{V;W}([(g_1 \oplus g_2)^{-1} \otimes Id_W] \circ k_{VW}^{-1} \circ (h_1 \oplus h_2)) \\ &= Tr_{V;W}([(Q_1 \circ g_1^{-1} \circ Q_1^*) \otimes Id_W] \circ k_{VW}^{-1} \circ \mathrm{Hom}(P_1, Id_W) \circ h_1 \circ P_1 \\ &\quad +[(Q_1 \circ g_1^{-1} \circ Q_1^*) \otimes Id_W] \circ k_{VW}^{-1} \circ \mathrm{Hom}(P_2, Id_W) \circ h_2 \circ P_2 \\ &\quad +[(Q_2 \circ g_2^{-1} \circ Q_2^*) \otimes Id_W] \circ k_{VW}^{-1} \circ \mathrm{Hom}(P_1, Id_W) \circ h_1 \circ P_1 \\ &\quad +[(Q_2 \circ g_2^{-1} \circ Q_2^*) \otimes Id_W] \circ k_{VW}^{-1} \circ \mathrm{Hom}(P_2, Id_W) \circ h_2 \circ P_2) \\ &= Tr_{V;W}([(Q_1 \circ g_1^{-1}) \otimes Id_W] \circ k_{V_1W}^{-1} \circ h_1 \circ P_1) \\ &\quad +Tr_{V;W}([(Q_2 \circ g_2^{-1}) \otimes Id_W] \circ k_{V_2W}^{-1} \circ h_2 \circ P_2) \\ &= Tr_{V_1;W}([(P_1 \circ Q_1 \circ g_1^{-1}) \otimes Id_W] \circ k_{V_1W}^{-1} \circ h_1) \\ &\quad +Tr_{V_2;W}([(P_2 \circ Q_2 \circ g_2^{-1}) \otimes Id_W] \circ k_{V_2W}^{-1} \circ h_2) = RHS. \end{aligned}$$

The last steps used Corollary 2.58 and Lemma 1.36. ∎

THEOREM 4.37. *For metrics $g_1$, $g_2$ on $V_1$, $V_2$, and vector valued forms $h_1 : V_1 \to \mathrm{Hom}(V_1, W_1)$, $h_2 : V_2 \to \mathrm{Hom}(V_2, W_2)$,*

$$Tr_{\{g_1 \otimes g_2\};W_1 \otimes W_2}(j \circ [h_1 \otimes h_2]) = (Tr_{g_1;W_1}(h_1)) \otimes (Tr_{g_2;W_2}(h_2)) \in W_1 \otimes W_2.$$

PROOF. The following diagram is commutative.

$$\begin{tikzcd} V_1 \otimes V_2 \otimes W_1 \otimes W_2 \arrow[d, "{[[g_1 \otimes g_2] \otimes Id_{W_1 \otimes W_2}]}"] & V_1 \otimes W_1 \otimes V_2 \otimes W_2 \arrow[l, "s_1"'] \arrow[d, "{[[g_1 \otimes Id_{W_1}] \otimes [g_2 \otimes Id_{W_2}]]}"'] \\ V_1^* \otimes V_2^* \otimes W_1 \otimes W_2 \arrow[r, "s"] \arrow[d, "{[j \otimes Id_{W_1 \otimes W_2}]}"] & V_1^* \otimes W_1 \otimes V_2^* \otimes W_2 \arrow[d, "{[k_{V_1W_1} \otimes k_{V_2W_2}]}"] \\ \mathrm{Hom}(V_2 \otimes V_2, \mathbb{K} \otimes \mathbb{K}) \otimes W_1 \otimes W_2 \arrow[d, "{[\mathrm{Hom}(Id_{V_1 \otimes V_2}, l) \otimes Id_{W_1 \otimes W_2}]}"] & \mathrm{Hom}(V_1, W_1) \otimes \mathrm{Hom}(V_2, W_2) \arrow[d, "j"] \\ (V_1 \otimes V_2)^* \otimes W_1 \otimes W_2 \arrow[r, "k_{V_1 \otimes V_2, W_1 \otimes W_2}"] & \mathrm{Hom}(V_1 \otimes V_2, W_1 \otimes W_2) \end{tikzcd}$$

The commutativity of the lower part is exactly Lemma 2.32. The top square is easy to check, where the $s_1$ map is as in Theorem 2.40. The statement of the Theorem follows from Corollary 2.64, using Lemma 1.36 and the formula for $\{g_1 \otimes g_2\}^{-1}$ from Corollary 3.19:

$$\begin{aligned}
LHS = \; & Tr_{V_1\otimes V_2;W_1\otimes W_2}([([g_1^{-1}\otimes g_2^{-1}]\circ j^{-1}\circ \mathrm{Hom}(Id_{V_1\otimes V_2}, l^{-1}))\otimes Id_{W_1\otimes W_2}] \\
& \circ k^{-1}_{V_1\otimes V_2,W_1\otimes W_2}\circ j\circ [h_1\otimes h_2]) \\
= \; & Tr_{V_1\otimes V_2;W_1\otimes W_2}(s_1\circ [[g_1^{-1}\otimes Id_{W_1}]\otimes [g_2^{-1}\otimes Id_{W_2}]] \\
& \circ [k^{-1}_{V_1W_1}\otimes k^{-1}_{V_2W_2}]\circ [h_1\otimes h_2]) \\
= \; & Tr_{V_1\otimes V_2;W_1\otimes W_2}(s_1\circ (j_3'(([g_1^{-1}\otimes Id_{W_1}]\circ k^{-1}_{V_1W_1}\circ h_1) \\
& \otimes ([g_2^{-1}\otimes Id_{W_2}]\circ k^{-1}_{V_2W_2}\circ h_2)))) \\
= \; & (Tr_{V_1;W_1}([g_1^{-1}\otimes Id_{W_1}]\circ k^{-1}_{V_1W_1}\circ h_1)) \\
& \otimes (Tr_{V_2;W_2}([g_2^{-1}\otimes Id_{W_2}]\circ k^{-1}_{V_2W_2}\circ h_2)) = RHS.
\end{aligned}$$

■

COROLLARY 4.38. *For metrics $g_1$, $g_2$ on $V_1$, $V_2$, a scalar valued form $h_1 : V_1 \to V_1^*$, and a $W$-valued form $h_2 : V_2 \to \mathrm{Hom}(V_2, W)$,*

$$Tr_{\{g_1\otimes g_2\};W}(\mathrm{Hom}(Id_{V_1\otimes V_2}, l_W)\circ j\circ [h_1\otimes h_2]) = Tr_{g_1}(h_1)\cdot Tr_{g_2;W}(h_2).$$

PROOF. Using Theorem 4.34, the previous Theorem, and Theorem 4.29,

$$\begin{aligned}
LHS &= l_W(Tr_{\{g_1\otimes g_2\};\mathbb{K}\otimes W}(j\circ [h_1\otimes h_2])) \\
&= l_W((Tr_{g_1;\mathbb{K}}(h_1))\otimes (Tr_{g_2;W}(h_2))) = RHS.
\end{aligned}$$

■

THEOREM 4.39. *If $\frac{1}{2}\in\mathbb{K}$, and $g$ and $y$ are metrics on $V$ and $W$, then the direct sum $Sym(V;W)\oplus Alt(V;W)$ is orthogonal with respect to the induced metric.*

PROOF. Since $V$ is finite-dimensional, all the arrows in the diagram for Lemma 4.5 are invertible. Let $H = \mathrm{Hom}(Id_V, k_{VW})\circ n_1$, so $H$ and $H^{-1}$ are isometries by Theorem 3.41, Theorem 3.45, and Lemma 3.73. Also, $[T_V\otimes Id_W]$ is an isometry by Corollary 3.48 and Theorem 3.28, so by Lemma 4.5 and Definition 4.2, $T_{V;W}$ is an isometry, and an involution by Lemma 4.4. Then Lemma 3.55 applies to the direct sum produced by $T_{V;W}$. ■

By Theorem 3.56 and Theorem 3.60, $\mathrm{Hom}(V, V^*)\otimes W = (Sym(V)\otimes W)\oplus (Alt(V)\otimes W)$ is an orthogonal direct sum. Since $H$ respects the direct sums, by Lemma 4.5 and Lemma 1.126, it follows from Theorem 3.61 that the maps between $Sym(V)\otimes W$ and $Sym(V;W)$, and between $Alt(V)\otimes W$ and $Alt(V;W)$, are isometries.

THEOREM 4.40. *If $Tr_V(Id_V)\neq 0$, and $g$ is a metric on $V$, then there is a direct sum $\mathrm{Hom}(V,\mathrm{Hom}(V,W)) = W\oplus \ker(Tr_{g;W})$. If $y$ is a metric on $W$, then the direct sum is orthogonal with respect to the induced metric.*

PROOF. By Theorem 3.53, $\mathrm{Hom}(V,V^*) = \mathbb{K}\oplus\ker(Tr_g)$ is an orthogonal direct sum, with data $P_1'' = \alpha\cdot Tr_g$, $Q_1'' : \lambda\mapsto\lambda\cdot\beta\cdot g$, with $\alpha\cdot\beta\cdot Tr_V(Id_V) = 1$ as in Example 2.9. Also, $P_2'' = Id_{\mathrm{Hom}(V,V^*)} - Q_1''\circ P_1''$, and $Q_2''$ is just the inclusion of the subspace $\ker(Tr_g)$ in $\mathrm{Hom}(V,V^*)$. By Example 1.81, $\mathrm{Hom}(V,V^*)\otimes W$ is a direct sum of $\mathbb{K}\otimes W$ and $(\ker(Tr_g))\otimes W$, with data $P_i = [P_i''\otimes Id_W]$, $Q_i = [Q_i''\otimes Id_W]$.

Let $H = \mathrm{Hom}(Id_V, k_{VW}) \circ n_1$, and let $P_1' = \alpha \cdot Tr_{g;W}$, so that the following diagram is commutative by Lemma 4.26.

$$\begin{tikzcd}
\mathrm{Hom}(V, \mathrm{Hom}(V, W)) \arrow[d, "\alpha \cdot Tr_{g;W}"] & \mathrm{Hom}(V, V^*) \otimes W \arrow[l, "H"'] \arrow[d, "{[(\alpha \cdot Tr_g) \otimes Id_W]}"] \\
W & \mathbb{K} \otimes W \arrow[l, "l_W"]
\end{tikzcd}$$

Let $Q_2'$ be the inclusion of $\ker(Tr_{g;W})$ in $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$, which is a linear monomorphism so that $P_1' \circ Q_2' = 0_{\mathrm{Hom}(\ker(Tr_{g;W}), W)}$. Define $H_2 : (\ker(Tr_g)) \otimes W \to \ker(Tr_{g;W})$ by $H_2 = H \circ Q_2$; the image of $H_2$ is contained in $\ker(Tr_{g;W})$ by Lemma 4.26, so $Q_2' \circ H_2 = H \circ Q_2$. Theorem 1.102 applies, so that $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ has a direct sum structure $W \oplus \ker(Tr_{g;W})$, and $H$ respects the direct sums. By Theorem 3.60, if $W$ has a metric $y$, then the direct sum $\mathrm{Hom}(V, V^*) \otimes W = (\mathbb{K} \otimes W) \oplus ((\ker(Tr_g)) \otimes W)$ is orthogonal with respect to the induced metric. Since $H$ is an isometry (as mentioned in the proof of the previous Theorem), it follows from Theorem 3.61 that $W \oplus \ker(Tr_{g;W})$ is orthogonal with respect to the induced metric. ∎

COROLLARY 4.41. *The metric induced on $W$ by the direct sum from the previous Theorem is $\beta^2 \cdot Tr_V(Id_V) \cdot y$.*

PROOF. The induced metric on $\mathbb{K} \otimes W$ is $\{h^\nu \otimes y\}$, for $\nu = \beta^2 \cdot Tr_V(Id_V)$ by Theorem 3.60 and Lemma 3.71. By Theorem 3.61, $P_1' \circ H \circ Q_1 = l_W \circ P_1 \circ Q_1 = l_W : \mathbb{K} \otimes W \to W$ is an isometry, so by Lemma 3.69, the metric in the target must be $\nu \cdot y$. ∎

COROLLARY 4.42. *Given a metric $g$ on $V$, if both $\frac{1}{2} \in \mathbb{K}$ and $Tr_V(Id_V) \neq 0$, then there is a direct sum $\mathrm{Hom}(V, \mathrm{Hom}(V, W)) = W \oplus Sym_0(g; W) \oplus Alt(V; W)$, where $Sym_0(g; W)$ is the kernel of the restriction of $Tr_{g;W}$ to $Sym(V; W)$. If $W$ has a metric $y$, then there is an orthogonal direct sum.* ∎

EXERCISE 4.43. For $K : V \to \mathrm{Hom}(V, W)$, an orthogonal direct sum $V = V_1 \oplus V_2$ with respect to a metric $g$ on $V$, and the induced metrics $g_1$, $g_2$, on $V_1$, $V_2$,

$$Tr_{g;W}(K) = Tr_{g_1;W}(\mathrm{Hom}(Q_1, Id_W) \circ K \circ Q_1) + Tr_{g_2;W}(\mathrm{Hom}(Q_2, Id_W) \circ K \circ Q_2).$$

HINT. In analogy with Exercise 3.101, Lemma 1.62 and Corollary 2.58 apply:

$$\begin{aligned}
LHS &= Tr_{V^*;W}(k_{VW}^{-1} \circ K \circ g^{-1}) \\
&= Tr_{V^*;W}(k_{VW}^{-1} \circ K \circ (Q_1 \circ P_1 + Q_2 \circ P_2) \circ g^{-1} \circ (Q_1 \circ P_1 + Q_2 \circ P_2)^*) \\
&= Tr_{V^*;W}(k_{VW}^{-1} \circ K \circ Q_1 \circ P_1 \circ g^{-1} \circ P_1^* \circ Q_1^*) \\
&\quad + Tr_{V^*;W}(k_{VW}^{-1} \circ K \circ Q_2 \circ P_2 \circ g^{-1} \circ P_2^* \circ Q_2^*) \\
&= Tr_{V_1;W}([Q_1^* \otimes Id_W] \circ k_{VW}^{-1} \circ K \circ Q_1 \circ g_1^{-1}) \\
&\quad + Tr_{V_2;W}([Q_2^* \otimes Id_W] \circ k_{VW}^{-1} \circ K \circ Q_2 \circ g_2^{-1}) \\
&= Tr_{V_1;W}(k_{V_1W}^{-1} \circ \mathrm{Hom}(Q_1, Id_W) \circ K \circ Q_1 \circ g_1^{-1}) \\
&\quad + Tr_{V_2;W}(k_{V_2W}^{-1} \circ \mathrm{Hom}(Q_2, Id_W) \circ K \circ Q_2 \circ g_2^{-1}) = RHS.
\end{aligned}$$

∎

## 4.4. Revisiting the generalized trace

We return to some notions introduced in Section 2.4. Recall, from Notation 2.68, the map $\eta_V = s \circ k^{-1} \circ Q_1^1 : \mathbb{K} \to V \otimes V^*$.

NOTATION 4.44. For finite-dimensional $V$, consider the following diagram.

$$\begin{tikzcd}
 & \mathrm{End}(V) \otimes U \arrow[dddrr, "n_1"] & & & V^* \otimes V \otimes U \arrow[lll, "{[k \otimes Id_U]}"'] \arrow[dl, "{[s \otimes Id_U]}"'] \arrow[dddl, "k_{V,V\otimes U}"] \\
\mathbb{K} \otimes U \arrow[ur, "{[Q_1^1 \otimes Id_U]}"] \arrow[rrr, "{[\eta_V \otimes Id_U]}"] \arrow[d, "l_U"] & & & V \otimes V^* \otimes U \arrow[ddlll, "{[Id_V \otimes k_{VU}]}"] & \\
U \arrow[d, "\eta_{VU}"] & & & & \\
V \otimes \mathrm{Hom}(V,U) \arrow[rrr, "n_2"] & & & \mathrm{Hom}(V, V \otimes U) &
\end{tikzcd}$$

The top block is from (2.13), the back triangle is commutative by Lemma 1.65, and the right block is also commutative by a variation on Lemma 1.65. So, a map $\eta_{VU} : U \to V \otimes \mathrm{Hom}(V,U)$ can be defined by the following equal formulas:

$$\begin{aligned}
\eta_{VU} &= [Id_V \otimes k_{VU}] \circ [\eta_V \otimes Id_U] \circ l_U^{-1} \\
&= n_2^{-1} \circ n_1 \circ [Q_1^1 \otimes Id_U] \circ l_U^{-1} : \\
u : &\mapsto n_2^{-1}(n_1(Id_V \otimes u)).
\end{aligned}$$

This map $\eta_{UV}$ is a <u>generalized coevaluation</u>.

With the above notation, Theorem 2.77 can be re-stated in terms of $\eta_{VU}$.

COROLLARY 4.45. *For finite-dimensional $V$, $n_2$ as in the above diagram, any $F : V \otimes U \to V \otimes W$, and $u \in U$,*

$$(Tr_{V;U,W}(F))(u) = Tr_{V;W}(F \circ (n_2(\eta_{VU}(u)))).$$

PROOF. The following diagram is a modification of the diagram from the Proof of Theorem 2.77.

$$\begin{tikzcd}
\mathrm{End}(V) \otimes U & V^* \otimes V \otimes U \arrow[l, "{[k \otimes Id_U]}"] \arrow[d, "{[s \otimes Id_U]}"'] \arrow[ddr, "k_{V,V\otimes U}"] \arrow[r, "{[Id_{V^*} \otimes F]}"] & V^* \otimes V \otimes W \arrow[r, "{[Ev_V \otimes Id_W]}"] \arrow[d, "k_{V,V\otimes W}"] & \mathbb{K} \otimes W \arrow[d, "l_W"] \\
\mathbb{K} \otimes U \arrow[u, "{[Q_1^1 \otimes Id_U]}"] \arrow[r, "{[\eta_V \otimes Id_U]}"] \arrow[d, "l_U"] & V \otimes V^* \otimes U \arrow[d, "{[Id_V \otimes k_{VU}]}"'] & \mathrm{Hom}(V, V \otimes W) \arrow[r, "Tr_{V;W}"] & W \\
U \arrow[r, "\eta_{VU}"] & V \otimes \mathrm{Hom}(V,U) \arrow[r, "n_2"] & \mathrm{Hom}(V, V \otimes U) \arrow[u, "{\mathrm{Hom}(Id_V, F)}"'] & \mathrm{Hom}(V,W) \otimes V \arrow[ul, "n'"'] \arrow[u, "Ev_{VW}"']
\end{tikzcd}$$

The diagram is commutative; the left blocks and lower middle triangle by the construction of $\eta_V$ and $\eta_{VU}$ in Notation 4.44, the upper middle triangle by Lemma 1.62, and the right block copied from the Proof of Theorem 2.77. The path from $U$ to $W$ along the top row is $Tr_{V;U,W}(F)$ by Theorem 2.69, and equals the same composite map from $U$ to $W$ along the lower row, so

$$
\begin{aligned}
(4.9)\qquad Tr_{V;U,W}(F) : u &\mapsto Ev_{VW}((n')^{-1}(F \circ (n_2(\eta_{VU}(u))))) \\
&= Tr_{V;W}(F \circ (n_2(\eta_{VU}(u)))) \\
&= Tr_{V;W}(F \circ (n_1(Id_V \otimes u))).
\end{aligned}
$$

$\blacksquare$

The composition in (4.9) from the lower path in the diagram, or equivalently

$$
(4.10)\qquad Tr_{V;U,W}(F) = Ev_{VW} \circ (n')^{-1} \circ \mathrm{Hom}(Id_V, F) \circ n_2 \circ \eta_{VU} : U \to W,
$$

has two interesting properties: it does not involve scalar multiplication or duals (except in the construction of $\eta_{VU}$), and the maps $\eta_{VU}$ and $Ev_{VW}$ appear in symmetric roles.

Notation 4.44 also allows for a comparison between (4.10) and the formula from Corollary 2.76,

$$
Tr_{V;U,W}(F) = Ev_{VW} \circ (n')^{-1} \circ (q^{-1}(F)).
$$

As in (2.19), using $n_2 \circ \eta_{VU} : u \mapsto n_1(Id_V \otimes u)$, the composite $\mathrm{Hom}(Id_V, F) \circ n_2 \circ \eta_{VU}$ is equal to $q^{-1}(F)$:

$$
\begin{aligned}
\mathrm{Hom}(Id_V, F) \circ n_2 \circ \eta_{VU} : u &\mapsto F \circ (n_1(Id_V \otimes u)) : \\
v &\mapsto F(v \otimes u) = ((q^{-1}(F))(u))(v).
\end{aligned}
$$

EXERCISE 4.46. In the case $U = \mathbb{K}$, these maps are equal: $\eta_{VU} = \eta_{V\mathbb{K}} = \eta_V$.

$\blacksquare$

LEMMA 4.47. *For finite-dimensional $V$ and $V'$, and maps $A : U \to U'$, $B : V \to V'$, the following diagram is commutative.*

$$
\begin{tikzcd}
U \arrow[r, "\eta_{VU}"] \arrow[dd, "A"'] & V \otimes \mathrm{Hom}(V, U) \arrow[d, "{[B \otimes \mathrm{Hom}(Id_V, A)]}"] \\
& V' \otimes \mathrm{Hom}(V, U') \\
U' \arrow[r, "\eta_{V'U'}"] & V' \otimes \mathrm{Hom}(V', U') \arrow[u, "{[Id_{V'} \otimes \mathrm{Hom}(B, Id_{U'})]}"']
\end{tikzcd}
$$

PROOF. First consider this diagram, with switching maps $s$, $s'$, and $s''$, and where the $k$ maps are invertible by Lemma 1.64.

$$\begin{tikzcd}
V^* \otimes V \arrow[r, "s"] \arrow[d, "k_{VV}"'] \arrow[ddr, "{[Id_{V^*} \otimes B]}"] & V \otimes V^* \arrow[ddr, "{[B \otimes Id_{V^*}]}"] & \\
\mathrm{End}(V) \arrow[d, "\mathrm{Hom}(Id_V, B)"'] & & \\
\mathrm{Hom}(V, V') & V^* \otimes V' \arrow[l, "k_{VV'}"'] \arrow[r, "s''"] & V' \otimes V^* \\
\mathrm{End}(V') \arrow[u, "\mathrm{Hom}(B, Id_{V'})"] & & \\
V'^* \otimes V' \arrow[u, "k_{V'V'}"] \arrow[uur, "{[B^* \otimes Id_{V'}]}"'] \arrow[r, "s'"] & V' \otimes V'^* \arrow[uur, "{[Id_{V'} \otimes B^*]}"'] &
\end{tikzcd}$$

The right blocks are commutative by Lemma 1.39 and the left blocks by Lemma 1.62. The distinguished elements $Id_V$ and $Id_{V'}$ have the same image in $\mathrm{Hom}(V, V')$:

$$\mathrm{Hom}(Id_V, B)(Id_V) = B \circ Id_V = B = Id_{V'} \circ B = \mathrm{Hom}(B, Id_{V'})(Id_{V'}).$$

By the commutativity of the blocks in in the diagram,

$$\begin{aligned}
(s'' \circ k_{VV'}^{-1})(B) &= ([B \otimes Id_{V^*}] \circ s \circ k_{VV}^{-1})(Id_V) \\
&= ([Id_{V'} \otimes B^*] \circ s' \circ k_{V'V'}^{-1})(Id_{V'}).
\end{aligned} \tag{4.11}$$

In this second diagram, the commutativity of the center left block is the claim of the Theorem.

$$\begin{tikzcd}
 & \mathrm{End}(V)\otimes U & \\
\mathbb{K}\otimes U \arrow[ur, "{[Q_1^1\otimes Id_U]}"] \arrow[dd, "l_U"'] & V^*\otimes V\otimes U \arrow[u, "{[k_{VV}\otimes Id_U]}"'] \arrow[d, "{[s\otimes Id_U]}"] & \\
 & V\otimes V^*\otimes U \arrow[d, "{[Id_V\otimes k_{VU}]}"'] \arrow[ddr, "{[B\otimes[Id_{V^*}\otimes A]]}"] & \\
U \arrow[r, "\eta_{UV}"'] \arrow[dd, "A"'] & V\otimes \mathrm{Hom}(V,U) \arrow[d, "{[B\otimes \mathrm{Hom}(Id_V,A)]}"'] & \\
 & V'\otimes \mathrm{Hom}(V,U') & V'\otimes V^*\otimes U' \arrow[l, "{[Id_{V'}\otimes k_{VU'}]}"] \\
U' \arrow[r, "\eta_{U'V'}"] & V'\otimes \mathrm{Hom}(V',U') \arrow[u, "{[Id_{V'}\otimes \mathrm{Hom}(B,Id_{U'})]}"] & \\
 & V'\otimes V'^*\otimes U' \arrow[u, "{[Id_{V'}\otimes k_{V'U'}]}"] \arrow[uur, "{[Id_{V'}\otimes[B^*\otimes Id_{U'}]]}"'] & \\
\mathbb{K}\otimes U' \arrow[uu, "l_{U'}"] \arrow[dr, "{[\tilde{Q}_1^1\otimes Id_{U'}]}"'] & V'^*\otimes V'\otimes U' \arrow[u, "{[s'\otimes Id_{U'}]}"'] \arrow[d, "{[k_{V'V'}\otimes Id_{U'}]}"] & \\
 & \mathrm{End}(V')\otimes U' &
\end{tikzcd}$$

The upper and lower left blocks are from the definition of the $\eta_{UV}$, $\eta_{U'V'}$ maps from Notation 4.44, and the right blocks are commutative by Lemma 1.36 and Lemma 1.62. So to establish the claim it is enough to check that these two paths from $U$ to $V'\otimes V^*\otimes U'$ define the same composite.

$$\begin{aligned}
&(4.12) && [B\otimes[Id_{V^*}\otimes A]]\circ[s\otimes Id_U]\circ[k_{VV}^{-1}\otimes Id_U]\circ[Q_1^1\otimes Id_U]\circ l_U^{-1}: \\
&u && \mapsto ([B\otimes[Id_{V^*}\otimes A]]\circ[s\otimes Id_U]\circ[k_{VV}^{-1}\otimes Id_U])(Id_V\otimes u) \\
& && = (([B\otimes Id_{V^*}]\circ s\circ k_{VV}^{-1})(Id_V))\otimes(A(u)), \\
&(4.13) && [Id_{V'}\otimes[B^*\otimes Id_{U'}]]\circ[s'\circ Id_{U'}]\circ[k_{V'V'}^{-1}\otimes Id_{U'}]\circ[\tilde{Q}_1^1\otimes Id_{U'}]\circ l_{U'}^{-1}\circ A: \\
&u && \mapsto ([Id_{V'}\otimes[B^*\otimes Id_{U'}]]\circ[s'\circ Id_{U'}]\circ[k_{V'V'}^{-1}\otimes Id_{U'}])(Id_{V'}\otimes(A(u))) \\
& && = (([Id_{V'}\otimes B^*]\circ s'\circ k_{V'V'}^{-1})(Id_{V'}))\otimes(A(u)).
\end{aligned}$$

The maps (4.12) and (4.13) are equal by (4.11). ∎

The following pair of Theorems are analogues of Theorem 2.96; the idea is that $\eta_{VU}$ and $Ev_{VW}$ satisfy identities analogous to the abstractly defined evaluation and coevaluation maps as in Definition 2.97. Theorem 4.49 uses the transpose for vector valued forms from Definition 4.2.

THEOREM 4.48. *For any $U$ and $V$, if $V$ is finite-dimensional then the following composite is equal to a switching map:*

$$[Id_V \otimes Ev_{VU}] \circ [\eta_{VU} \otimes Id_V] = s_0 : U \otimes V \to V \otimes U.$$

PROOF. The claim is analogous to the first identity from Theorem 2.96, and the proof is also analogous. The labeling $V = V_1 = V_2 = V_3$ is introduced to track the action of the switching $s$ maps. The upper and middle left squares are from the diagram from Notation 4.44. The claim is that the lower left triangle in the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(V_2, V_1) \otimes U \otimes V_3 & V_2^* \otimes V_1 \otimes U \otimes V_3 \arrow[l, "{[[k \otimes Id_U] \otimes Id_V]}"'] \arrow[r, "{[Id_{V^*} \otimes s_4^{-1}]}"'] \arrow[d, "{[[s \otimes Id_U] \otimes Id_V]}"] & V_2^* \otimes V_3 \otimes V_1 \otimes U \arrow[dd, "{[Ev_V \otimes Id_{V \otimes U}]}"'] \\
\mathbb{K} \otimes U \otimes V_3 \arrow[u, "{[[Q_1^1 \otimes Id_U] \otimes Id_V]}"'] \arrow[r, "{[[\eta_V \otimes Id_U] \otimes Id_V]}"] \arrow[d, "{[l_U \otimes Id_V]}"] & V_1 \otimes V_2^* \otimes U \otimes V_3 \arrow[d, "{[[Id_V \otimes k_{VU}] \otimes Id_V]}"] & \\
U \otimes V_3 \arrow[r, "{[\eta_{VU} \otimes Id_V]}"] \arrow[dr, "s_0"] & V_1 \otimes \mathrm{Hom}(V_2, U) \otimes V_3 \arrow[d, "{[Id_V \otimes Ev_{VU}]}"] & \mathbb{K} \otimes V_1 \otimes U \arrow[dl, "l_{V \otimes U}"] \\
 & V \otimes U &
\end{tikzcd}$$

The commutativity of the right block is easy to check, where the switching map $s_4$ is as in Theorem 2.87 and Corollary 2.88. The composition starting at $U \otimes V$ in the lower left and going all the way around the diagram clockwise to $V \otimes U$ is the trace, as in Theorem 2.69, of $s_4^{-1}$, and the computation of Example 2.92 applies.

$$\begin{aligned}
& [Id_V \otimes Ev_{VU}] \circ [\eta_{VU} \otimes Id_V] \\
= \ & l_{V \otimes U} \circ [Ev_V \otimes Id_{V \otimes U}] \circ [Id_{V^*} \otimes s_4^{-1}] \circ [(k^{-1} \circ Q_1^1) \otimes Id_{U \otimes V}] \circ [l_U^{-1} \otimes Id_V] \\
= \ & Tr_{V;U \otimes V, V \otimes U}(s_4^{-1}) = s_3^{-1} = s_0.
\end{aligned}$$

∎

THEOREM 4.49. *For any $U$ and $V$, if $V$ is finite-dimensional, then the $n$ maps indicated in the following diagram are invertible,*

$$\begin{tikzcd}
\mathrm{Hom}(V, U) \arrow[r, "{\mathrm{Hom}(Id_V, \eta_{VU})}"] & \mathrm{Hom}(V, V \otimes \mathrm{Hom}(V, U)) & \mathrm{Hom}(V, \mathrm{Hom}(V, U)) \otimes V \arrow[l, "n_1"'] \arrow[d, "{[T_{V;U} \otimes Id_V]}"'] \\
 & \mathrm{Hom}(V, \mathrm{Hom}(V, U) \otimes V) \arrow[ul, "{\mathrm{Hom}(Id_V, Ev_{VU})}"] & \mathrm{Hom}(V, \mathrm{Hom}(V, U)) \otimes V \arrow[l, "n_2"']
\end{tikzcd}$$

*and the diagram is commutative in the sense that this composite map is equal to the identity map:*

$$\mathrm{Hom}(Id_V, Ev_{VU}) \circ n_2 \circ [T_{V;U} \otimes Id_V] \circ n_1^{-1} \circ \mathrm{Hom}(Id_V, \eta_{VU}) = Id_{\mathrm{Hom}(V,U)}.$$

PROOF. The claim is analogous to the second identity from Theorem 2.96, and the overall proof is also analogous. As in the Proof of Theorem 4.48, the labeling $V = V_1 = V_2 = V_3$ is introduced to track the action of the $n$, $p$, and $s$ maps in this "main diagram."

$$\begin{tikzcd}
 & & V_1^* \otimes U \arrow[ddll, "k_{VU}"'] \\
 & & \mathbb{K} \otimes V_1^* \otimes U \arrow[u, "l_{V^*\otimes U}"'] \arrow[d, "{[Q_1^1 \otimes Id_{V^*\otimes U}]}"'] \\
\mathrm{Hom}(V_1, U) \arrow[d, "{\mathrm{Hom}(Id_V, \eta_{VU})}"] & V_2^* \otimes V_3 \otimes V_1^* \otimes U \arrow[r, "{[k \otimes Id_{V^*\otimes U}]}"'] \arrow[ddr, "{[p \otimes Id_{V^*\otimes U}]}"] & \mathrm{Hom}(V_2, V_3) \otimes V_1^* \otimes U \arrow[d, "{[t \otimes Id_{V^*\otimes U}]}"] \\
\mathrm{Hom}(V_1, V_3 \otimes \mathrm{Hom}(V_2, U)) & V_1^* \otimes V_3 \otimes V_2^* \otimes U \arrow[l, "a_1"] \arrow[u, "{[s'' \otimes Id_U]}"] \arrow[d, "s_1"'] & \mathrm{Hom}(V_3^*, V_2^*) \otimes V_1^* \otimes U \\
\mathrm{Hom}(V_1, \mathrm{Hom}(V_2, U)) \otimes V_3 \arrow[u, "n_1"] \arrow[d, "{[T_{V;U} \otimes Id_V]}"] & V_2^* \otimes V_1^* \otimes U \otimes V_3 \arrow[r, "p'"'] \arrow[ddl, "a_2"] & V_3^{**} \otimes V_2^* \otimes V_1^* \otimes U \arrow[u, "{[k' \otimes Id_{V^*\otimes U}]}"'] \arrow[d, "{[Id_{V^{**}} \otimes s_4]}"'] \\
\mathrm{Hom}(V_2, \mathrm{Hom}(V_1, U)) \otimes V_3 \arrow[d, "n_2"] & & V_3^{**} \otimes V_1^* \otimes U \otimes V_2^* \arrow[d, "{[Ev_{V^*} \otimes Id_{U\otimes V^*}]}"'] \\
\mathrm{Hom}(V_2, \mathrm{Hom}(V_1, U) \otimes V_3) \arrow[d, "{\mathrm{Hom}(Id_V, Ev_{VU})}"] & & \mathbb{K} \otimes U \otimes V_2^* \arrow[d, "l_{U\otimes V^*}"'] \\
\mathrm{Hom}(V_2, U) & V_2^* \otimes U \arrow[l, "k_{VU}"] & U \otimes V_2^* \arrow[l, "s_3^{-1}"]
\end{tikzcd}$$

The abbreviations $t_{VV} = t$, $k_{VV} = k$, $k_{V^*V^*} = k'$, and the map $p$ are as in the notation from Lemma 2.5. The switching involution $s''$ appeared in Lemma 2.84. The composite $t \circ Q_1^1 : \mathbb{K} \to \mathrm{End}(V^*)$ is equal to another inclusion, temporarily denoted $\tilde{Q}_1^1$, that maps 1 to $Id_{V^*} = Id_V^* = t(Id_V)$ as in Equation (2.4). So, the composition in the right column is the trace, as in Theorem 2.69, of $s_4$, and the computation of Example 2.91 applies:

$$\begin{aligned}
& l_{U\otimes V^*} \circ [Ev_{V^*} \otimes Id_{U\otimes V^*}] \circ [Id_{V^{**}} \otimes s_4] \circ [((k')^{-1} \circ \tilde{Q}_1^1) \otimes Id_{V^*\otimes U}] \circ l_{V^*\otimes U}^{-1} \\
= \quad & Tr_{V;V^*\otimes U, U\otimes V^*}(s_4) = s_3.
\end{aligned}$$

The $p$, $k$, and $n$ maps are invertible by the finite-dimensionality of $V$; in the right center block, the triangle with $k$, $t$, and $p$ is commutative by Lemma 1.75, and it is easy to check that the other triangle with $p'$ is also commutative. The claim of the Theorem is that the composition in the left column gives the identity map; this will follow if we can find $a_1$ and $a_2$ as indicated that make the main diagram commutative.

The following maps, and $s_1$ in the above diagram, are from Lemma 4.10, in the case $V = V_1 = V_2 = V_3$, $U = W$:

$$\begin{aligned}
a_1 &= \mathrm{Hom}(Id_V, [Id_V \otimes k_{VU}]) \circ k_{V,V\otimes V^*\otimes U} \\
a_2 &= \mathrm{Hom}(Id_V, [k_{VU} \otimes Id_V]) \circ k_{V,V^*\otimes U\otimes V}.
\end{aligned}$$

The commutativity of the left center block then follows from Lemma 4.10, so to prove the Theorem it only remains to show that these maps $a_1$, $a_2$ make the upper and lower blocks of the main diagram commutative.

To check the lower block, start with $\psi \otimes \phi \otimes u \otimes v \in V^* \otimes V^* \otimes U \otimes V$ at its top, and $w \in V$.

$$\begin{array}{rcl}
 & & \mathrm{Hom}(Id_V, Ev_{VU}) \circ \mathrm{Hom}(Id_V, [k_{VU} \otimes Id_V]) \circ k_{V,V^* \otimes U \otimes V} : \\
\psi \otimes \phi \otimes u \otimes v & \mapsto & Ev_{VU} \circ [k_{VU} \otimes Id_V] \circ (k_{V,V^* \otimes U \otimes V}(\psi \otimes \phi \otimes u \otimes v)) : \\
w & \mapsto & Ev_{VU}([k_{VU} \otimes Id_V](\psi(w) \cdot \phi \otimes u \otimes v)) \\
 & = & Ev_{VU}(\psi(w) \cdot (k_{VU}(\phi \otimes u)) \otimes v) = \psi(w) \cdot \phi(v) \cdot u, \\
 & & k_{VU} \circ s_3^{-1} \circ l_{U \otimes V^*} \circ [Ev_{V^*} \otimes Id_{U \otimes V^*}] \circ [Id_{V^{**}} \otimes s_4] \circ p' : \\
\psi \otimes \phi \otimes u \otimes v & \mapsto & (k_{VU} \circ s_3^{-1} \circ l_{U \otimes V^*} \circ [Ev_{V^*} \otimes Id_{U \otimes V^*}])((d_V(v)) \otimes \phi \otimes u \otimes \psi) \\
 & = & (k_{VU} \circ s_3^{-1})(\phi(v) \cdot u \otimes \psi) = \phi(v) \cdot k_{VU}(\psi \otimes u) : \\
w & \mapsto & \phi(v) \cdot \psi(w) \cdot u.
\end{array}$$

In the following diagram, the two lower right commutative squares are from the definition of $\eta_{VU}$.

$$\begin{tikzcd}
 & V_2^* \otimes V_3 \otimes V_1^* \otimes U \arrow[d, "{[k \otimes Id_{V^* \otimes U}]}"] & \\
 & \mathrm{Hom}(V_2, V_3) \otimes V_1^* \otimes U \arrow[d, "{[s_5 \otimes Id_U]}"] & V_1^* \otimes V_3 \otimes V_2^* \otimes U \arrow[ul, "{[s'' \otimes Id_U]}"'] \arrow[d, "k_{V,V \otimes V^* \otimes U}"'] \\
\mathbb{K} \otimes V_1^* \otimes U \arrow[ur, "{[Q_1^1 \otimes Id_{V^* \otimes U}]}"] \arrow[dd, "l_{V^* \otimes U}"] & V_1^* \otimes \mathrm{Hom}(V_2, V_3) \otimes U \arrow[d, "k_{V,\mathrm{End}(V) \otimes U}"] & \mathrm{Hom}(V, V_3 \otimes V_2^* \otimes U) \\
 & \mathrm{Hom}(V_1, \mathrm{Hom}(V_2, V_3) \otimes U) & \mathrm{Hom}(V_1, V_2^* \otimes V_3 \otimes U) \arrow[l, "{\mathrm{Hom}(Id_V, [k \otimes Id_U])}"'] \arrow[u, "{\mathrm{Hom}(Id_V, [s \otimes Id_U])}"'] \arrow[d, "{\mathrm{Hom}(Id_V, [s \otimes Id_U])}"'] \\
V_1^* \otimes U \arrow[dr, "k_{VU}"'] & \mathrm{Hom}(V_1, \mathbb{K} \otimes U) \arrow[u, "{\mathrm{Hom}(Id_V, [Q_1^1 \otimes Id_U])}"] \arrow[r, "{\mathrm{Hom}(Id_V, [\eta_V \otimes Id_U])}"'] \arrow[d, "{\mathrm{Hom}(Id_V, l_U)}"] & \mathrm{Hom}(V_1, V_3 \otimes V_2^* \otimes U) \arrow[d, "{\mathrm{Hom}(Id_V, [Id_V \otimes k_{VU}])}"'] \\
 & \mathrm{Hom}(V_1, U) \arrow[r, "{\mathrm{Hom}(Id_V, \eta_{VU})}"'] & \mathrm{Hom}(V_1, V_3 \otimes \mathrm{Hom}(V_2, U))
\end{tikzcd}$$

To check the commutativity of the upper right block, start with $\phi \otimes v \otimes \psi \otimes u \in V_1^* \otimes V_3 \otimes V_2^* \otimes U$ and $w \in V$:

$$\begin{array}{rcl}
 & & \mathrm{Hom}(Id_V, [k \otimes Id_U]) \circ \mathrm{Hom}(Id_V, [s^{-1} \otimes Id_U]) \circ k_{V,V \otimes V^* \otimes U} : \\
\phi \otimes v \otimes \psi \otimes u & \mapsto & [(k \circ s^{-1}) \otimes Id_U] \circ (k_{V,V \otimes V^* \otimes U}(\phi \otimes v \otimes \psi \otimes u)) : \\
w & \mapsto & [(k \circ s^{-1}) \otimes Id_U](\phi(w) \cdot v \otimes \psi \otimes u) = \phi(w) \cdot (k(\psi \otimes v)) \otimes u, \\
 & & k_{V,\mathrm{End}(V) \otimes U} \circ [s_5 \otimes Id_U] \circ [k \otimes Id_{V^* \otimes U}] \circ [s'' \otimes Id_U] : \\
\phi \otimes v \otimes \psi \otimes u & \mapsto & k_{V,\mathrm{End}(V) \otimes U}(\phi \otimes (k(\psi \otimes v)) \otimes u) : \\
w & \mapsto & \phi(w) \cdot (k(\psi \otimes v)) \otimes u.
\end{array}$$

To check the commutativity of the left block, start with $\alpha\otimes\phi\otimes u\in\mathbb{K}\otimes V^*\otimes U$:

$$\begin{array}{rcl} & & k_{V,\mathrm{End}(V)\otimes U}\circ[s_5\otimes Id_U]\circ[Q_1^1\otimes Id_{V^*\otimes U}]: \\ \alpha\otimes\phi\otimes u & \mapsto & k_{V,\mathrm{End}(V)\otimes U}(\phi\otimes(\alpha\cdot Id_V)\otimes u): \\ v & \mapsto & (\phi(v))\cdot(\alpha\cdot Id_V)\otimes u, \\ & & \mathrm{Hom}(Id_V,[Q_1^1\otimes Id_U])\circ\mathrm{Hom}(Id_V,l_U^{-1})\circ k_{VU}\circ l_{V^*\otimes U}: \\ \alpha\otimes\phi\otimes u & \mapsto & [Q_1^1\otimes Id_U]\circ l_U^{-1}\circ(k_{VU}(\alpha\cdot\phi\otimes u)): \\ v & \mapsto & (\alpha\cdot\phi(v))\cdot Id_V\otimes u. \end{array}$$

The downward composite of the four arrows in the right column equals the previously defined $a_1$. So, the above calculation is enough to establish the commutativity of the top block in the main diagram:

$$\mathrm{Hom}(Id_V,\eta_{VU})\circ k_{VU}=a_1\circ[(s'')^{-1}\otimes Id_U]\circ[(k^{-1}\circ Q_1^1)\otimes Id_{V^*\otimes U}]\circ l_{V^*\otimes U}^{-1}.$$

As mentioned earlier, this proves the claim of the Theorem. ∎

## 4.5. Topics and applications

### 4.5.1. Quadratic forms.

PROPOSITION 4.50. *Given vector spaces $V$, $W$, and a function $\mathfrak{q} : V \rightsquigarrow W$, if $\frac{1}{2} \in \mathbb{K}$ then the following are equivalent.*

(1) *There exists a symmetric $W$-valued form $h_1 : V \to \mathrm{Hom}(V, W)$ such that for all $v \in V$,*
$$\mathfrak{q}(v) = (h_1(v))(v).$$
(2) *There exists a $W$-valued bilinear form $h_2 : V \to \mathrm{Hom}(V, W)$ such that for all $v \in V$,*
$$\mathfrak{q}(v) = (h_2(v))(v).$$
(3) *There exists a bilinear function $B_1 : V \times V \rightsquigarrow W$ such that for all $v \in V$,*
$$\mathfrak{q}(v) = B_1(v, v).$$
(4) *For any $\alpha \in \mathbb{K}$ and $v \in V$, $\mathfrak{q}(\alpha \cdot v) = \alpha^2 \cdot \mathfrak{q}(v)$, and the function $B_2 : V \times V \rightsquigarrow W$ defined by*
$$B_2(u, v) = \mathfrak{q}(u + v) - \mathfrak{q}(u) - \mathfrak{q}(v)$$
*is bilinear.*
(5) *For any $\alpha \in \mathbb{K}$ and $v \in V$, $\mathfrak{q}(\alpha \cdot v) = \alpha^2 \cdot \mathfrak{q}(v)$, and the function $B_3 : V \times V \rightsquigarrow W$ defined by*
$$B_3(u, v) = \mathfrak{q}(u) + \mathfrak{q}(v) - \mathfrak{q}(u - v)$$
*is bilinear.*
(6) *For any $\alpha \in \mathbb{K}$ and $v \in V$, $\mathfrak{q}(\alpha \cdot v) = \alpha^2 \cdot \mathfrak{q}(v)$, and the function $B_4 : V \times V \rightsquigarrow W$ defined by*
$$B_4(u, v) = \mathfrak{q}(u + v) - \mathfrak{q}(u - v)$$
*is bilinear.*
(7) *For all $u, v \in V$, $\mathfrak{q}$ satisfies:*
$$\mathfrak{q}(u + v) + \mathfrak{q}(u - v) = 2 \cdot \mathfrak{q}(u) + 2 \cdot \mathfrak{q}(v), \tag{4.14}$$
*and the function $B_4 : V \times V \rightsquigarrow W$ defined by*
$$B_4(u, v) = \mathfrak{q}(u + v) - \mathfrak{q}(u - v)$$
*satisfies, for all $\alpha \in \mathbb{K}$, $B_4(\alpha \cdot u, v) = \alpha \cdot B_4(u, v)$.*

PROOF. As in Notation 0.41, the $\rightsquigarrow$ arrow symbol refers to functions which are not necessarily linear. The functions $B_1, \ldots, B_4$ are bilinear as in Definition 1.23.

The implication (1) $\implies$ (2) is trivial. For (2) $\implies$ (3), define $B_1$ by $B_1(u, v) = (h_2(u))(v)$, and similarly for (3) $\implies$ (2), define $h_2$ by the formula $h_2(u) : v \mapsto B_1(u, v)$. The correspondence between $h_2$ and $B_1$ is a $W$-valued version of the construction from Example 1.55.

Now, assuming (3), so that $B_1$ is bilinear and $\mathfrak{q}(v) = B_1(v, v)$, the first property from (4), (5) and (6) is immediate:
$$\mathfrak{q}(\alpha \cdot v) = B_1(\alpha \cdot v, \alpha \cdot v) = \alpha^2 \cdot B_1(v, v) = \alpha^2 \cdot \mathfrak{q}(v).$$

Expanding $B_2$, $B_3$, $B_4$ in terms of $B_1$:

$$\begin{aligned}
B_2(u,v) &= \mathfrak{q}(u+v)-\mathfrak{q}(u)-\mathfrak{q}(v)\\
&= B_1(u+v,u+v)-B_1(u,u)-B_1(v,v)\\
&= B_1(u,v)+B_1(v,u).\\
B_3(u,v) &= \mathfrak{q}(u)+\mathfrak{q}(v)-\mathfrak{q}(u-v)\\
&= B_1(u,u)+B_1(v,v)-B_1(u-v,u-v)\\
&= B_1(u,v)+B_1(v,u).\\
B_4(u,v) &= \mathfrak{q}(u+v)-\mathfrak{q}(u-v)\\
&= B_1(u+v,u+v)-B_1(u-v,u-v)\\
&= B_1(u,v)+B_1(v,u)+B_1(u,v)+B_1(v,u). \qquad (4.15)
\end{aligned}$$

The bilinearity of $B_1$ implies the bilinearity of $B_1(u,v)+B_1(v,u)$, so (3) implies (4), (5), and (6). The relation $B_4 = B_2 + B_3$ also shows that any two of (4), (5), and (6) together imply the third.

For (3) $\implies$ (7), expanding (4.14) in terms of $B_1$ shows $LHS = RHS$ (a related quantity already appeared in Proposition 3.125), and $B_4$ is bilinear as in (4.15).

So far, the implications have not yet used $\frac{1}{2} \in \mathbb{K}$. To show (6) $\implies$ (3), given the bilinear form $B_4$, define $B_1(u,v) = \left(\frac{1}{2}\right)^2 \cdot B_4(u,v)$, so that $B_1$ is bilinear, and for any $v \in V$,

$$\begin{aligned}
B_1(v,v) &= \left(\frac{1}{2}\right)^2 \cdot (\mathfrak{q}(v+v)-\mathfrak{q}(v-v))\\
&= \left(\frac{1}{2}\right)^2 \cdot (\mathfrak{q}((1+1)\cdot v)-\mathfrak{q}(0\cdot v))\\
&= \left(\frac{1}{2}\right)^2 \cdot \big((1+1)^2\cdot\mathfrak{q}(v)-0^2\cdot\mathfrak{q}(v)\big) = \mathfrak{q}(v). \qquad (4.16)
\end{aligned}$$

Similar calculations using $\frac{1}{2} \in \mathbb{K}$ would directly show (4) $\implies$ (3) and (5) $\implies$ (3).

For (7) $\implies$ (3), Equation (4.14) with $u = v = 0_V$ gives $\mathfrak{q}(0_V) = 0_W$, and with $u = 0_V$ gives $\mathfrak{q}(-v) = \mathfrak{q}(v)$. Then

$$B_4(v,u) = \mathfrak{q}(v+u)-\mathfrak{q}(v-u) = \mathfrak{q}(v+u)-\mathfrak{q}(u-v) = B_4(u,v).$$

It follows that $B_4(u,\alpha\cdot v) = B_4(\alpha\cdot v,u) = \alpha\cdot B_4(v,u) = \alpha\cdot B_4(u,v)$. The following calculation shows that $2\cdot B_4$ is additive in the first entry, using (4.14) in steps (4.18) and (4.20) and some add-and-subtract steps in (4.17) and (4.19).

$$\begin{aligned}
& 2 \cdot B_4(u_1+u_2,v) \\
= \;& 2\cdot\mathfrak{q}(u_1+u_2+v) - 2\cdot\mathfrak{q}(u_1+u_2-v) \\
= \;& 2\cdot\mathfrak{q}(u_1+u_2+v) + 2\cdot\mathfrak{q}(u_1-v) - 2\cdot\mathfrak{q}(u_1+u_2-v) - 2\cdot\mathfrak{q}(u_1+v) \\
& +2\cdot\mathfrak{q}(u_1+v) - 2\cdot\mathfrak{q}(u_1-v) \qquad (4.17) \\
= \;& \mathfrak{q}(2\cdot u_1+u_2) + \mathfrak{q}(u_2+2\cdot v) - \mathfrak{q}(2\cdot u_1+u_2) - \mathfrak{q}(u_2-2\cdot v) \\
& +2\cdot\mathfrak{q}(u_1+v) - 2\cdot\mathfrak{q}(u_1-v) \qquad (4.18) \\
= \;& \mathfrak{q}(u_2+2\cdot v) + \mathfrak{q}(u_2) - \mathfrak{q}(u_2-2\cdot v) - \mathfrak{q}(u_2) \\
& +2\cdot\mathfrak{q}(u_1+v) - 2\cdot\mathfrak{q}(u_1-v) \qquad (4.19) \\
= \;& 2\cdot\mathfrak{q}(u_2+v) + 2\cdot\mathfrak{q}(v) - 2\cdot\mathfrak{q}(u_2-v) - 2\cdot\mathfrak{q}(-v) \\
& +2\cdot\mathfrak{q}(u_1+v) - 2\cdot\mathfrak{q}(u_1-v) \qquad (4.20) \\
= \;& 2\cdot B_4(u_1,v) + 2\cdot B_4(u_2,v).
\end{aligned}$$

By symmetry again, $2\cdot B_4$ is bilinear, and so is $B_1 = \left(\frac{1}{2}\right)^2 \cdot B_4$. The following calculation establishing (3) using (4.14) is different from (4.16):

$$\begin{aligned}
B_1(v,v) &= \left(\frac{1}{2}\right)^2 \cdot (\mathfrak{q}(v+v) - \mathfrak{q}(v-v)) \\
&= \left(\frac{1}{2}\right)^2 \cdot (\mathfrak{q}(v+v) + \mathfrak{q}(v-v)) \\
&= \left(\frac{1}{2}\right)^2 \cdot (2\cdot\mathfrak{q}(v) + 2\cdot\mathfrak{q}(v)) = \mathfrak{q}(v).
\end{aligned}$$

Finally, to show (2) $\implies$ (1) using $\frac{1}{2}\in\mathbb{K}$, let $h_1$ be the symmetric part of $h_2$ as in (3.2) and (4.6):

$$\begin{aligned}
(h_1(u))(v) &= \frac{1}{2}\cdot((h_2(u))(v) + (h_2(v))(u)) \\
\implies (h_1(v))(v) &= \frac{1}{2}\cdot(\mathfrak{q}(v)+\mathfrak{q}(v)) = \mathfrak{q}(v).
\end{aligned}$$

∎

DEFINITION 4.51. Assuming $\frac{1}{2}\in\mathbb{K}$, a function $\mathfrak{q}: V \rightsquigarrow W$ satisfying any of the equivalent properties from Proposition 4.50 is a $W$-valued <u>quadratic form</u>.

REMARK 4.52. The equations from (4), (5), and (6) are known as <u>polarization formulas</u>. Equation (4.14) from (7) is the <u>parallelogram law</u> for $\mathfrak{q}$. The case where $\frac{1}{2}\notin\mathbb{K}$ is more complicated and not considered here; the remaining statements here in Section 4.5.1 will all assume $\frac{1}{2}\in\mathbb{K}$.

EXERCISE 4.53. Given a quadratic form $\mathfrak{q}$, the symmetric form $h_1$ from Proposition 4.50 is unique.

HINT. This is a statement about symmetric forms rather than quadratic forms: the claim is that if $h_0$ and $h_1$ are both symmetric forms and $(h_0(v))(v) = (h_1(v))(v)$ for all $v\in V$, then $h_0 = h_1$. The hint is to expand $(h_0(u+v))(u+v)$ and use $\frac{1}{2}\in\mathbb{K}$. ∎

EXERCISE 4.54. The set $\mathfrak{Q}(V;W)$ of $W$-valued quadratic forms on $V$ is a vector space. The map $\mathfrak{f}_{VW}$ defined by $\mathfrak{f}_{VW}:\mathfrak{Q}(V;W)\to Sym(V;W):\mathfrak{q}\mapsto h_1$ as in Exercise 4.53 is linear and invertible.

HINT. $\mathfrak{Q}(V;W)$ is a subspace of $\mathcal{F}(V,W)$ as in Example 6.28. ∎

DEFINITION 4.55. For vector spaces $U$, $V$, $W$, define a linear map $\mathfrak{t}_{UV}^W$ by:

$$\begin{array}{rcl} \mathfrak{t}_{UV}^W:\mathrm{Hom}(U,V) & \to & \mathrm{Hom}(\mathfrak{Q}(V;W),\mathfrak{Q}(U;W)) \\ H & \mapsto & (\mathfrak{q}\mapsto\mathfrak{q}\circ H). \end{array}$$

EXERCISE 4.56. There are a few things to check in Definition 4.55: first, that for $H:U\to V$, the composite $\mathfrak{q}\circ H$ is a quadratic form, second, that $\mathfrak{t}_{UV}^W(H)$ is linear, and third, that $\mathfrak{t}_{UV}^W$ is linear. Further, the following diagram is commutative.

$$\begin{tikzcd}
\mathfrak{Q}(V;W) \arrow[rr, "\mathfrak{t}_{UV}^W(H)"] \arrow[d, "\mathfrak{f}_{VW}"] & & \mathfrak{Q}(U;W) \arrow[d, "\mathfrak{f}_{UW}"] \\
Sym(V;W) \arrow[rr] \arrow[d, hook] & & Sym(U;W) \arrow[d, hook] \\
\mathrm{Hom}(V,\mathrm{Hom}(V,W)) \arrow[rr, "\mathrm{Hom}(H,\mathrm{Hom}(H,Id_W))"] & & \mathrm{Hom}(U,\mathrm{Hom}(U,W))
\end{tikzcd}$$

The middle horizontal arrow is the map induced by $\mathrm{Hom}(H,\mathrm{Hom}(H,Id_W))$ as in Lemma 4.17.

HINT. The linearity of $\mathfrak{t}_{UV}^W(H)$ can be checked directly, but also follows from the commutativity of the upper block in the diagram. It is enough to check the commutativity of the large block from upper left to lower right; temporarily denote the left inclusion $Q_V$ and the right inclusion $Q_U$. For $\mathfrak{q}$ with $\mathfrak{f}_{VW}(\mathfrak{q})=h_1$,

$$\begin{array}{rrcl} & \mathrm{Hom}(H,Id_W)\circ(Q_V(\mathfrak{f}_{VW}(\mathfrak{q})))\circ H:v & \mapsto & (h_1(H(v)))\circ H: \\ (4.21) & u & \mapsto & (h_1(H(v)))(H(u)). \end{array}$$

Then $Q_U\circ\mathfrak{f}_{UW}\circ(\mathfrak{t}_{UV}^W(H)):\mathfrak{q}\mapsto Q_U(\mathfrak{f}_{UW}(\mathfrak{q}\circ H))=h_0$ is the unique $W$-valued bilinear form that is symmetric and that has the property $(\mathfrak{q}\circ H)(u)=(h_0(u))(u)$. However, $\mathfrak{q}(H(u))=(h_1(H(u)))(H(u))$, so $\mathrm{Hom}(H,\mathrm{Hom}(H,Id_W))(Q_V(h_1))$ from (4.21) has both these properties and is equal to $h_0$ by uniqueness, proving the commutativity of the diagram. ∎

EXERCISE 4.57. $\mathfrak{t}_{VV}^W(Id_V)=Id_{\mathfrak{Q}\ (V;W)}$. For $H:U\to V$ and $A:V\to X$,

$$\mathfrak{t}_{UX}^W(A\circ H)=\mathfrak{t}_{UV}^W(H)\circ\mathfrak{t}_{VX}^W(A).$$

In particular, if $H$ has a linear left (or right) inverse, then $\mathfrak{t}_{UV}^W(H)$ has a linear right (or left) inverse. ∎

PROPOSITION 4.58. ([HK] §10.2) *Suppose $U$ is finite-dimensional and $W\neq\{0_W\}$. For $H:U\to U$, the following are equivalent.*

(1) $\mathfrak{t}_{UU}^W(H):\mathfrak{Q}(U;W)\to\mathfrak{Q}(U;W)$ *is one-to-one.*
(2) $H$ *is invertible.*

PROOF. The (2) $\implies$ (1) direction follows from Exercise 4.57.

For the other direction, suppose, contrapositively, that $H$ is not invertible; then by Lemma 1.19 (which uses the finite-dimensional property of $U$), $H^* : U^* \to U^*$ is not invertible. By Claim 0.56 (using the finite-dimensional property of $U^*$), $H^*$ is not one-to-one and there exists $\phi \in U^*$ so that $\phi \neq 0_{U^*}$ and $H^*(\phi) = \phi \circ H = 0_{U^*}$. Pick any $w \in W$ with $w \neq 0_W$, and define a bilinear form

$$g = k_{U,\mathrm{Hom}(U,W)}(\phi \otimes (k_{UW}(\phi \otimes w))) \in \mathrm{Hom}(U, \mathrm{Hom}(U, W)).$$

$g \neq 0_{\mathrm{Hom}(U,\mathrm{Hom}(U,W))}$ because there is some $x \in U$ with $\phi(x) \neq 0$, and

$$(g(x))(x) = (\phi(x) \cdot k_{UW}(\phi \otimes w))(x) = \phi(x) \cdot \phi(x) \cdot w \neq 0_W.$$

Also, $g$ is symmetric:

$$\begin{aligned} (g(v_1))(v_2) &= (\phi(v_1) \cdot k_{UW}(\phi \otimes w))(v_2) = \phi(v_1) \cdot \phi(v_2) \cdot w, \\ (g(v_2))(v_1) &= (\phi(v_2) \cdot k_{UW}(\phi \otimes w))(v_1) = \phi(v_2) \cdot \phi(v_1) \cdot w. \end{aligned}$$

$\mathfrak{q} = \mathfrak{f}_{UW}^{-1}(g)$ is the quadratic form $\mathfrak{q}(v) = \phi(v) \cdot \phi(v) \cdot w$, and again using $v = x$, $\mathfrak{q} \neq 0_{\mathfrak{Q}\ (U;W)}$.

However, $\mathfrak{t}_{UU}^W(H) : \mathfrak{q} \mapsto \mathfrak{q} \circ H$, and for any $u \in U$,

$$(\mathfrak{q} \circ H)(u) = \mathfrak{q}(H(u)) = \phi(H(u)) \cdot \phi(H(u)) \cdot w = 0 \cdot 0 \cdot w = 0_W.$$

So $\mathfrak{q} \circ H = 0_{\mathfrak{Q}\ (U;W)}$, and $\mathfrak{t}_{UU}^W(H)$ is not one-to-one. ∎

### 4.5.2. Algebras.

DEFINITION 4.59. A vector space $V$ together with a $V$-valued bilinear form $h : V \to \mathrm{End}(V)$ is an algebra $(V, h)$.

DEFINITION 4.60. For any algebra $(V, h)$ with $V$ finite-dimensional, the canonical metric $(k^*)^{-1} \circ e$ on $\mathrm{End}(V)$ from Example 3.143 (and Equation (3.15)) pulls back by $h$ to give a scalar bilinear form on $V$, the Cartan-Killing form

$$\kappa = h^* \circ (k^*)^{-1} \circ e \circ h : v \mapsto \kappa(v) : u \mapsto Tr_V((h(v)) \circ (h(u))).$$

The CK form $\kappa$ is symmetric by Lemma 4.17 (or Lemma 3.8 or Lemma 2.6), but is not necessarily a metric on $V$.

THEOREM 4.61. *Given an algebra $(V, h)$, suppose $Q : U \to V$ satisfies, for any $u \in U$, $v \in V$,*

$$(h(Q(u)))(v) \in Q(U). \tag{4.22}$$

*If $Q$ is one-to-one, then for any left inverse of $Q$, $P : V \rightsquigarrow U$,*

$$h_1 : U \to \mathrm{End}(U) : u \mapsto P \circ (h(Q(u))) \circ Q \tag{4.23}$$

*defines an algebra $(U, h_1)$, and $h_1$ does not depend on the choice of $P$. If, further, $V$ is finite-dimensional, with CK form $\kappa$, then the CK form $\kappa_1$ of $(U, h_1)$ is the pullback of $\kappa$ by $Q$.*

PROOF. Using $P \circ Q = Id_U$ and the property (4.22), for any $u \in U$ and $v \in V$,

$$\begin{aligned} (h(Q(u)))(v) &\in Q(U) \\ \implies Q(P((h(Q(u)))(v))) &= (h(Q(u)))(v), \\ \implies Q \circ P \circ (h(Q(u))) &= h(Q(u)). \end{aligned} \tag{4.24}$$

Note that if $P' : V \rightsquigarrow U$ is any other (not necessarily linear) left inverse of $Q$, then composing $P'$ with both sides of (4.24) gives, for all $u \in U$:

$$P \circ (h(Q(u))) = P' \circ (h(Q(u))),$$

so the expression (4.23) does not depend on the choice of $P$ and defines $h_1$ uniquely. Further, because $h(Q(u)) \in \mathrm{End}(V)$ has image contained in $Q(U)$ by (4.22), the composite $P \circ (h(Q(u))) : V \to U$ is linear by Exercise 0.51, which justifies the use of $\mathrm{End}(U)$ as the target space in (4.23). It remains to check that $h_1$ is linear; for $u_1, u_2, u_3 \in U$,

$$\begin{aligned} h_1(u_1+u_2) &= P \circ (h(Q(u_1+u_2))) \circ Q \\ &= P \circ (h(Q(u_1)) + h(Q(u_2))) \circ Q : \\ u_3 &\mapsto P((h(Q(u_1)))(Q(u_3)) + (h(Q(u_2)))(Q(u_3))) \\ &= P((h(Q(u_1)))(Q(u_3))) + P((h(Q(u_2)))(Q(u_3))), \qquad (4.25) \\ &= (h_1(u_1) + h_1(u_2))(u_3). \end{aligned}$$

where step (4.25) is from Equation (0.1) in Exercise 0.51. The scaling property for $h_1$ similarly follows from Exercise 0.51. The conclusion is that if $Q : U \to V$ is one-to-one, then some left inverse exists (as in Exercise 0.48) and $(U, h_1)$ is an algebra as claimed. If $P$ is a linear left inverse, then the expression (4.23) can be denoted $h_1 = \mathrm{Hom}(Q, P) \circ h \circ Q$.

For finite-dimensional $V$, $U$ is also finite-dimensional (Exercise 0.50). The CK form on $U$ can be computed for $u_1, u_2 \in U$, using the linearity of the composites $P \circ (h(Q(u_1)))$ and $P \circ (h(Q(u_2)))$ so that Lemma 2.6 applies in step (4.26), and using Equation (4.24) in step (4.27):

$$\begin{aligned} (\kappa_1(u_1))(u_2) &= Tr_U((P \circ (h(Q(u_1))) \circ Q) \circ (P \circ (h(Q(u_2))) \circ Q)) \\ &= Tr_V((Q \circ P \circ (h(Q(u_1)))) \circ (Q \circ P \circ (h(Q(u_2))))) \qquad (4.26) \\ &= Tr_V((h(Q(u_1))) \circ (h(Q(u_2)))) \qquad (4.27) \\ &= ((Q^* \circ \kappa \circ Q)(u_1))(u_2). \end{aligned}$$

In particular, $\kappa_1 = Q^* \circ \kappa \circ Q$ also does not depend on $P$. ∎

Remark 4.62. The above property (4.22) represents the notion of an <u>ideal</u> of the algebra $(V, h)$. The next Theorem describes an algebra which is a direct sum of ideals.

Theorem 4.63. *Given an algebra $(V, h)$, suppose there is a direct sum $V = U_1 \oplus U_2$, with projections $(P_1, P_2)$ and inclusions $(Q_1, Q_2)$. The following are equivalent.*

(1) *$h$ respects the direct sums*

$$U_1 \oplus U_2 \to \mathrm{Hom}(V, U_1) \oplus \mathrm{Hom}(V, U_2).$$

(2) *For both $i = 1, 2$, and all $u \in U_i$, $v \in V$,*

$$(h(Q_i(u)))(v) \in Q_i(U_i).$$

Proof. In (1), the direct sum is as in Example 1.82, so the assumption is

$$h \circ Q_i \circ P_i = \mathrm{Hom}(Id_V, Q_i) \circ \mathrm{Hom}(Id_V, P_i) \circ h.$$

For any $u \in U_i$, $v \in V$,

$$\begin{aligned} h(Q_i(u)) &= h(Q_i(P_i(Q_i(u)))) \\ &= Q_i \circ P_i \circ (h(Q_i(u))) : \\ v &\mapsto Q_i(P_i((h(Q_i(u)))(v))) \in Q_i(U_i), \end{aligned}$$

so (1) $\implies$ (2). Conversely, assuming (2), for $u \in U_i$, $v \in V$, and different indices $i \neq I$,

$$\begin{aligned} (h(Q_i(u)))(v) &\in Q_i(U_i) \\ \implies (h(Q_i(u)))(v) &= Q_i(P_i((h(Q_i(u)))(v))) \\ \implies h(Q_i(u)) &= Q_i \circ P_i \circ (h(Q_i(u))) \\ \implies h \circ Q_i &= \mathrm{Hom}(Id_V, Q_i \circ P_i) \circ h \circ Q_i \\ \implies \mathrm{Hom}(Id_V, P_I) \circ h \circ Q_i &= \mathrm{Hom}(Id_V, P_I) \circ \mathrm{Hom}(Id_V, Q_i \circ P_i) \circ h \circ Q_i \\ &= 0_{\mathrm{Hom}(U_i, \mathrm{Hom}(V, U_I))}, \end{aligned}$$

so by Lemma 1.87, $h$ respects the direct sums as in (1). ∎

For a direct sum as in Theorem 4.63, Theorem 4.61 applies, so that each $U_i$ has an algebra structure $(U_i, h_i)$, and if $V$ is finite-dimensional, then each $(U_i, h_i)$ has CK form $\kappa_i = Q_i^* \circ \kappa \circ Q_i$.

THEOREM 4.64. *For an algebra $(V, h)$, the following are equivalent.*

(1) *For all $u$, $v$, $w \in V$,*

$$(h(u))((h(v))(w)) = (h((h(u))(v)))(w).$$

(2) *For any $u \in V$, this diagram is commutative.*

$$\begin{tikzcd} V \arrow[r, "h(u)"] \arrow[d, "h"] & V \arrow[d, "h"] \\ \mathrm{End}(V) \arrow[r, "\mathrm{Hom}(Id_V, h(u))"] & \mathrm{End}(V) \end{tikzcd}$$

PROOF. (1) is equivalent to: for all $u$, $v$,

$$(h(u)) \circ (h(v)) = h((h(u))(v)),$$

which is equivalent to $\mathrm{Hom}(Id_V, h(u)) \circ h = h \circ (h(u))$ for all $u$, which is (2). ∎

DEFINITION 4.65. An algebra $(V, h)$ satisfying either equivalent property from Theorem 4.64 is an associative algebra.

EXAMPLE 4.66. The generalized transpose from Definition 1.7 and Example 1.53,

$$t_{VV}^V : \mathrm{End}(V) \to \mathrm{End}(\mathrm{End}(V)) : A \mapsto \mathrm{Hom}(A, Id_V) : B \mapsto B \circ A,$$

defines an associative algebra $(\mathrm{End}(V), t_{VV}^V)$. Using Corollary 2.37,

$$\begin{aligned} (\kappa(A))(B) &= Tr_{\mathrm{End}(V)}((t_{VV}^V(A)) \circ (t_{VV}^V(B))) \\ &= Tr_{\mathrm{End}(V)}(\mathrm{Hom}(A, Id_V) \circ \mathrm{Hom}(B, Id_V)) \\ &= Tr_{\mathrm{End}(V)}(\mathrm{Hom}(B \circ A, Id_V)) \\ &= Tr_V(B \circ A) \cdot Tr_V(Id_V), \end{aligned}$$

so the form $\kappa$ for the algebra $(\mathrm{End}(V), t_{VV}^V)$ is a scalar multiple of the canonical metric from Example 3.143.

DEFINITION 4.67. For any vector space $V$, define the linear map $ad$,

$$\begin{array}{rcl} ad : \mathrm{End}(V) & \to & \mathrm{End}(\mathrm{End}(V)) : \\ A & \mapsto & \mathrm{Hom}(Id_V, A) - \mathrm{Hom}(A, Id_V) : \\ B & \mapsto & A \circ B - B \circ A. \end{array}$$

THEOREM 4.68. *For an algebra $(V, h)$, if $h$ is antisymmetric then the following are equivalent.*

(1) *For all $u$, $v$, $w \in V$,*

$$(h((h(v))(w)))(u) + (h((h(w))(u)))(v) + (h((h(u))(v)))(w) = 0_V.$$

(2) *For any $v \in V$, this diagram is commutative.*

$$\begin{tikzcd} V \arrow[r, "h(v)"] \arrow[d, "h"] & V \arrow[d, "h"] \\ \mathrm{End}(V) \arrow[r, "ad(h(v))"] & \mathrm{End}(V) \end{tikzcd}$$

PROOF. The first property is the Jacobi identity for $h$. In (2), the map $ad : \mathrm{End}(V) \to \mathrm{End}(\mathrm{End}(V))$ is as in Definition 4.67. Using the antisymmetric property, (1) is equivalent to

$$(h((h(v))(w)))(u) = (h(v))((h(w))(u)) - (h(w))((h(v))(u))$$

for all $u$, $v$, $w$, which is equivalent to, for all $v$, $w$,

$$\begin{array}{rcl} h((h(v))(w)) & = & (h(v)) \circ (h(w)) - (h(w)) \circ (h(v)) \\ & = & (ad(h(v)))(h(w)). \end{array}$$

This is equivalent to $h \circ (h(v)) = (ad(h(v))) \circ h$ for all $v$, which is (2). ∎

DEFINITION 4.69. An algebra $(V, h)$ with $h$ antisymmetric and satisfying either equivalent property from Theorem 4.68 is a Lie algebra.

EXERCISE 4.70. If $\frac{1}{2} \in \mathbb{K}$ and $h : V \to \mathrm{End}(V)$ satisfies (1) from Theorem 4.64, then its antisymmetric part $\frac{1}{2}(h - T_{V;V}(h))$ from (4.7) satisfies (1) from Theorem 4.68. Similarly, $h - T_{V;V}(h)$ satisfies the Jacobi identity even without assuming $\frac{1}{2} \in \mathbb{K}$. So for any associative algebra $(V, h)$, there is a Lie algebra $(V, h - T_{V;V}(h))$.

EXAMPLE 4.71. $(\mathrm{End}(V), ad)$ is a Lie algebra. This uses the construction of Exercise 4.70 applied to $(\mathrm{End}(V), t_{VV}^V)$ from Example 4.66, although with the opposite sign, so that $ad = T_{V;V}(t_{VV}^V) - t_{VV}^V$. For finite-dimensional $V$, the form $\kappa$ is the pullback of the canonical metric on $\mathrm{End}(\mathrm{End}(V))$ (from Equation (3.16)) by $ad$ (or by its opposite, $-ad$),

$$\kappa = (\pm ad)^* \circ (k_{\mathrm{End}(V),\mathrm{End}(V)}^*)^{-1} \circ e_{\mathrm{End}(V),\mathrm{End}(V)} \circ (\pm ad).$$

Using Equation (3.15), Lemma 2.6, and Corollary 2.37,

$$
\begin{aligned}
(4.28)\quad (\kappa(A))(B) &= Tr_{\mathrm{End}(V)}((ad(A))\circ(ad(B)))\\
&= Tr_{\mathrm{End}(V)}((\mathrm{Hom}(Id_V,A)-\mathrm{Hom}(A,Id_V))\circ(\mathrm{Hom}(Id_V,B)-\mathrm{Hom}(B,Id_V)))\\
&= Tr_{\mathrm{End}(V)}(\mathrm{Hom}(Id_V,A\circ B)-\mathrm{Hom}(A,B)-\mathrm{Hom}(B,A)+\mathrm{Hom}(B\circ A,Id_V))\\
&= 2\cdot Tr_V(Id_V)\cdot Tr_V(A\circ B)-2\cdot Tr_V(A)\cdot Tr_V(B).
\end{aligned}
$$

EXAMPLE 4.72. Direct sums of ideals as in Theorem 4.63 are considered in the Lie algebra case by [**Humphreys**] §II.5. If $V$ is finite-dimensional and $Tr_V(Id_V)\neq 0$, then Theorem 4.61 applies to the Lie algebra $(\mathrm{End}(V),ad)$ and the direct sum $\mathrm{End}(V)=\mathbb{K}\oplus\mathrm{End}_0(V)$ from Example 2.9. The corresponding CK forms are $\kappa_1=0_{\mathrm{Hom}(\mathbb{K},\mathbb{K}^*)}$ and $\kappa_2=Q_2^*\circ\kappa\circ Q_2$, so that for trace 0 elements $A$, $B\in\mathrm{End}_0(V)$, the pullback of (4.28) by the inclusion $Q_2$ gives $(\kappa_2(A))(B)=2\cdot Tr_V(Id_V)\cdot Tr_V(A\circ B)$.

EXAMPLE 4.73. For a vector space $U$, with $\frac{1}{2}\in\mathbb{K}$, and a metric $g:U\to U^*$, recall from Definition 3.116 the direct sum on $\mathrm{End}(U)$ produced by the involution $\mathrm{Hom}(g,g^{-1})\circ t$, defining subspaces of self-adjoint and skew-adjoint endomorphisms, temporarily denoted here by:

$$
\begin{aligned}
\mathrm{End}(U) &= \mathfrak{sa}\oplus\mathfrak{so}\\
&= \{A=g^{-1}\circ A^*\circ g\}\oplus\{A=-g^{-1}\circ A^*\circ g\}.
\end{aligned}
$$

Let $(P_1,P_2)$, $(Q_1,Q_2)$ denote the projection and inclusions for the direct sum. Then with $\mathrm{Hom}(Q_2,P_2)\circ ad\circ Q_2:\mathfrak{so}\to\mathrm{End}(\mathfrak{so})$ as in (4.23), $(\mathfrak{so},\mathrm{Hom}(Q_2,P_2)\circ ad\circ Q_2)$ is a Lie algebra. It is a subalgebra, but not an ideal, of $(\mathrm{End}(U),ad)$, so Theorem 4.61 and Theorem 4.63 do not apply.

DEFINITION 4.74. An algebra $(V,h)$ is a <u>one-sided division algebra</u> means that for every $v\neq 0_V$, $h(v)$ is invertible. $(V,h)$ is a <u>two-sided division algebra</u> means that for every $v\neq 0_V$, both $h(v)$ and $(T_{V;V}(h))(v)$ are invertible.

### 4.5.3. Curvature tensors.

EXAMPLE 4.75. Consider $V$ with a metric $g$, and an $\mathrm{End}(V)$-valued bilinear form $R\in\mathrm{Hom}(V,\mathrm{Hom}(V,\mathrm{End}(V)))$.

If $\frac{1}{2}\in\mathbb{K}$ and $R$ satisfies

$$T_{V;\mathrm{End}(V)}(R)=-R \tag{4.29}$$

so that $R\in Alt(V;\mathrm{End}(V))$, then by Corollary 4.32,

$$Tr_{g;\mathrm{End}(V)}(R)=0_{\mathrm{End}(V)}.$$

If $\frac{1}{2}\in\mathbb{K}$ and $R$ satisfies

$$\mathrm{Hom}(Id_V,\mathrm{Hom}(Id_V,\mathrm{Hom}(g,g^{-1})\circ t))(R)=-R \tag{4.30}$$

so that for any $u,v\in V$,

$$(R(u))(v)=-g^{-1}\circ((R(u))(v))^*\circ g\in\mathrm{End}(V),$$

then $(R(u))(v)$ is skew-adjoint with respect to $g$ and, as in Exercise 3.119,

$$
\begin{aligned}
\mathrm{Hom}(Id_V,\mathrm{Hom}(Id_V,Tr_V))(R):u &\mapsto \\
((\mathrm{Hom}(Id_V,Tr_V))\circ R)(u):v &\mapsto (Tr_V\circ(R(u)))(v)\\
&= Tr_V((R(u))(v))=0\\
(4.31)\qquad \implies \mathrm{Hom}(Id_V,\mathrm{Hom}(Id_V,Tr_V))(R) &= 0_{\mathrm{Hom}(V,V^*)}.
\end{aligned}
$$

There is another trace that is not necessarily zero even under both of the above conditions. Define:

$$Ric = (\mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V)) \circ \mathrm{Hom}(Id_V, T_{V;V}))\,(R) \in \mathrm{Hom}(V, V^*).$$

If $Tr_V(Id_V) \neq 0$, then Theorem 3.35 and Theorem 3.53 apply — there is a canonical, orthogonal decomposition:

$$Ric = \frac{Tr_g(Ric)}{Tr_V(Id_V)} \cdot g + \left( Ric - \frac{Tr_g(Ric)}{Tr_V(Id_V)} \cdot g \right). \tag{4.32}$$

Remark 4.76. For a smooth manifold where $V$ is a tangent vector space over $\mathbb{K} = \mathbb{R}$ and $g$ is a pseudo-Riemannian metric, the linear algebra properties of the Riemann curvature tensor $R^i_{qkl}$ at a point are modeled by a form $R$ as in Example 4.75, with the symmetries (4.29), (4.30), and (4.33). Its trace $Ric$ is the Ricci curvature tensor, and $Tr_g(Ric)$ from (4.32) is the scalar curvature. See [**DFN**] §30.

Remark 4.77. The second term from (4.32) is the trace-free Ricci tensor. There are other interesting linear combinations of $Ric$ and $g$, including the trace-reversed Ricci tensor,

$$Ric - 2 \cdot \frac{Tr_g(Ric)}{Tr_V(Id_V)} \cdot g,$$

and the Einstein tensor,

$$Ric - \frac{1}{2} \cdot Tr_g(Ric) \cdot g.$$

Proposition 4.78. *If $\frac{1}{2} \in \mathbb{K}$ and $R \in \mathrm{Hom}(V, \mathrm{Hom}(V, \mathrm{End}(V)))$ satisfies (4.29) and (4.31), and additionally has the property*

$$(T_{V;\mathrm{End}(V)} \circ \mathrm{Hom}(Id_V, T_{V;V}) + \mathrm{Hom}(Id_V, T_{V;V}) \circ T_{V;\mathrm{End}(V)})(R) = -R, \tag{4.33}$$

*then $Ric$ is symmetric.*

Proof. Using Lemma 4.6 and Equations (4.33), (4.31), and (4.29),

$$\begin{aligned}
T_V(Ric) &= (T_{V;\mathbb{K}} \circ \mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V)) \circ \mathrm{Hom}(Id_V, T_{V;V}))(R) \\
&= (\mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V)) \circ T_{V;\mathrm{End}(V)} \circ \mathrm{Hom}(Id_V, T_{V;V}))(R) \\
&= -\mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V))(R) \\
&\quad -(\mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V)) \circ \mathrm{Hom}(Id_V, T_{V;V}) \circ T_{V;\mathrm{End}(V)})(R) \\
&= -0_{\mathrm{Hom}(V,V^*)} \\
&\quad -(\mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, Tr_V)) \circ \mathrm{Hom}(Id_V, T_{V;V}))(-R) = Ric.
\end{aligned}$$

If, further, $Tr_V(Id_V) \neq 0$, then it follows that the second, trace-free term in (4.32) is also symmetric. ∎

Remark 4.79. The lowered-index curvature tensor $R_{iqkl}$ is modeled by the multilinear form $R'$ in the following Example 4.80. Then Proposition 4.81 demonstrates the symmetry property $R_{iqkl} = R_{kliq}$.

Example 4.80. For $V$, $g$, and $R \in \mathrm{Hom}(V, \mathrm{Hom}(V, \mathrm{End}(V)))$ as in Example 4.75, define

$$R' = \mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, \mathrm{Hom}(Id_V, g)))(R) \in \mathrm{Hom}(V, \mathrm{Hom}(V, \mathrm{Hom}(V, V^*))),$$

so that for $u, v \in V$, $(R'(u))(v) = g \circ ((R(u))(v)) : V \to V^*$. As in Theorem 3.115 and Example 3.117, if $\frac{1}{2} \in \mathbb{K}$ and $(R(u))(v)$ is skew-adjoint then $(R'(u))(v)$ is an antisymmetric form, so if $R$ has property (4.30) then $R'$ satisfies

$$\text{Hom}(Id_V, \text{Hom}(Id_V, T_V))(R') = -R'. \tag{4.34}$$

By Lemma 4.6,

$$\begin{aligned} & \text{Hom}(Id_V, \text{Hom}(Id_V, \text{Hom}(Id_V, g))) \circ T_{V;\text{End}(V)} \\ = \quad & T_{V;\text{Hom}(V,V^*)} \circ \text{Hom}(Id_V, \text{Hom}(Id_V, \text{Hom}(Id_V, g))), \end{aligned}$$

so if $R$ has property (4.29), then $R'$ satisfies

$$T_{V;\text{Hom}(V,V^*)}(R') = -R'. \tag{4.35}$$

Similarly by Lemma 4.6,

$$\begin{aligned} & \text{Hom}(Id_V, \text{Hom}(Id_V, \text{Hom}(Id_V, g))) \circ \text{Hom}(Id_V, T_{V;V}) \\ = \quad & \text{Hom}(Id_V, T_{V;V^*}) \circ \text{Hom}(Id_V, \text{Hom}(Id_V, \text{Hom}(Id_V, g))), \end{aligned}$$

so if $R$ has property (4.33), then $R'$ satisfies

$$(T_{V;\text{Hom}(V,V^*)} \circ \text{Hom}(Id_V, T_{V;V^*}) + \text{Hom}(Id_V, T_{V;V^*}) \circ T_{V;\text{Hom}(V,V^*)})(R') = -R'. \tag{4.36}$$

Proposition 4.81. *If $\frac{1}{2} \in \mathbb{K}$ and $R' \in \text{Hom}(V, \text{Hom}(V, \text{Hom}(V, V^*)))$ satisfies (4.34), (4.35), and (4.36), then $R'$ is a fixed point of the involution*

$$\text{Hom}(Id_V, T_{V;V^*}) \circ T_{V;\text{Hom}(V,V^*)} \circ \text{Hom}(Id_V, \text{Hom}(Id_V, T_V)) \circ \text{Hom}(Id_V, T_{V;V^*}). \tag{4.37}$$

Proof. Temporarily denote the involutions:

$$\begin{aligned} a_1 &= T_{V;\text{Hom}(V,V^*)} \\ a_2 &= \text{Hom}(Id_V, T_{V;V^*}) \\ a_3 &= \text{Hom}(Id_V, \text{Hom}(Id_V, T_V)). \end{aligned}$$

Properties (4.34) and (4.35) then can be stated:

$$a_1(R') = a_3(R') = -R'. \tag{4.38}$$

By Lemma 4.6, $a_1 \circ a_3 = a_3 \circ a_1$, and this is enough to show that the composite $a_2 \circ a_1 \circ a_3 \circ a_2$ from (4.37) is an involution. Lemma 4.12 gives the relations:

$$\begin{aligned} a_2 \circ a_1 \circ a_2 &= a_1 \circ a_2 \circ a_1 \\ a_3 \circ a_2 \circ a_3 &= a_2 \circ a_3 \circ a_2. \end{aligned}$$

Starting with property (4.36), applying $a_3$ to both sides, and then using (4.38) gives:

$$\begin{aligned} (a_1 \circ a_2 + a_2 \circ a_1)(R') &= -R' \qquad (4.39) \\ \implies (a_3 \circ a_1 \circ a_2 + a_3 \circ a_2 \circ a_1)(R') &= a_3(-R') \\ (a_1 \circ a_2 \circ a_3 \circ a_2 \circ a_3 + a_2 \circ a_3 \circ a_2 \circ a_3 \circ a_1)(R') &= R' \\ \implies (a_1 \circ a_2 \circ a_3 \circ a_2 + a_2 \circ a_3 \circ a_2 \circ a_3)(R') &= -R'. \qquad (4.40) \end{aligned}$$

Applying $a_2 \circ a_1 \circ a_3 \circ a_2$ to both sides of (4.39) gives:

$$\begin{aligned} & (a_2 \circ a_1 \circ a_3 \circ a_2 \circ a_1 \circ a_2)(R') \\ & +(a_2 \circ a_1 \circ a_3 \circ a_2 \circ a_2 \circ a_1)(R') = (a_2 \circ a_1 \circ a_3 \circ a_2)(-R') \\ & (a_2 \circ a_3 \circ a_2 \circ a_1 + a_2 \circ a_1)(R') = -(a_2 \circ a_1 \circ a_3 \circ a_2)(R'). \qquad (4.41) \end{aligned}$$

Applying $a_1$ to both sides of (4.41) gives:

$$\begin{aligned}
(a_1\circ a_2\circ a_3\circ a_2\circ a_1 + a_1\circ a_2\circ a_1)(R') &= -(a_1\circ a_2\circ a_1\circ a_3\circ a_2)(R')\\
(a_1\circ a_2\circ a_3\circ a_2 + a_1\circ a_2)(-R') &= -(a_2\circ a_1\circ a_2\circ a_3\circ a_2)(R')\\
\implies (a_1\circ a_2\circ a_3\circ a_2 + a_1\circ a_2)(R') &= -(a_2\circ a_1\circ a_3\circ a_2)(R'). \qquad (4.42)
\end{aligned}$$

Adding (4.39) and (4.40) and subtracting (4.41) and (4.42), there are cancellations using (4.38) again, to get:

$$0_{\mathrm{Hom}(V,\mathrm{Hom}(V,\mathrm{Hom}(V,V^*)))} = -2\cdot R' + 2\cdot(a_2\circ a_1\circ a_3\circ a_2)(R'),$$

which proves the claim. ∎

**4.5.4. Partially symmetric forms.** A map $A\otimes V\otimes V\to F$ is called a "trilinear $F$-form" in [**EHM**], and it is "partially symmetric" means that it is invariant under switching of the $V$ factors. Such forms, of course, lie in the scope of these notes, and it will also be convenient to consider maps of the form

$$V\otimes U\to\mathrm{Hom}(V,W),$$

as in the vector valued forms of Sections 4.2 and 4.3, but with the domain twisted by $U$. The two notions are related, as already seen in the Proof of Lemma 4.12.

Some of the statements in this Section will use variant $q$ maps as in Notation 1.49. For arbitrary $V$, $U$, $W$, $X$, define

$$q:\mathrm{Hom}(X\otimes U,\mathrm{Hom}(V,W))\to\mathrm{Hom}(V\otimes X\otimes U,W) \qquad (4.43)$$

so that for $G:X\otimes U\to\mathrm{Hom}(V,W)$, $v\in V$, $x\in X$, and $u\in U$,

$$q(G):v\otimes x\otimes u\mapsto(G(x\otimes u))(v).$$

In the following Lemma, $T_V$ is the transpose map from Definition 3.2, and $q_1$ is the $q$ map from (4.43) in the case $X=V$.

Lemma 4.82. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Hom}(V,V^*)\otimes\mathrm{Hom}(U,W) \arrow[r, "{[T_V\otimes Id_{\mathrm{Hom}(U,W)}]}"] \arrow[d, "j"] & \mathrm{Hom}(V,V^*)\otimes\mathrm{Hom}(U,W) \arrow[d, "j"] \\
\mathrm{Hom}(V\otimes U,V^*\otimes W) \arrow[d, "{\mathrm{Hom}(Id_{V\otimes U},k_{VW})}"'] & \mathrm{Hom}(V\otimes U,V^*\otimes W) \arrow[d, "{\mathrm{Hom}(Id_{V\otimes U},k_{VW})}"] \\
\mathrm{Hom}(V\otimes U,\mathrm{Hom}(V,W)) \arrow[d, "q_1"'] & \mathrm{Hom}(V\otimes U,\mathrm{Hom}(V,W)) \arrow[d, "q_1"] \\
\mathrm{Hom}(V\otimes V\otimes U,W) \arrow[r, "{\mathrm{Hom}([s\otimes Id_U],Id_W)}"] & \mathrm{Hom}(V\otimes V\otimes U,W)
\end{tikzcd}$$

Proof. Without stating all the details, the upper part of the diagram is analogous to the diagram from Lemma 4.5, and the lower part is analogous to the diagram from Lemma 4.1. ∎

Definition 4.83. Define $T_{V;U,W}\in\mathrm{End}(\mathrm{Hom}(V\otimes U,\mathrm{Hom}(V,W)))$ by

$$T_{V;U,W}=q_1^{-1}\circ\mathrm{Hom}([s\otimes Id_U],Id_W)\circ q_1.$$

With this construction, the $T_{V;U,W}$ maps are analogous to, but not a special case of, the maps $T_{V_1,V_2;W}$ from Lemma 4.1 and Definition 4.2. If $V$ is finite-dimensional, then $j$ and $k$ in the above diagram are invertible, and by the Lemma,

$$T_{V;U,W} = \mathrm{Hom}(Id_{V\otimes U}, k_{VW}) \circ j \circ [T_V \otimes Id_{\mathrm{Hom}(U,W)}] \circ j^{-1} \circ \mathrm{Hom}(Id_{V\otimes U}, k_{VW}^{-1}).$$

As in Section 4.2, $T_{V;U,W}$ is an involution, and if $\frac{1}{2} \in \mathbb{K}$, then it produces a direct sum structure on $\mathrm{Hom}(V \otimes U, \mathrm{Hom}(V, W))$, by Lemma 1.119. The other two involutions in the above diagram also produce direct sums, and by Lemma 1.126, the maps $q_1$ and $\mathrm{Hom}(Id_{V\otimes U}, k_{VW}) \circ j$ respect these direct sums, although the comments about Example 1.145 in Remark 4.14 apply here also.

EXERCISE 4.84. With respect to induced metrics, $T_{V;U,W}$ is an isometry, and if $\frac{1}{2} \in \mathbb{K}$, then it produces an orthogonal direct sum.

HINT. Use Definition 4.83 to show $T_{V;U,W}$ is a composition of isometries. Then Lemma 3.55 applies, as in Theorem 4.39. ∎

DEFINITION 4.85. A map $G : V \otimes U \to \mathrm{Hom}(V, W)$ is <u>partially symmetric</u> means: $T_{V;U,W}(G) = G$. More generally, a map $G : X \otimes U \to \mathrm{Hom}(V, W)$ is partially symmetric with respect to a map $H : V \to X$ means that $G \circ [H \otimes Id_U] : V \otimes U \to \mathrm{Hom}(V, W)$ is partially symmetric.

LEMMA 4.86. ([EHM]) *For any $V$, $U$, $W$, $X$, and $G : X \otimes U \to \mathrm{Hom}(V, W)$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathbb{K} \otimes X \otimes U \arrow[rr, "l"] \arrow[d, "{[Q_1^1 \otimes Id_{X\otimes U}]}"] & & X \otimes U \arrow[dd, "G"] \\
\mathrm{End}(V) \otimes X \otimes U \arrow[d, "n"] & & \\
\mathrm{Hom}(V, V \otimes X \otimes U) \arrow[rr, "{\mathrm{Hom}(Id_V, q(G))}"] & & \mathrm{Hom}(V, W) \\
V^* \otimes V \otimes X \otimes U \arrow[u, "k_{V,V\otimes X\otimes U}"] \arrow[rr, "{[Id_{V^*} \otimes (q(G))]}"] & & V^* \otimes W \arrow[u, "k_{VW}"']
\end{tikzcd}$$

PROOF. The lower square uses Lemma 1.62. In the upper square, the maps are $n$ as in Definition 1.40, and an inclusion $Q_1^1 : \lambda \to \lambda \cdot Id_V$ as in Example 2.9 and Equation (2.4).

$$\begin{aligned}
\lambda \otimes x \otimes u &\mapsto (\mathrm{Hom}(Id_V, q(G)) \circ n \circ [Q_1^1 \otimes Id_{X\otimes U}])(\lambda \otimes x \otimes u) \\
&= (q(G)) \circ (n(\lambda \cdot Id_V \otimes x \otimes u)) : \\
v &\mapsto (q(G))(\lambda \cdot v \otimes x \otimes u) \\
&= \lambda \cdot (G(x \otimes u))(v) = ((G \circ l)(\lambda \otimes x \otimes u))(v).
\end{aligned}$$

∎

THEOREM 4.87. *For any spaces $U$, $V$, $W$, $X$, $Y$, and any maps $G : X \otimes U \to \mathrm{Hom}(V, W)$, $M : Y \to X \otimes U$, the following are equivalent.*

$$\begin{aligned}
(q(G)) \circ [Id_V \otimes M] &= 0_{\mathrm{Hom}(V\otimes Y, W)} \\
\iff G \circ M &= 0_{\mathrm{Hom}(Y, \mathrm{Hom}(V,W))}.
\end{aligned}$$

PROOF. Let $q$ be the map from Equation (4.43), and let $q_2$ be another such map in the following diagram.

$$\begin{tikzcd}
\mathrm{Hom}(X\otimes U,\mathrm{Hom}(V,W)) \arrow[r, "q"] \arrow[d, "\mathrm{Hom}(M,Id_{\mathrm{Hom}(V,W)})"'] & \mathrm{Hom}(V\otimes X\otimes U,W) \arrow[d, "\mathrm{Hom}([Id_V\otimes M],Id_W)"] \\
\mathrm{Hom}(Y,\mathrm{Hom}(V,W)) \arrow[r, "q_2"] & \mathrm{Hom}(V\otimes Y,W)
\end{tikzcd}$$

The diagram is commutative by Lemma 1.50. So, $q_2(G\circ M)=(q(G))\circ[Id_V\otimes M]$. Since $q_2$ is invertible (Lemma 1.47), $G\circ M$ is zero if and only if its output under $q_2$ is also zero. ∎

EXAMPLE 4.88. Suppose there is some direct sum $V\otimes X\otimes U=W_1\oplus W_2$, with projections $P_i:V\otimes X\otimes U\to W_i$. Then, for $q_i:\mathrm{Hom}(X\otimes U,\mathrm{Hom}(V,W_i))\to\mathrm{Hom}(V\otimes X\otimes U,W_i)$ and $M:Y\to X\otimes U$,

$$P_i\circ[Id_V\otimes M]=0_{\mathrm{Hom}(V\otimes Y,W_i)}\iff(q_i^{-1}(P_i))\circ M=0_{\mathrm{Hom}(Y,\mathrm{Hom}(V,W_i))}.$$

The following Theorem uses the the switching involution $s$ as in Lemma 4.82, and the direct sum $V\otimes V=S^2V\oplus\Lambda^2V$ produced by $s$ as in Example 1.124, with projections $(P_1,P_2)$ and inclusions $(Q_1,Q_2)$.

THEOREM 4.89. *Let $H:V\to X$, and suppose there is some direct sum*

$$V\otimes X=Z_1\oplus Z_2$$

*with operators $P_i'$, $Q_i'$, such that $[Id_V\otimes H]:V\otimes V\to V\otimes X$ respects the direct sums and $P_2'\circ[Id_V\otimes H]\circ Q_2:\Lambda^2V\to Z_2$ is invertible. If $G:X\otimes U\to\mathrm{Hom}(V,W)$ is partially symmetric with respect to $H$, then*

$$q(G)=(q(G))\circ[(Q_1'\circ P_1')\otimes Id_U].$$

PROOF. The following diagram is commutative by Lemma 1.50, where $q_1$ is as in Lemma 4.82.

$$\begin{tikzcd}
\mathrm{Hom}(X\otimes U,\mathrm{Hom}(V,W)) \arrow[r, "q"] \arrow[d, "\mathrm{Hom}([H\otimes Id_U],Id_{\mathrm{Hom}(V,W)})"'] & \mathrm{Hom}(V\otimes X\otimes U,W) \arrow[d, "\mathrm{Hom}([Id_V\otimes[H\otimes Id_U]],Id_W)"] \\
\mathrm{Hom}(V\otimes U,\mathrm{Hom}(V,W)) \arrow[r, "q_1"] & \mathrm{Hom}(V\otimes V\otimes U,W)
\end{tikzcd}$$

By the assumption about respecting the direct sum, $P_I'\circ[Id_V\otimes H]\circ Q_i$ is zero for $i\neq I$. Let $P_2''$ denote the projection $\frac{1}{2}\cdot(Id_{\mathrm{Hom}(V\otimes U,\mathrm{Hom}(V,W))}-T_{V;U,W})$, so that if $G$ is partially symmetric with respect to $H$, then

$$\begin{aligned}
0_{\mathrm{Hom}(\Lambda^2V\otimes U,W)} &= (q_1(P_2''(G\circ[H\otimes Id_U])))\circ[Q_2\otimes Id_U]\\
&= (\frac{1}{2}\cdot q_1(G\circ[H\otimes Id_U]))\circ[Q_2\otimes Id_U]\\
&\quad -(\frac{1}{2}\cdot q_1(G\circ[H\otimes Id_U]))\circ[s\otimes Id_U]\circ[Q_2\otimes Id_U]\\
&= (q_1(G\circ[H\otimes Id_U]))\circ[Q_2\otimes Id_U]\circ[P_2\otimes Id_U]\circ[Q_2\otimes Id_U]\\
&= (q(G))\circ[Id_V\otimes[H\otimes Id_U]]\circ[Q_2\otimes Id_U]\\
&= (q(G))\circ([Q_1'\otimes Id_U]\circ[P_1'\otimes Id_U]+[Q_2'\otimes Id_U]\circ[P_2'\otimes Id_U])\\
&\quad \circ[[Id_V\otimes H]\otimes Id_U]\circ[Q_2\otimes Id_U]\\
&= (q(G))\circ[Q_2'\otimes Id_U]\circ[P_2'\otimes Id_U]\circ[[Id_V\otimes H]\otimes Id_U]\circ[Q_2\otimes Id_U].
\end{aligned}$$

Then, since $[P_2' \otimes Id_U] \circ [[Id_V \otimes H] \otimes Id_U] \circ [Q_2 \otimes Id_U]$ is invertible,

$$0_{\mathrm{Hom}(Z_2 \otimes U, W)} = (q(G)) \circ [Q_2' \otimes Id_U],$$

and it follows that

$$q(G) = (q(G)) \circ [(Q_1' \circ P_1' + Q_2' \circ P_2') \otimes Id_U] = (q(G)) \circ [(Q_1' \circ P_1') \otimes Id_U]. \quad \blacksquare$$

It follows from the previous two Theorems that if $M : Y \to X \otimes U$ and $G$ is partially symmetric with respect to $H$, then

$$q_2(G \circ M) = (q(G)) \circ [(Q_1' \circ P_1') \otimes Id_U] \circ [Id_V \otimes M].$$

Remark 4.90. The map $H : V \to X$ could be an inclusion of a vector subspace, in which case the above $Z_1$ corresponds to the space denoted by $H.V$ in [**EHM**].

Big Exercise 4.91. Given a metric $g$ on $V$, there exists a trace operator

$$Tr_{g;U,W} : \mathrm{Hom}(V \otimes U, \mathrm{Hom}(V, W)) \to \mathrm{Hom}(U, W)$$

having nice properties which follow as corollaries of the results in Section 2.2. $\blacksquare$

CHAPTER 5

# Complex Structures

At this point we abandon the general field $\mathbb{K}$ and work exclusively with real number scalars and vector spaces over the field $\mathbb{R}$. Some of the objects could be considered vector spaces over the field of complex numbers, but in this Chapter, complex numbers will not be used as scalars or for any other purpose. The objects will instead be real vector spaces paired with some additional structure, and the maps are all $\mathbb{R}$-linear, although some of the $\mathbb{R}$-linear maps will respect the additional structure.

## 5.1. Complex Structure Operators

DEFINITION 5.1. Given a real vector space $V$, an endomorphism $J \in \mathrm{End}(V)$ is a complex structure operator means: $J \circ J = -Id_V$.

NOTATION 5.2. A complex structure operator is more briefly called a CSO. Sometimes a pair $(V, J)$ will be denoted by a matching boldface letter, $\mathbf{V}$. Expressions such as $v \in \mathbf{V}$, $A : \mathbf{U} \to \mathbf{V}$, etc., refer to the underlying real space, so that $v \in V$, $A : U \to V$, etc.

EXAMPLE 5.3. Given $\mathbf{V} = (V, J_V)$ and another vector space $U$, $[Id_U \otimes J_V] \in \mathrm{End}(U \otimes V)$ is a canonical CSO on $U \otimes V$, so we may denote $U \otimes \mathbf{V} = (U \otimes V, [Id_U \otimes J_V])$. Similarly, denote $\mathbf{V} \otimes U = (V \otimes U, [J_V \otimes Id_U])$.

EXAMPLE 5.4. Given a vector space $V$ with CSO $J_V$ and another vector space $U$, $\mathrm{Hom}(Id_U, J_V) : A \mapsto J_V \circ A$ is a canonical CSO on $\mathrm{Hom}(U, V)$, so we may denote $\mathrm{Hom}(U, \mathbf{V}) = (\mathrm{Hom}(U, V), \mathrm{Hom}(Id_U, J_V))$. Similarly, denote $\mathrm{Hom}(\mathbf{V}, U) = (\mathrm{Hom}(V, U), \mathrm{Hom}(J_V, Id_U))$.

EXAMPLE 5.5. Given $V$, a CSO $J$ induces a CSO $J^* = \mathrm{Hom}(J, Id_{\mathbb{R}})$ on $V^* = \mathrm{Hom}(V, \mathbb{R})$.

EXAMPLE 5.6. Given $V$, a CSO $J \in \mathrm{End}(V)$, and any involution $N$ that commutes with $J$ (i.e., $N \in \mathrm{End}(V)$ such that $N \circ N = Id_V$ and $N \circ J = J \circ N$), $N \circ J$ is a CSO.

EXAMPLE 5.7. Given $V \neq \{0_V\}$, any CSO $J \in \mathrm{End}(V)$ is not unique, since $-J$ is a different CSO. This is the $N = -Id_V$ case from Example 5.6.

EXAMPLE 5.8. Given $V = V_1 \oplus V_2$, suppose there is an invertible map $A : V_2 \to V_1$, as in (3) from Theorem 1.136. Then, $V$ also admits a CSO,

$$J_V = Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2, \tag{5.1}$$

and its opposite, $-J_V$.

EXAMPLE 5.9. In the special case of Example 5.8 where $V = U \oplus U$, $A = Id_U$ (Case (1) from Theorem 1.136), the CSO (5.1) is $J_V = Q_2 \circ P_1 - Q_1 \circ P_2$.

EXAMPLE 5.10. For $V = V_1 \oplus V_2$ and $A : V_2 \to V_1$ as in Example 5.8, the implication (3) $\implies$ (8) from Theorem 1.136 showed that there exist anticommuting involutions $K_1$ and $K_2$ on $V$. Conversely, for any pair of anticommuting involutions $K_1$ and $K_2$, the composite $K_1 \circ K_2$ is a CSO on $V$ (and so is its opposite $-K_1 \circ K_2 = K_2 \circ K_1$). Using $K_1$ to produce a direct sum $V = V_1 \oplus V_2$ and using $K_2$ to define a map $A = P_1 \circ K_2 \circ Q_2 : V_2 \to V_1$ as in (1.20), the construction (5.1) from Example 5.8 gives the same pair of CSOs: $\{\pm K_1 \circ K_2\} = \{\pm J_V\}$.

LEMMA 5.11. *Given $V$ with CSO $J$ and $v \in V$, if $v \neq 0_V$, then the ordered list $(v, J(v))$ is linearly independent.*

PROOF. Linear independence as in Definition 0.32 refers to the scalar field $\mathbb{R}$ in this Chapter. If $J(v) = \alpha \cdot v$ for some $\alpha \in \mathbb{R}$, then $J(J(v)) = (-1) \cdot v = \alpha^2 \cdot v$. There are no solutions for $v \neq 0_V$ and $\alpha \in \mathbb{R}$. ∎

EXERCISE 5.12. Not every real vector space admits a CSO. ∎

LEMMA 5.13. *Given $V$ with CSO $J$ and $v_1, \ldots, v_\ell \in V$, if the ordered list*

$$(v_1, \ldots, v_{\ell-1}, v_\ell, J(v_1), \ldots, J(v_{\ell-1}))$$

*is linearly independent, then so is the ordered list*

$$(v_1, \ldots, v_{\ell-1}, v_\ell, J(v_1), \ldots, J(v_{\ell-1}), J(v_\ell)) .$$

PROOF. The $\ell = 1$ case is Lemma 5.11. For $\ell \geq 2$, suppose there are real scalars $\alpha_1, \ldots, \alpha_\ell, \beta_1, \ldots, \beta_\ell$ such that

$$\alpha_1 \cdot v_1 + \ldots + \alpha_\ell \cdot v_\ell + \beta_1 \cdot J(v_1) + \ldots + \beta_\ell \cdot J(v_\ell) = 0_V . \tag{5.2}$$

Then,

$$\alpha_1 \cdot J(v_1) + \ldots + \alpha_\ell \cdot J(v_\ell) - \beta_1 \cdot v_1 - \ldots - \beta_\ell \cdot v_\ell = 0_V . \tag{5.3}$$

Subtracting $\alpha_\ell$ times (5.2) minus $\beta_\ell$ times (5.3), the $J(v_\ell)$ terms cancel, and

$$\begin{aligned} &(\alpha_1\alpha_\ell + \beta_1\beta_\ell) \cdot v_1 + \ldots + (\alpha_{\ell-1}\alpha_\ell + \beta_{\ell-1}\beta_\ell) \cdot v_{\ell-1} + (\alpha_\ell^2 + \beta_\ell^2) \cdot v_\ell \\ &+(\alpha_\ell\beta_1 - \alpha_1\beta_\ell) \cdot J(v_1) + \ldots + (\alpha_\ell\beta_{\ell-1} - \alpha_{\ell-1}\beta_\ell) \cdot J(v_{\ell-1}) = 0_V . \end{aligned}$$

By the linear independence of the ordered list with $2\ell - 1$ elements, $\alpha_\ell^2 + \beta_\ell^2 = 0 \implies \alpha_\ell = \beta_\ell = 0$. Then (5.2) and the independence hypothesis again (or Lemma 5.11 if $\ell = 2$) imply $\alpha_1 = \ldots = \alpha_{\ell-1} = \beta_1 = \ldots = \beta_{\ell-1} = 0$. ∎

DEFINITION 5.14. Given $V$ with CSO $J$, a subspace $H$ of $V$ is <u>$J$-invariant</u> means: $J(H) \subseteq H$. Equivalently, because $J$ is invertible, $J(H) = H$.

LEMMA 5.15. *Given $V$ with CSO $J$, and a $J$-invariant subspace $H$ of $V$, if $H \neq \{0_V\}$ and $H$ is finite-dimensional, then $H$ admits an ordered basis of the form*

$$(v_1, \ldots, v_{\ell-1}, v_\ell, J(v_1), \ldots, J(v_{\ell-1}), J(v_\ell)) .$$

PROOF. By hypothesis, there is some $v_1 \in H$ with $v_1 \neq 0_V$, and by $J$-invariance, $J(v) \in H$, so by Lemma 5.11, $(v_1, J(v_1))$ is a linearly independent ordered list of vectors in $H$. Suppose inductively that

$$(v_1, \ldots, v_{\ell-1}, v_\ell, J(v_1), \ldots, J(v_{\ell-1}), J(v_\ell))$$

is a linearly independent ordered list of vectors in $H$. If the ordered list spans $H$, it is an ordered basis; otherwise, there is some $v_{\ell+1} \in H$ not in its span, so

$$(v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, J(v_1), \ldots, J(v_{\ell-1}), J(v_\ell))$$

is a linearly independent ordered list. $J(v_{\ell+1}) \in H$ and Lemma 5.13 applies, so

$$(v_1, \ldots, v_{\ell-1}, v_\ell, v_{\ell+1}, J(v_1), \ldots, J(v_{\ell-1}), J(v_\ell), J(v_{\ell+1}))$$

is a linearly independent ordered list of elements in $H$. The construction eventually terminates by Definition 0.28. ∎

Definition 5.16. Given $V$ with CSO $J$, a subspace $H$ of $V$ which is equal to a span of a two-element set and is also $J$-invariant, $H = J(H) \subseteq V$, will be called a $J$-complex line in $V$.

By Lemma 5.15, a $J$-complex line $H$ must be of the form $\operatorname{span}\{v, J(v)\}$ for some non-zero $v \in H$.

Lemma 5.17. *Given $V$ with CSO $J$, a $J$-complex line $L$, and a $J$-invariant subspace $H \subseteq V$, if there is a non-zero element $v \in L \cap H$, then $L \subseteq H$. In particular, if $H$ is a $J$-complex line, then $L = H$.*

Proof. By $J$-invariance, $\{v, J(v)\} \subseteq L \cap H$. By Lemma 5.11, $(v, J(v))$ is a linearly independent ordered list, so it is an ordered basis of $L$ and its span is contained in $H$. Comment: the contrapositive can be stated: Given $V$ with CSO $J$, $v \in V$, and a $J$-invariant subspace $H$, if $v \notin H$, then $H \cap \operatorname{span}\{v, J(v)\} = \{0_V\}$. ∎

Lemma 5.18. *Given $V$ with CSO $J$, if $L^1$, $L^2$ are distinct $J$-complex lines in $V$, then $\operatorname{span}(L^1 \cup L^2)$ has an ordered basis with 4 elements. In particular, a subspace $H \subseteq V$ that does not have 4 elements forming a linearly independent list can contain at most one $J$-complex line.*

Proof. Suppose $L_1$ is a $J$-complex line in $V$, with $L_1 = \operatorname{span}\{v, J(v)\}$. If $L_2$ is a $J$-complex line with $L_2 \neq L_1$, then $L_2 \not\subseteq L_1$, so there is some $u \in L_2 \setminus L_1$, and $(v, J(v), u)$ is a linearly independent list. Because $L_2$ is $J$-invariant, $J(u) \in L_2$, so by Lemma 5.13, $(v, J(v), u, J(u))$ is a linearly independent ordered list of elements of $L^1 \cup L^2$, and an ordered basis of $\operatorname{span}(L^1 \cup L^2)$. ∎

Lemma 5.19. *Given $V$ with CSO $J$, and a subspace $H$ of $V$, if $H = \{0_V\}$ or $H$ has an ordered basis of the form $(u_1, J(u_1), \ldots, u_\nu, J(u_\nu))$, then either $H = V$ or there exists a subspace $U$ of $V$ such that $U$ is $J$-invariant, $H \subseteq U$, and $U$ admits an ordered basis with $2(\nu + 1)$ elements.*

Proof. This is trivial for $H = \{0_V\}$ or $H = V$; otherwise, the ordered basis for $H$ can be extended by two more elements to an ordered basis of a $J$-invariant subspace as in the Proof of Lemma 5.15. Note that this Lemma then applies to $U$ and can be repeated to get another subspace containing $U$. ∎

Lemma 5.20. *Given a vector space $V_1$ with CSO $J_1$ and an element $v \in V_1$, another vector space $V_2$ with CSO $J_2$, and a real linear map $A : V_1 \to V_2$, the following are equivalent.*

(1) $A(J_1(v)) \in \operatorname{span}\{A(v), J_2(A(v))\}$.
(2) *$A$ maps the subspace $\operatorname{span}\{v, J_1(v)\} \subseteq V_1$ to the subspace*

$$\operatorname{span}\{A(v), J_2(A(v))\} \subseteq V_2.$$

*Further, if $A$ and $v$ satisfy (1) and $A(J_1(v)) \neq 0_{V_2}$, then $A$ and $J_1(v)$ satisfy (1):*

$$A(J_1(J_1(v))) \in \operatorname{span}\{A(J_1(v)), J_2(A(J_1(v)))\}.$$

PROOF. (2) $\iff$ (1) is straightforward. If $A(J_1(v)) = \alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))$ for some $(\alpha_1, \alpha_2) \neq (0,0)$, then there is this linear combination.

$$\begin{aligned}
& \frac{-\alpha_1}{\alpha_1^2+\alpha_2^2} \cdot A(J_1(v)) + \frac{\alpha_2}{\alpha_1^2+\alpha_2^2} \cdot J_2(A(J_1(v))) \\
= & \frac{-\alpha_1}{\alpha_1^2+\alpha_2^2} \cdot (\alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))) \\
& + \frac{\alpha_2}{\alpha_1^2+\alpha_2^2} \cdot J_2(\alpha_1 \cdot A(v) + \alpha_2 \cdot J_2(A(v))) \\
= & -A(v) = A(J_1(J_1(v))). \qquad \blacksquare
\end{aligned}$$

The above notion for a real linear map $A$ is slightly stronger than the statement that $A$ maps the $J_1$-complex line $\text{span}\{v, J_1(v)\}$ into some $J_2$-complex line; if $A(v) = 0_{V_2}$, condition (2) implies $A$ maps $\text{span}\{v, J_1(v)\}$ to the zero subspace.

EXERCISE 5.21. Given $V$ with CSO $J$, if $V \neq \{0_V\}$, then the ordered list $(Id_V, J)$ is linearly independent in $\text{End}(V)$. $\blacksquare$

EXERCISE 5.22. Given $V$ with two CSOs $J_1$ and $J_2$, if $J_2 \neq \pm J_1$, then the ordered list $(Id_V, J_1, J_2, J_1 \circ J_2)$ is linearly independent in $\text{End}(V)$.

HINT. The hypothesis implies $V \neq \{0_V\}$, so Exercise 5.21 applies and $(Id_V, J_1)$ is a linearly independent list. The next step is to show that $(Id_V, J_1, J_2)$ is a linearly independent list. If there are real scalars such that $\alpha_1 \cdot Id_V + \alpha_2 \cdot J_1 + \alpha_3 \cdot J_2 = 0_{\text{End}(V)}$, then:

$$\begin{aligned}
(\alpha_1 \cdot Id_V + \alpha_2 \cdot J_1) \circ (\alpha_1 \cdot Id_V + \alpha_2 \cdot J_1) &= (-\alpha_3 \cdot J_2) \circ (-\alpha_3 \cdot J_2) \\
(\alpha_1^2 - \alpha_2^2) \cdot Id_V + 2\alpha_1\alpha_2 \cdot J_1 &= -\alpha_3^2 \cdot Id_V \\
\implies \alpha_1\alpha_2 &= 0.
\end{aligned}$$

If $\alpha_2 = 0$ then by the linear independence of $(Id_V, J_2)$ from Exercise 5.21, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. If $\alpha_1 = 0$ then $(\alpha_2 \cdot J_1) \circ (\alpha_2 \cdot J_1) = (-\alpha_3 \cdot J_2) \circ (-\alpha_3 \cdot J_2) \implies -\alpha_2^2 = -\alpha_3^2 \implies \alpha_2 \cdot (J_1 \pm J_2) = 0_{\text{End}(V)}$, and the hypothesis $J_2 \neq \pm J_1$ implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

The claimed independence of the list $(Id_V, J_1, J_2, J_1 \circ J_2)$ then follows from applying Lemma 5.13 to the list $(Id_V, J_1, J_2)$ and the CSO $\text{Hom}(J_2, Id_V)$ on $\text{End}(V)$ from Example 5.4. $\blacksquare$

REMARK 5.23. The results in this Section give some details omitted from [**C**$_1$] §2.

## 5.2. Complex linear and antilinear maps

LEMMA 5.24. *Given $U$ with CSO $J_U$, $V$ with CSO $J_V$, and a real linear map $A : U \to V$, the set*

$$\{u \in U : J_V(A(u)) = A(J_U(u))\}$$

*is a $J_U$-invariant subspace of $U$, and its image*

$$\{A(u) : J_V(A(u)) = A(J_U(u))\}$$

*is a $J_V$-invariant subspace of $V$.*

PROOF. The first set is a real linear subspace of $U$ because it is the kernel of the real linear map $J_V \circ A - A \circ J_U$. To verify Definition 5.14, check that if $u$ is in the set, then so is $J_U(u)$; the claim for the image set follows. ∎

EXAMPLE 5.25. Given $V$ with two CSOs $J_1$ and $J_2$, the set

$$\{v \in V : J_1(v) = J_2(v)\}$$

is a $J_1$-invariant subspace of $V$ and also a $J_2$-invariant subspace. This is the $A = Id_V$ special case of Lemma 5.24.

DEFINITION 5.26. For $\mathbf{U} = (U, J_U)$, $\mathbf{V} = (V, J_V)$, a real linear map $A \in \mathrm{Hom}(U, V)$ is <u>c-linear</u> means: $A \circ J_U = J_V \circ A$. A map $A \in \mathrm{Hom}(U, V)$ is <u>a-linear</u> means: $A \circ J_U = -J_V \circ A$.

Because some vector spaces can admit several complex structures, it will sometimes be more clear to specifically refer to $A : U \to V$ as c-linear (or a-linear) with respect to the pair $(J_U, J_V)$.

LEMMA 5.27. *If $A : \mathbf{U} \to \mathbf{V}$ is c-linear (or a-linear) and invertible, then $A^{-1}$ is also c-linear (or a-linear). The composite of two c-linear maps (or two a-linear maps) is c-linear.* ∎

LEMMA 5.28. *If $A : U \to V$ is c-linear (or a-linear) with respect to $J_U$, $J_V$, then the kernel $\ker(A)$ is a $J_U$-invariant subspace of $U$ and the image $A(U)$ is a $J_V$-invariant subspace of $V$.* ∎

LEMMA 5.29. *Given $\mathbf{V} = (V, J_V)$ and $\mathbf{V}' = (V', J'_V)$, any map $A : U \to U'$, and a c-linear map $B : \mathbf{V} \to \mathbf{V}'$, the maps $[A \otimes B] : U \otimes \mathbf{V} \to U' \otimes \mathbf{V}'$ and $[B \otimes A] : \mathbf{V} \otimes U \to \mathbf{V}' \otimes U'$ are c-linear with respect to the induced CSOs from Example 5.3.* ∎

LEMMA 5.30. *Given $\mathbf{V} = (V, J_V)$ and $\mathbf{V}' = (V', J'_V)$, any map $A : U' \to U$, and a c-linear map $B : \mathbf{V} \to \mathbf{V}'$, the maps $\mathrm{Hom}(A, B) : \mathrm{Hom}(U, \mathbf{V}) \to \mathrm{Hom}(U', \mathbf{V}')$ and $\mathrm{Hom}(B, A) : \mathrm{Hom}(\mathbf{V}', U') \to \mathrm{Hom}(\mathbf{V}, U)$ are c-linear with respect to the induced CSOs from Example 5.4.* ∎

EXERCISE 5.31. Given $\mathbf{V} = (V, J_V)$, the canonical maps $l_1 : \mathbb{R} \otimes \mathbf{V} \to \mathbf{V}$ and $l_2 : \mathbf{V} \otimes \mathbb{R} \to \mathbf{V}$ from Example 1.28 are c-linear.

HINT. The CSO on $\mathbb{R} \otimes V$ is as in Example 5.3, and the claim for $l_1$ follows from Lemma 1.38. The claim for $l_2 : V \otimes \mathbb{R} \to V$ is analogous. ∎

EXERCISE 5.32. Given $\mathbf{W} = (W, J)$, the map $m : \mathbf{W} \to \mathrm{Hom}(\mathbb{K}, \mathbf{W})$ from Definition 1.20 is c-linear.

HINT. The CSO on $\mathrm{Hom}(\mathbb{K}, W)$ is as in Example 5.4, and the claim follows from Lemma 1.21. ∎

EXERCISE 5.33. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J)$, the canonical map (Definition 1.7) $t_{UV}^{W} : \mathrm{Hom}(\mathbf{U}, V) \to \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(\mathbf{U}, W))$ is c-linear with respect to the induced CSOs as in Example 5.4. ∎

EXERCISE 5.34. Given $U$, $V$, $W$, with $\mathbf{V} = (V, J)$, $t_{UV}^{W} : \mathrm{Hom}(U, \mathbf{V}) \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W), \mathrm{Hom}(U, W))$ is c-linear with respect to the induced CSOs. ∎

EXERCISE 5.35. Given $U$, $V$, $W$, with $\mathbf{W} = (W, J)$, and any $A \in \mathrm{Hom}(U, V)$, the map

$$t_{UV}^{W}(A) \in \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(U, W))$$

is c-linear $\mathrm{Hom}(V, \mathbf{W}) \to \mathrm{Hom}(U, \mathbf{W})$.

HINT. This claim, Exercise 5.33, and Exercise 5.34 all follow from Lemma 1.8. ∎

EXERCISE 5.36. Given $V$ and $W$, with $\mathbf{V} = (V, J_V)$, the canonical map (Definition 1.13) $d_{VW} : \mathbf{V} \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W), W)$ is c-linear with respect to the induced CSO as in Example 5.4. ∎

EXERCISE 5.37. Given $V$ and $W$, with $\mathbf{W} = (W, J_W)$, and any $v \in V$, the map $d_{VW}(v) \in \mathrm{Hom}(\mathrm{Hom}(V, W), W)$ is c-linear $\mathrm{Hom}(V, \mathbf{W}) \to \mathbf{W}$.

HINT. This claim and Exercise 5.36 both follow from Lemma 1.14. ∎

EXERCISE 5.38. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J)$, the canonical map (Definition 1.56) $e_{UV}^{W} : \mathrm{Hom}(\mathbf{U}, V) \to \mathrm{Hom}(\mathrm{Hom}(V, W) \otimes \mathbf{U}, W)$ is c-linear with respect to the induced CSOs. ∎

EXERCISE 5.39. Given $U$, $V$, $W$, with $\mathbf{V} = (V, J)$, $e_{UV}^{W} : \mathrm{Hom}(U, \mathbf{V}) \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W) \otimes U, W)$ is c-linear with respect to the induced CSOs. ∎

EXERCISE 5.40. Given $U$, $V$, $W$, with $\mathbf{W} = (W, J)$, and any $A \in \mathrm{Hom}(U, V)$, the map $e_{UV}^{W}(A) \in \mathrm{Hom}(\mathrm{Hom}(V, W) \otimes U, W)$ is c-linear $\mathrm{Hom}(V, \mathbf{W}) \otimes U \to \mathbf{W}$.

HINT. This claim, Exercise 5.38, and Exercise 5.39 all follow from Lemma 1.57. ∎

Recall from Definition 2.71 that $\mathrm{Hom}(\mathrm{Hom}(V, W) \otimes V, W)$ contains a distinguished element $Ev_{VW} : A \otimes v \mapsto A(v)$.

EXERCISE 5.41. Given $V$ and $W$, with $\mathbf{W} = (W, J_W)$, the canonical evaluation $Ev_{VW} : \mathrm{Hom}(V, \mathbf{W}) \otimes V \to \mathbf{W}$ is c-linear.

HINT. This claim can be checked directly; it also follows from Lemma 2.73, or the formula $Ev_{VW} = e_{VV}^{W}(Id_V)$ from Equation (2.16) and Exercise 5.40. ∎

EXERCISE 5.42. Given $U$ and $V$, with $V$ finite-dimensional and $\mathbf{U} = (U, J_U)$, the map $\eta_{VU} : \mathbf{U} \to V \otimes \mathrm{Hom}(V, \mathbf{U})$ from Notation 4.44 is c-linear.

HINT. The claim follows from Lemma 4.47. ∎

EXERCISE 5.43. Given $U$, $V$, $W$, with $\mathbf{W} = (W, J_W)$, the map

$$T_{U,V;W} : \mathrm{Hom}(U, \mathrm{Hom}(V, \mathbf{W})) \to \mathrm{Hom}(V, \mathrm{Hom}(U, \mathbf{W}))$$

from Definition 4.2 is c-linear.

HINT. The claim follows from Lemma 4.6. ∎

EXERCISE 5.44. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$, the map

$$T_{U,V;W} : \mathrm{Hom}(\mathbf{U}, \mathrm{Hom}(V, W)) \to \mathrm{Hom}(V, \mathrm{Hom}(\mathbf{U}, W)),$$

and its inverse from Equation (4.2) in Lemma 4.4,

$$T_{V,U;W} : \mathrm{Hom}(V, \mathrm{Hom}(\mathbf{U}, W)) \to \mathrm{Hom}(\mathbf{U}, \mathrm{Hom}(V, W)),$$

are c-linear.

HINT. Both claims follow from Lemma 4.6. Alternatively, the c-linearity of the composite formula from Definition 4.2,

$$T_{U,V;W} = \mathrm{Hom}(d_{VW}, Id_{\mathrm{Hom}(U,W)}) \circ t^{W}_{U,\mathrm{Hom}(V,W)},$$

and its inverse could be shown to follow from the c-linearity of the $t$ and $d$ maps, as in Exercise 5.33, Exercise 5.34, and Exercise 5.36. ∎

EXERCISE 5.45. Given $V$ and $A \in \mathrm{End}(V)$, if $V$ is finite-dimensional and there is some CSO $J$ on $V$ such that $A$ is a-linear with respect to $J$, then $Tr_V(A) = 0$. ∎

LEMMA 5.46. *Given $V$ and a CSO $J \in \mathrm{End}(V)$, for any invertible $A : U \to V$, the composite $A^{-1} \circ J \circ A \in \mathrm{End}(U)$ is a CSO. $A$ is c-linear with respect to $A^{-1} \circ J \circ A$ and $J$. If $B : U \to V$ is another invertible map and $A^{-1} \circ J \circ A = B^{-1} \circ J \circ B$, then $A \circ B^{-1}$ is a c-linear endomorphism of $(V, J)$.* ∎

The CSO $A^{-1} \circ J \circ A$ is the pullback of $J$.

LEMMA 5.47. *Given $V = V_1 \oplus V_2$ and a CSO $J \in \mathrm{End}(V)$, the following are equivalent.*

(1) *$J$ respects the direct sum.*
(2) *$Q_1 \circ P_1$ is c-linear.*
(3) *$Q_2 \circ P_2$ is c-linear.*
(4) *$P_1 \circ J \circ Q_1 = 0_{\mathrm{Hom}(V_1,V_2)}$ and $P_2 \circ J \circ Q_2 = 0_{\mathrm{Hom}(V_2,V_1)}$.*
(5) *There exists a CSO $J_1$ on $V_1$ and a CSO $J_2$ on $V_2$ so that*

$$J = Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2. \tag{5.4}$$

PROOF. The equivalence of the five properties follows from Lemma 1.87 and Definition 1.88. For (1) $\implies$ (5), it is easily checked that each induced map $J_i = P_i \circ J \circ Q_i$ is a CSO on $V_i$, and that $J$ is recovered by re-combining the induced maps as in (5.4):

$$Q_1 \circ (P_1 \circ J \circ Q_1) \circ P_1 + Q_2 \circ (P_2 \circ J \circ Q_2) \circ P_2 = J.$$

Conversely, given CSOs $J_1$ on $V_1$, $J_2$ on $V_2$, the map $J$ constructed as in (5.4) is a CSO on $V$, and each $J_i$ agrees with the CSO induced by this $J$:

$$P_i \circ (Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2) \circ Q_i = J_i. \quad ∎$$

LEMMA 5.48. *For $V = V_1 \oplus V_2$ and $V' = V_1' \oplus V_2'$ and invertible maps $A : V_2 \to V_1$, $A' : V_2' \to V_1'$, let $J$, $J'$ be CSOs on $V$ and $V'$ constructed as in Example 5.8. Then, for $H : V \to V'$, the following are equivalent.*

(1) *$H$ is c-linear with respect to $J$ and $J'$.*
(2) *$A' \circ P_2' \circ H \circ Q_2 = P_1' \circ H \circ Q_1 \circ A$ and $P_1' \circ H \circ Q_2 = -A' \circ P_2' \circ H \circ Q_1 \circ A$.*

HINT. To show (2) $\implies$ (1), expand

$$H \circ J = (Q_1' \circ P_1' + Q_2' \circ P_2') \circ H \circ (Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2)$$

and similarly $J' \circ H$.

For (1) $\implies$ (2), apply $\mathrm{Hom}(Q_1, P_2')$ to both sides of $H \circ J = J' \circ H$ to get one of the equations, and apply $\mathrm{Hom}(Q_2, P_2')$ to get the other. ∎

REMARK 5.49. Lemma 5.48 displays an algebraic pattern analogous to the Cauchy-Riemann equations.

EXAMPLE 5.50. For $V = V_1 \oplus V_2$, $V' = V_1' \oplus V_2'$, $A$, $A'$, $J$, $J'$ as in Lemma 5.48, any map $B : V_1 \to V_1'$, and any real scalars $\alpha$, $\beta$, the map $H : V \to V'$ defined by

$$\alpha \cdot Q_1' \circ B \circ P_1 + \beta \cdot Q_1' \circ B \circ A \circ P_2 - \beta \cdot Q_2' \circ (A')^{-1} \circ B \circ P_1 + \alpha \cdot Q_2' \circ (A')^{-1} \circ B \circ A \circ P_2 \tag{5.5}$$

is c-linear with respect to $J$, $J'$. In particular, in the case $\alpha = 1$, $\beta = 0$, we denote $H = B_c$:

$$B_c = Q_1' \circ B \circ P_1 + Q_2' \circ (A')^{-1} \circ B \circ A \circ P_2 \tag{5.6}$$

and consider $B_c : V \to V'$ a c-linear extension of $B : V_1 \to V_1'$ because it satisfies $P_1' \circ B_c \circ Q_1 = B$.

EXERCISE 5.51. If $(\alpha, \beta) \neq (0, 0)$ then for $B$ as in Example 5.50 and $H$ from (5.5), the following are equivalent.

(1) $B$ has a left inverse.
(2) $H$ has a left inverse.

Similarly, the following are equivalent.

(3) $B$ has a right inverse.
(4) $H$ has a right inverse.

Similarly, the following are equivalent.

(5) $B$ is invertible.
(6) $H$ is invertible.

HINT. For (1) $\implies$ (2), if $G : V_1' \to V_1$ is a left inverse of $B$ then a left inverse of $H$ is:

$$\frac{1}{\alpha^2 + \beta^2} \cdot (\alpha \cdot Q_1 \circ G \circ P_1' - \beta \cdot Q_1 \circ G \circ A' \circ P_2'$$
$$+ \beta \cdot Q_2 \circ A^{-1} \circ G \circ P_1' + \alpha \cdot Q_2 \circ A^{-1} \circ G \circ A' \circ P_2') .$$

The same formula works to show (3) $\implies$ (4) (assuming instead that $G$ is a right inverse of $B$) and (5) $\implies$ (6) (with $G = B^{-1}$).

For (2) $\implies$ (1), if $F : V' \to V$ is a left inverse of $H$ then a left inverse of $B$ is:

$$P_1 \circ F \circ (\alpha \cdot Q_1' - \beta \cdot Q_2' \circ (A')^{-1}).$$

For (4) $\implies$ (3) (and then (6) $\implies$ (5) as in Exercise 0.54), if $F'$ is a right inverse of $H$ then a right inverse of $B$ is

$$(\alpha \cdot P_1 + \beta \cdot A \circ P_2) \circ F' \circ Q_1'.$$

∎

EXERCISE 5.52. Given $(V_1, J_1)$ and $(V_2, J_2)$, a direct sum $V = V_1 \oplus V_2$, and a map $A : V_1 \to V_2$, the following are equivalent.

(1) $A$ is a-linear.
(2) $J = Q_1 \circ J_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ J_2 \circ P_2$ is a CSO on $V$.

For $A$ and $J$ as in the above statements (1) $\iff$ (2), the following are equivalent.

(3) $A = 0_{\mathrm{Hom}(V_1, V_2)}$.
(4) $J$ respects the direct sum $V = V_1 \oplus V_2$.
(5) $J$ is the CSO constructed in (5.4) from Lemma 5.47.

∎

EXERCISE 5.53. For $V$, $\mathbf{V_1}$, $\mathbf{V_2}$, $A$ as in Exercise 5.52, the CSO $J_A = Q_1 \circ J_1 \circ P_1 + Q_2 \circ A \circ P_1 + Q_2 \circ J_2 \circ P_2$ is similar to the direct sum CSO $J_0 = Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2$, in the sense that $J_A = G^{-1} \circ J_0 \circ G$ for some invertible $G \in \mathrm{End}(V)$, and $G$ can be chosen so that $P_2 \circ G \circ Q_2 = Id_{V_2}$.

HINT. Let $G = Id_V + \frac{1}{2} \cdot Q_2 \circ A \circ J_1 \circ P_1$, then check $G \circ J_A = J_0 \circ G$, or use $G^{-1} = Id_V - \frac{1}{2} \cdot Q_2 \circ A \circ J_1 \circ P_1$. ∎

EXERCISE 5.54. Given $V = V_1 \oplus V_2$, with CSOs $J_1$ on $V_1$, $J_2$ on $V_2$, and $J = Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2$ on $V$, let $A : V_1 \to V_2$ be any map, let

$$Q_1' = Q_1 + Q_2 \circ A : V_1 \to V,$$

and let

$$H = Q_1'(V_1) \cap J(Q_1'(V_1)).$$

$H$ is the largest $J$-invariant subspace of $V$ contained in the image of $Q_1'$. For any $u \in V_1$, the following are equivalent:

(1) $A(J_1(u)) = J_2(A(u))$.
(2) $Q_1'(u) \in H$.

HINT. $J$ is the CSO constructed in (5.4) from Lemma 5.47, and the same as $J_0$ from Exercise 5.53. The map $Q_1'$ is an inclusion in the construction of the graph of $A$, from Exercise 1.107. The property (1) was considered in Lemma 5.24.

Assuming (1),

$$\begin{aligned} J(Q_1'(u)) &= (Q_1 \circ J_1 \circ P_1 + Q_2 \circ J_2 \circ P_2)(Q_1(u) + Q_2(A(u))) \\ &= Q_1(J_1(u)) + Q_2(J_2(A(u))) \\ &= Q_1(J_1(u)) + Q_2(A(J_1(u))) = Q_1'(J_1(u)). \end{aligned} \tag{5.7}$$

So $Q_1'(u) = -J(Q_1'(J_1(u))) \in J(Q_1'(V_1))$.

Assuming (2), if $Q_1'(u) \in J(Q_1'(V_1))$, then $J(Q_1'(u)) = Q_1'(v)$ for some $v \in V_1$. From (5.7),

$$Q_1(J_1(u)) + Q_2(J_2(A(u))) = Q_1(v) + Q_2(A(v)),$$

and it follows that $v = J_1(u)$ and $J_2(A(u)) = A(v) = A(J_1(u))$, showing (1). ∎

PROPOSITION 5.55. *Given $V$, consider three elements $A, J_1, J_2 \in \mathrm{End}(V)$. The following two statements are equivalent.*

(1) $(J_1 + J_2) \circ A = J_1 - J_2$.
(2) $J_2 \circ (Id_V + A) = J_1 \circ (Id_V - A)$.

*The following two statements are also equivalent to each other.*

(1′) $A \circ (J_1 + J_2) = J_1 - J_2$.
(2′) $(Id_V + A) \circ J_2 = (Id_V - A) \circ J_1$.

*If $A, J_1, J_2$ satisfy either condition (1) or (1′), then any two of the following imply the remaining third.*

(3) *$J_1$ is invertible.*
(4) *$J_1 + J_2$ is invertible.*
(5) *$Id_V + A$ is invertible.*

*If $A, J_1, J_2$ satisfy (1) or (1′), and also (5), then any two of the following imply the remaining third.*

(6) $A \circ J_1 = -J_1 \circ A$.

(7) *$J_1$ is a CSO.*
(8) *$J_2$ is a CSO.*

PROOF. (1) $\iff$ (2) by an elementary algebraic manipulation. The $(1')$ $\iff$ $(2')$ and subsequent implications are analogous and left as an exercise. For (3), (4), (5), use (1) or (2) to establish

$$(J_1 + J_2) \circ (Id_V + A) = 2 \cdot J_1,$$

and the claim follows.

For (6), (7), (8), use (2) to establish

$$\begin{aligned} (5.8) \qquad & (J_1 + J_2) \circ (A \circ J_1 + J_1 \circ A) \\ = \quad & J_1 \circ A \circ J_1 + J_2 \circ A \circ J_1 + J_1 \circ J_1 \circ A + J_2 \circ J_1 \circ A \\ = \quad & J_1 \circ A \circ J_1 + J_2 \circ (Id_V + A) \circ J_1 + J_1 \circ J_1 \circ A - J_2 \circ J_1 \circ (Id_V - A) \\ = \quad & J_1 \circ A \circ J_1 + J_1 \circ (Id_V - A) \circ J_1 + J_1 \circ J_1 \circ A - J_2 \circ J_2 \circ (Id_V + A) \\ = \quad & (J_1 \circ J_1 - J_2 \circ J_2) \circ (Id_V + A). \end{aligned}$$

Given (6), (8), the equation (5.8) becomes $0_{\mathrm{End}(V)} = (J_1 \circ J_1 + Id_V) \circ (Id_V + A)$, and (7) follows from (5). Similarly, given (6), (7), (5.8) becomes $0_{\mathrm{End}(V)} = -(Id_V + J_2 \circ J_2) \circ (Id_V + A)$, and (8) follows from (5). For (7), (8), note that (7) implies (3) and then (5) implies (4), so by (7) and (8), (5.8) becomes $LHS = 0_{\mathrm{End}(V)}$, and (6) follows from (4). ∎

Given $V$ with a CSO $J_1$, Proposition 5.55 establishes a bijective correspondence between the set of CSOs $J_2$ on $V$ with $J_1 + J_2$ invertible and the set of $A \in \mathrm{End}(V)$ with $A$ a-linear (with respect to $J_1$) and $Id_V + A$ invertible, as follows. Since $J_1$ is a CSO, (3) and (7) hold. For any a-linear $A \in \mathrm{End}(V)$ with $Id_V + A$ invertible, (5) and (6) hold. If we define $J_2$ by the similarity relation $J_2 = (Id_V + A) \circ J_1 \circ (Id_V + A)^{-1}$, then $J_2 = J_1 \circ (Id_V - A) \circ (Id_V + A)^{-1}$, so (2) holds, (1), (4), and (8) follow as consequences, and $A$ satisfies $A = (J_1 + J_2)^{-1} \circ (J_1 - J_2)$. Conversely, for any CSO $J_2$ with $J_1 + J_2$ invertible, (4) and (8) hold. If we define

$$A = (J_1 + J_2)^{-1} \circ (J_1 - J_2), \tag{5.9}$$

then (1) holds, (2), (5), and (6) follow as consequences, and $J_2$ satisfies $J_2 = J_1 \circ (Id_V - A) \circ (Id_V + A)^{-1} = (Id_V + A) \circ J_1 \circ (Id_V + A)^{-1}$.

EXERCISE 5.56. Given $V$ and any two CSOs $J_1$, $J_2$, the map $J_1 + J_2$ is c-linear with respect to $J_1$ and $J_2$, and the maps $\pm(J_1 - J_2)$ are a-linear with respect to $J_1$ and $J_2$. ∎

BIG EXERCISE 5.57. Given $V_1$, $V_2$ with CSOs $J_1$, $J_2$, and a real linear map $A : V_1 \to V_2$, if the image subspace $A(V_1)$ admits a linearly independent list of 3 or more elements of $V_2$, then the following are equivalent.

(1) For each $v \in V_1$, $A(J_1(v)) \in \mathrm{span}\{A(v), J_2(A(v))\}$.
(2) For each $v \in V_1$, $A$ maps the subspace $\mathrm{span}\{v, J_1(v)\} \subseteq V_1$ to the subspace $\mathrm{span}\{A(v), J_2(A(v))\} \subseteq V_2$.
(3) For each $v \in V_1$, either $A(J_1(v)) = J_2(A(v))$ or $A(J_1(v)) = -J_2(A(v))$.
(4) $A \circ J_1 = J_2 \circ A$ or $A \circ J_1 = -J_2 \circ A$.

HINT. The idea is that $A$ takes $J_1$-complex lines to $J_2$-complex lines, and that this is equivalent to $A$ being c-linear or a-linear. See also [C1] for other properties of $A$ equivalent to (4). ∎

## 5.3. Commuting Complex Structure Operators

### 5.3.1. Two Commuting Complex Structure Operators.

LEMMA 5.58. *Given $V$ and two CSOs $J_1$, $J_2$, the following are equivalent.*

(1) *$J_1$ and $J_2$ commute (i.e., $J_1 \circ J_2 = J_2 \circ J_1$).*
(2) *The composite $J_1 \circ J_2$ is an involution.*
(3) *The composite $-J_1 \circ J_2$ is an involution.*

∎

EXAMPLE 5.59. Given $V$ with commuting CSOs $J_1$, $J_2$, Lemma 1.119 applies to the involution $-J_1 \circ J_2$ as in Lemma 5.58: there is a direct sum $V = V_c \oplus V_a$ produced by $-J_1 \circ J_2$, where

$$\begin{aligned} V_c &= \{v \in V : (-J_1 \circ J_2)(v) = v\} = \{v \in V : J_2(v) = J_1(v)\} \\ V_a &= \{v \in V : (-J_1 \circ J_2)(v) = -v\} = \{v \in V : J_2(v) = -J_1(v)\}, \end{aligned}$$

with projections

$$P_c = \frac{1}{2} \cdot (Id_V - J_1 \circ J_2) : V \twoheadrightarrow V_c, \ P_a = \frac{1}{2} \cdot (Id_V + J_1 \circ J_2) : V \twoheadrightarrow V_a.$$

As remarked in the Proof of Lemma 1.119, the same formulas are also used for $Q_c \circ P_c$, $Q_a \circ P_a \in \text{End}(V)$, where $Q_c$ and $Q_a$ are the corresponding subspace inclusions.

Note that applying Lemma 1.119 to the involution $J_1 \circ J_2$ would give the direct sum in the other order, $V = V_a \oplus V_c$. The subspace $V_c$ from Example 5.59 is the same as in Example 5.25, where $J_1$ and $J_2$ did not necessarily commute.

LEMMA 5.60. *For $V$, $J_1$, $J_2$ as in Lemma 5.58, and another space $V'$ with commuting CSOs $J_1'$, $J_2'$, a map $H : V \to V'$ respects the direct sums $V_c \oplus V_a$ and $V_c' \oplus V_a'$ if and only if $H \circ J_1 \circ J_2 = J_1' \circ J_2' \circ H$.*

PROOF. This is an example of Lemma 1.126. ∎

EXAMPLE 5.61. Lemma 5.60 applies to $V' = V$, $J_1' = J_1$, $J_2' = J_2$, and either $H = J_1$ or $H = J_2$, each of which induces a CSO on $V_c$ and on $V_a$ by Lemma 5.47. The subspace $V_c$ has a canonical CSO, induced by either $J_1$ or $J_2$:

$$P_c \circ J_1 \circ Q_c = P_c \circ J_2 \circ Q_c \in \text{End}(V_c).$$

The maps $P_c : V \twoheadrightarrow V_c$ and $Q_c : V_c \hookrightarrow V$ are c-linear with respect to either CSO on $V$. The induced CSOs on the subspace $V_a$ are opposite, and generally distinct:

$$P_a \circ J_1 \circ Q_a = -P_a \circ J_2 \circ Q_a \in \text{End}(V_a).$$

LEMMA 5.62. *For $V$, $J_1$, $J_2$ as in Lemma 5.58, $\mathbf{U} = (U, J_U)$, and a map $H : U \to V$, if $H$ is c-linear with respect to both $(J_U, J_1)$ and $(J_U, J_2)$, then the image of $H$ is contained in $V_c$.*

PROOF. This can be checked by showing $P_a \circ H = 0_{\text{Hom}(U, V_a)}$. ∎

LEMMA 5.63. *For $V$, $J_1$, $J_2$ as in Lemma 5.58, $\mathbf{U} = (U, J_U)$, and a map $H : V \to U$, if $H$ is c-linear with respect to either $(J_1, J_U)$ or $(J_2, J_U)$, then $H \circ Q_c : V_c \to \mathbf{U}$ is c-linear.*

PROOF. By the construction of the CSO on $V_c$ from Example 5.61, the LHS quantities in the following two equations are equal to each other:

$$\begin{aligned}(H \circ Q_c) \circ (P_c \circ J_1 \circ Q_c) &= H \circ J_1 \circ Q_c,\\ (H \circ Q_c) \circ (P_c \circ J_2 \circ Q_c) &= H \circ J_2 \circ Q_c.\end{aligned}$$

The equalities hold because $J_1$ and $J_2$ commute with $Q_c \circ P_c$. On the RHS, either $H \circ J_1$ or $H \circ J_2$ is equal to $J_U \circ H$ by hypothesis, so the claim $(H \circ Q_c) \circ (P_c \circ J_1 \circ Q_c) = J_U \circ (H \circ Q_c)$ follows. ∎

LEMMA 5.64. *For $V$ with commuting CSOs $J_1$, $J_2$, and $V'$ with commuting CSOs $J'_1$, $J'_2$ as in Lemma 5.60, if $H : V \to V'$ is c-linear with respect to any of the pairs $(J_1, J'_1)$ or $(J_1, J'_2)$ or $(J_2, J'_1)$ or $(J_2, J'_2)$, then $P'_c \circ H \circ Q_c : V_c \to V'_c$ is c-linear with respect to the canonical CSOs.*

PROOF. The CSOs on $V_c$ and $V'_c$ are as in Example 5.61. ∎

LEMMA 5.65. *Given $V$ with commuting CSOs $J_V^1$, $J_V^2$, and $U$ with commuting CSOs $J_U^1$, $J_U^2$, if $H : U \to V$ satisfies both $H \circ J_U^1 = J_V^1 \circ H$ and $H \circ J_U^2 = J_V^2 \circ H$, then $H : U_c \oplus U_a \to V_c \oplus V_a$ respects the direct sums and the induced map $P_c^V \circ H \circ Q_c^U : U_c \to V_c$ is c-linear with respect to the induced CSOs. If, also, $H$ is invertible, then for $i = c, a$, the induced map $P_i^V \circ H \circ Q_i^U : U_i \to V_i$ is invertible.*

PROOF. This follows from Lemma 5.60, Lemma 5.64, and Lemma 1.89. ∎

We remark that the direct summands in Lemma 5.65 are all subspaces; if $u = Q_c^U(u) \in U_c \subseteq U$, then $H(u) = H(Q_c^U(u)) \in V_c \subseteq V$, so $H(u)$ is in the fixed point set of the idempotent $P_c^V : V \to V$, as follows:

$$\begin{aligned}u \in U_c \implies H(u) &= H(Q_c^U(u)) = H(Q_c^U(P_c^U(Q_c^U(u)))) = Q_c^V(P_c^V(H(Q_c^U(u))))\\ &= (P_c^V \circ H \circ Q_c^U)(u) = P_c^V(H(u)) \in V_c \subseteq V.\end{aligned}$$

The induced map $P_c^V \circ H \circ Q_c^U : U_c \to V_c$ is just the restriction of $H : U \to V$ to the subspace $U_c$, with image contained in the subspace $V_c$ of the target.

LEMMA 5.66. *Given $V$ with commuting CSOs $J_V^1$, $J_V^2$, and $U$ with commuting CSOs $J_U^1$, $J_U^2$, if $H : U \to V$ satisfies both $H \circ J_U^1 = -J_V^1 \circ H$ and $H \circ J_U^2 = -J_V^2 \circ H$, then $H : U_c \oplus U_a \to V_c \oplus V_a$ respects the direct sums and the induced map $P_c^V \circ H \circ Q_c^U : U_c \to V_c$ is a-linear with respect to the induced CSOs.*

PROOF. This is straightforward to check directly, or follows from Lemma 5.65 with $J_V^1$, $J_V^2$ replaced by the opposite CSOs $-J_V^1$, $-J_V^2$. ∎

The following Theorem weakens the hypotheses of Lemma 5.65.

THEOREM 5.67. *Given $V$ with commuting CSOs $J_V^1$, $J_V^2$, and $U$ with commuting CSOs $J_U^1$, $J_U^2$, if $H : U \to V$ is c-linear with respect to $(J_U^2, J_V^2)$, then the kernel of the composite $P_c^V \circ H \circ Q_a^U : U_a \to V_c$ is equal to the set $\{u \in U_a : (H \circ J_U^1 \circ Q_a^U)(u) = (J_V^1 \circ H \circ Q_a^U)(u)\}$.*

PROOF. Composing with $Q_c^V$ does not change the kernel, so using the equalities $J_U^1 \circ Q_a^U = -J_U^2 \circ Q_a^U$ and $J_V^2 \circ H = H \circ J_U^2$,

$$\begin{aligned}Q_c^V \circ P_c^V \circ H \circ Q_a^U &= \frac{1}{2} \cdot (Id_V - J_V^1 \circ J_V^2) \circ H \circ Q_a\\ &= \frac{1}{2} \cdot (H - J_V^1 \circ H \circ J_U^2) \circ Q_a^U = \frac{1}{2} \cdot (H + J_V^1 \circ H \circ J_U^1) \circ Q_a^U,\end{aligned}$$

and the composite with the invertible map $2\cdot J_V^1$ has the same kernel:

$$2\cdot J_V^1\circ Q_c^V\circ P_c^V\circ H\circ Q_a^U=J_V^1\circ H\circ Q_a^U-H\circ J_U^1\circ Q_a^U.$$

∎

EXAMPLE 5.68. For a vector space $V$ with a CSO $J$ and involution $N$ that commute as in Example 5.6, the CSO $N\circ J$ commutes with $J$ and $N$. The involution $N$ produces a direct sum $V=V_1\oplus V_2$ with projections $(P_1,P_2)$ and inclusions $(Q_1,Q_2)$ as in Lemma 1.119, and because $N=-(N\circ J)\circ J$, $V_1\oplus V_2$ is the same as the direct sum $V=V_c\oplus V_a$ as in Example 5.59. Lemma 1.126 and Lemma 5.47 apply to both $J$ and $N\circ J$: they respect the direct sum $V_1\oplus V_2$, and the induced maps $P_1\circ J\circ Q_1$ and $P_1\circ N\circ J\circ Q_1$ are commuting CSOs on $V_1$, and similarly for $V_2$. More specifically, the CSOs on $V_1$ are equal: $P_1\circ J\circ Q_1=P_1\circ N\circ J\circ Q_1$, while the CSOs $P_2\circ J\circ Q_2$ and $P_2\circ N\circ J\circ Q_2$ on $V_2$ are opposite.

EXAMPLE 5.69. Given $V=V_1\oplus V_2$ and CSOs $J_i\in\operatorname{End}(V_i)$ for $i=1,2$, $V$ admits four CSOs: $\pm J$ and $\pm J'$, where:

$$\begin{aligned}J&=Q_1\circ J_1\circ P_1+Q_2\circ J_2\circ P_2,\\ J'&=Q_1\circ J_1\circ P_1-Q_2\circ J_2\circ P_2.\end{aligned}$$

$J$ is as in (5.4), and all four CSOs respect the direct sum as in Lemma 5.47. $J$ and $J'$ commute with each other, with $-J\circ J'=Q_1\circ P_1-Q_2\circ P_2$, so the given direct sum is equivalent to the direct sum produced by the involution, $V=V_c\oplus V_a$, as in Example 1.122. This construction is a special case of Example 5.68, with the involution $N=Q_1\circ P_1-Q_2\circ P_2$ that commutes with $J$.

EXAMPLE 5.70. Given $U$ with commuting CSOs $J_1$, $J_2$, if $H\in\operatorname{End}(U)$ is an involution such that $H\circ J_1\circ J_2=J_1\circ J_2\circ H$, then $U$ admits two direct sums: $U=U_1\oplus U_2$ produced by $H$ as in Lemma 1.119, and $U=U_c\oplus U_a$ produced by $-J_1\circ J_2$. The composite $(-J_1\circ J_2)\circ H$ is a third involution, which produces a direct sum $U=U_5\oplus U_6$, as denoted in Theorem 1.130. Both $H$ and $-J_1\circ J_2\circ H$ respect the direct sum $U_c\oplus U_a$ as in Lemma 1.128 and Lemma 5.60; they both induce involutions $U_c\to U_c$ and $U_a\to U_a$, with the involutions being equal on $U_c$ as in (1.15), and opposite on $U_a$ as in (1.16), producing a direct sum $U_c=U_c'\oplus U_c''$, and an unordered pair of subspaces of $U_a$:

$$\begin{aligned}U_c'&=\{u\in U:u=-J_1(J_2(u))=H(u)\},\\ U_c''&=\{u\in U:u=-J_1(J_2(u))=-H(u)\},\\ U_a'&=\{u\in U:u=J_1(J_2(u))=H(u)\},\\ U_a''&=\{u\in U:u=J_1(J_2(u))=-H(u)\}.\end{aligned}\tag{5.10}$$

Similarly, both $-J_1\circ J_2$ and $-J_1\circ J_2\circ H$ respect the direct sum $U_1\oplus U_2$, and induce involutions on $U_1$ and $U_2$ that distinguish the same four subspaces:

$$\begin{aligned}U_1'&=\{u\in U:u=H(u)=-J_1(J_2(u))\},\\ U_1''&=\{u\in U:u=H(u)=J_1(J_2(u))\},\\ U_2'&=\{u\in U:u=-H(u)=-J_1(J_2(u))\},\\ U_2''&=\{u\in U:u=-H(u)=J_1(J_2(u))\}.\end{aligned}$$

This configuration of subspaces gives an example of the results of Theorem 1.130, $U_1'=U_c'=U_1\cap U_c=U_1\cap U_c\cap U_5$, $U_c''=U_2'=U_2\cap U_6$, $U_1''=U_a'=U_a\cap U_6$, etc.

LEMMA 5.71. *Given $U$ with commuting CSOs $J_1$, $J_2$, if $H \in \mathrm{End}(U)$ is an involution such that $H \circ J_1 = J_1 \circ H$ and $H \circ J_2 = J_2 \circ H$, then $H$ respects the direct sum $U_c \oplus U_a \to U_c \oplus U_a$, and induces involutions on $U_c$ and $U_a$. The induced involution $P_c \circ H \circ Q_c$ on $U_c$ is c-linear, and its fixed point set $U_c'$ has a canonical CSO.*

PROOF. It follows from the c-linearity of $H$ that $H \circ J_1 \circ J_2 = J_1 \circ J_2 \circ H$. So, this is a special case of both Lemma 5.65 and Example 5.70: the induced involution on $U_c$ is c-linear with respect to the CSO on $U_c$ from Example 5.61, and produces a direct sum $U_c = U_c' \oplus U_c''$ as in (5.10). The subspace $U_c' = \{u \in U : u = -J_1(J_2(u)) = H(u)\}$ has a canonical CSO, from Example 5.68. ∎

LEMMA 5.72. *Given $U$ with commuting CSOs $J_1$, $J_2$, if $H \in \mathrm{End}(U)$ is an involution such that $H \circ J_1 = J_2 \circ H$, then $H$ respects the direct sum $U_c \oplus U_a \to U_c \oplus U_a$, and induces involutions on $U_c$ and $U_a$. The induced involution $P_c \circ H \circ Q_c$ on $U_c$ is c-linear, and its fixed point set $U_c'$ has a canonical CSO.*

PROOF. It follows from the involution property that $H \circ J_1 = J_2 \circ H \implies J_1 \circ H = H \circ J_2 \implies H \circ J_1 \circ J_2 = J_1 \circ J_2 \circ H$. So, this is similar to Lemma 5.71, and also a special case of both Lemma 5.65 and Example 5.70: the induced involution on $U_c$ is c-linear and produces a direct sum $U_c = U_c' \oplus U_c''$. The subspace $U_c' = \{u \in U : u = -J_1(J_2(u)) = H(u)\}$ has a canonical CSO, from Example 5.68. ∎

EXAMPLE 5.73. Given $U$ and $V$, suppose there are commuting CSOs $J_1$, $J_2$ on $V$. Then the CSOs $[J_1 \otimes Id_U]$ and $[J_2 \otimes Id_U]$ on $V \otimes U$, from Example 5.3, also commute. As in Example 5.59, this gives a direct sum $V \otimes U = (V \otimes U)_c \oplus (V \otimes U)_a$ with projections

$$\frac{1}{2} \cdot (Id_{V \otimes U} \pm [J_1 \otimes Id_U] \circ [J_2 \otimes Id_U]).$$

$V \otimes U$ also admits a direct sum $(V_c \otimes U) \oplus (V_a \otimes U)$ as in Example 1.81, with projections

$$[(\frac{1}{2} \cdot (Id_V \pm J_1 \circ J_2)) \otimes Id_U] = [P_i \otimes Id_U],$$

for $P_i : V \twoheadrightarrow V_i$, $i = c, a$, as in Example 5.59. This is a special case of Example 1.141; the pairs of projections are identical (using the linearity of $j$ and Lemma 1.36), so the direct sums are the same and we have the equalities of subspaces $(V \otimes U)_c = V_c \otimes U$ and $(V \otimes U)_a = V_a \otimes U$, with inclusions $[Q_i \otimes Id_U]$. Similarly, $(U \otimes V)_c = U \otimes V_c$ and $(U \otimes V)_a = U \otimes V_a$.

EXAMPLE 5.74. Given $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the two CSOs $[Id_U \otimes J_V]$, $[J_U \otimes Id_V] \in \mathrm{End}(U \otimes V)$ commute, so this is a special case of Example 5.59. The direct sum so produced is denoted

$$U \otimes V = (U \otimes_c V) \oplus (U \otimes_a V).$$

As in Example 5.61, the subspace

$$U \otimes_c V = \{w \in U \otimes V : [Id_U \otimes J_V](w) = [J_U \otimes Id_V](w)\}$$

has a canonical CSO, induced by either of the CSOs, so we may denote the space $\mathbf{U} \otimes_c \mathbf{V}$. The CSOs on $U \otimes V$ induce opposite CSOs on the subspace

$$U \otimes_a V = \{w \in U \otimes V : [Id_U \otimes J_V](w) = -[J_U \otimes Id_V](w)\}.$$

EXAMPLE 5.75. For c-linear maps $A : \mathbf{U} \to \mathbf{U}'$ and $B : \mathbf{V} \to \mathbf{V}'$, the map

$$[A \otimes B] : U \otimes V \to U' \otimes V'$$

satisfies the hypotheses of Lemma 5.65, and respects the direct sums from Example 5.74. The induced map

$$P_c' \circ [A \otimes B] \circ Q_c : \mathbf{U} \otimes_c \mathbf{V} \to \mathbf{U}' \otimes_c \mathbf{V}'$$

is c-linear.

NOTATION 5.76. For c-linear maps $A : \mathbf{U} \to \mathbf{U}'$ and $B : \mathbf{V} \to \mathbf{V}'$ as in Example 5.75, the bracket notation from Notation 1.35 is adapted in the following abbreviation:

$$[A \otimes_c B] = P_c' \circ [A \otimes B] \circ Q_c : \mathbf{U} \otimes_c \mathbf{V} \to \mathbf{U}' \otimes_c \mathbf{V}'.$$

As remarked after Lemma 5.65, this is exactly the restriction of the map $[A \otimes B]$ to the $\mathbf{U} \otimes_c \mathbf{V}$ subspace of the domain, with image contained in the $\mathbf{U}' \otimes_c \mathbf{V}'$ subspace of the target. The role of the $j$ map in this construction is considered in more detail by Example 5.136.

EXERCISE 5.77. For a-linear maps $A : \mathbf{U} \to \mathbf{U}'$ and $B : \mathbf{V} \to \mathbf{V}'$, the map

$$[A \otimes B] : U \otimes V \to U' \otimes V'$$

respects the direct sums. The induced map

$$P_c' \circ [A \otimes B] \circ Q_c : \mathbf{U} \otimes_c \mathbf{V} \to \mathbf{U}' \otimes_c \mathbf{V}'$$

is a-linear. ∎

EXAMPLE 5.78. Given $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the CSO $\mathrm{Hom}(Id_U, J_V)$ commutes with $\mathrm{Hom}(J_U, Id_V) \in \mathrm{End}(\mathrm{Hom}(U, V))$, so this is a special case of Example 5.59. The direct sum so produced is denoted

$$\mathrm{Hom}(U, V) = \mathrm{Hom}_c(U, V) \oplus \mathrm{Hom}_a(U, V). \tag{5.11}$$

The projection $P_c : \mathrm{Hom}(U, V) \twoheadrightarrow \mathrm{Hom}_c(U, V)$ is defined by

$$\frac{1}{2} \cdot (Id_{\mathrm{Hom}(U,V)} - \mathrm{Hom}(Id_U, J_V) \circ \mathrm{Hom}(J_U, Id_V)) = \frac{1}{2} \cdot (Id_{\mathrm{Hom}(U,V)} - \mathrm{Hom}(J_U, J_V)).$$

As in Example 5.61, the subspace

$$\begin{aligned} \mathrm{Hom}_c(U, V) &= \{A \in \mathrm{Hom}(U, V) : \mathrm{Hom}(Id_U, J_V)(A) = \mathrm{Hom}(J_U, Id_V)(A)\} \\ &= \{A \in \mathrm{Hom}(U, V) : J_V \circ A = A \circ J_U\} \end{aligned}$$

has a canonical CSO, induced by either one of the CSOs, so we may denote the space $\mathrm{Hom}_c(\mathbf{U}, \mathbf{V})$. This is exactly the set of c-linear maps $\mathbf{U} \to \mathbf{V}$ as in Section 5.2. The CSOs on $\mathrm{Hom}(U, V)$ induce opposite CSOs on the subspace of a-linear maps,

$$\begin{aligned} \mathrm{Hom}_a(U, V) &= \{A \in \mathrm{Hom}(U, V) : \mathrm{Hom}(Id_U, J_V)(A) = -\mathrm{Hom}(J_U, Id_V)(A)\} \\ &= \{A \in \mathrm{Hom}(U, V) : J_V \circ A = -A \circ J_U\}. \end{aligned}$$

EXAMPLE 5.79. Given $\mathbf{V} = (V, J_V)$, there is a direct sum

$$\mathrm{End}(V) = \mathrm{End}_c(V) \oplus \mathrm{End}_a(V)$$

as in Example 5.78, where $\mathrm{End}_c(\mathbf{V})$ admits a canonical CSO. The identity element $Id_V \in \mathrm{End}_c(\mathbf{V}) \subseteq \mathrm{End}(V)$ is c-linear, and so is $J_V$.

LEMMA 5.80. *For c-linear maps $A : \mathbf{U}' \to \mathbf{U}$ and $B : \mathbf{V} \to \mathbf{V}'$, the map*

$$\mathrm{Hom}(A, B) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U', V')$$

*respects the direct sums from Example 5.78. The induced map*

$$P'_c \circ \mathrm{Hom}(A, B) \circ Q_c : \mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}_c(\mathbf{U}', \mathbf{V}') : F \mapsto B \circ F \circ A,$$

*is c-linear.*

PROOF. $P'_c$ is the projection $\mathrm{Hom}(U', V') \twoheadrightarrow \mathrm{Hom}_c(\mathbf{U}', \mathbf{V}')$. Lemma 5.65 applies. ∎

NOTATION 5.81. The induced map from Lemma 5.80 is denoted

$$\mathrm{Hom}_c(A, B) = P'_c \circ \mathrm{Hom}(A, B) \circ Q_c : \mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}_c(\mathbf{U}', \mathbf{V}').$$

As in Notation 5.76, this is a restriction of the map $\mathrm{Hom}(A, B)$ to subspaces of its domain and target.

EXERCISE 5.82. For a-linear maps $A : \mathbf{U}' \to \mathbf{U}$ and $B : \mathbf{V} \to \mathbf{V}'$, the map

$$\mathrm{Hom}(A, B) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(U', V')$$

respects the direct sums. The induced map

$$P'_c \circ \mathrm{Hom}(A, B) \circ Q_c : \mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}_c(\mathbf{U}', \mathbf{V}') : F \mapsto B \circ F \circ A$$

is a-linear. ∎

EXAMPLE 5.83. Given $U$ and $V$, suppose there are commuting CSOs $J_1$, $J_2$ on $V$. Then the CSOs $\mathrm{Hom}(Id_U, J_1)$ and $\mathrm{Hom}(Id_U, J_2)$ on $\mathrm{Hom}(U, V)$ also commute. As in Example 5.59, this gives a direct sum temporarily denoted

$$\mathrm{Hom}(U, V) = (\mathrm{Hom}(U, V))_c \oplus (\mathrm{Hom}(U, V))_a \tag{5.12}$$

with projections

$$\frac{1}{2} \cdot (Id_{\mathrm{Hom}(U,V)} \pm \mathrm{Hom}(Id_U, J_1) \circ \mathrm{Hom}(Id_U, J_2)).$$

$\mathrm{Hom}(U, V)$ also admits a direct sum $\mathrm{Hom}(U, V_c) \oplus \mathrm{Hom}(U, V_a)$ as in Example 1.82, with projections

$$\mathrm{Hom}(Id_U, \frac{1}{2} \cdot (Id_V \pm J_1 \circ J_2)). \tag{5.13}$$

The pairs of projections are identical: this is a special case of Example 1.143. We can identify $\mathrm{Hom}(U, V_c) = (\mathrm{Hom}(U, V))_c$, and also identify $\mathrm{Hom}(Id_U, Q_c)$ with the inclusion of the subspace $(\mathrm{Hom}(U, V))_c$ in $\mathrm{Hom}(U, V)$; similarly, $(\mathrm{Hom}(U, V))_a = \mathrm{Hom}(U, V_a)$. More specifically, the set $(\mathrm{Hom}(U, V))_c$ is defined as $\{A \in \mathrm{Hom}(U, V) : J_1 \circ A = J_2 \circ A\}$, while elements of $\mathrm{Hom}(U, V_c)$ are maps $A$ such that for any $u \in U$, $A(u) \in V_c \subseteq V$, meaning $J_1(A(u)) = J_2(A(u))$.

EXAMPLE 5.84. Given $U$ and $V$, suppose there are commuting CSOs $J_1$, $J_2$ on $V$. Then the CSOs $\mathrm{Hom}(J_1, Id_U)$ and $\mathrm{Hom}(J_2, Id_U)$ on $\mathrm{Hom}(V, U)$ also commute. As in Example 5.59, this gives a direct sum $\mathrm{Hom}(V, U) = (\mathrm{Hom}(V, U))_c \oplus (\mathrm{Hom}(V, U))_a$ (the same notation as (5.12) but not the same subspaces) with projections

$$\frac{1}{2} \cdot (Id_{\mathrm{Hom}(V,U)} \pm \mathrm{Hom}(J_1, Id_U) \circ \mathrm{Hom}(J_2, Id_U)).$$

$\mathrm{Hom}(V, U)$ also admits a direct sum $\mathrm{Hom}(V_c, U) \oplus \mathrm{Hom}(V_a, U)$ as in Example 1.83, with projections $\mathrm{Hom}(Q_c, Id_U)$, $\mathrm{Hom}(Q_a, Id_U)$. Unlike Examples 5.73 and 5.83, these pairs of projections are not obviously identical, and in fact this is a special case of Example 1.144. The set $(\mathrm{Hom}(V, U))_c$ is defined as $\{A \in \mathrm{Hom}(V, U) : A \circ J_1 = A \circ J_2\}$, while elements of $\mathrm{Hom}(V_c, U)$ are maps $A$ defined only on the subspace of $v \in V$ such that $J_1(v) = J_2(v)$. The two direct sums are different but equivalent, as discussed in Example 1.144. Specifically, if, for $i = c, a$, $P_i''$, $Q_i''$ denote the projections and inclusions for the direct sum $\mathrm{Hom}(V, U) = (\mathrm{Hom}(V, U))_c \oplus (\mathrm{Hom}(V, U))_a$, then

$$Q_i'' \circ P_i'' = \mathrm{Hom}(P_i, Id_U) \circ \mathrm{Hom}(Q_i, Id_U) : \mathrm{Hom}(V, U) \to \mathrm{Hom}(V, U),$$

and as in Lemma 1.99,

$$P_i'' \circ \mathrm{Hom}(P_i, Id_U) : \mathrm{Hom}(V_i, U) \to (\mathrm{Hom}(V, U))_i$$

is invertible with inverse $\mathrm{Hom}(Q_i, Id_U) \circ Q_i''$, and for $i = c$, c-linear.

LEMMA 5.85. *Given $U$ and $V$, with commuting CSOs $J_1$, $J_2 \in \mathrm{End}(V)$, let $W$ be a space admitting a direct sum $W_1 \oplus W_2$. If $H : W \to \mathrm{Hom}(V, U)$ respects one of the two direct sums from Example 5.84, then $H$ also respects the other direct sum. If, further, $H$ is invertible, then both induced maps $W_1 \to (\mathrm{Hom}(V, U))_c$ and $W_1 \to \mathrm{Hom}(V_c, U)$ are also invertible. If the direct sum on $W$ is given by commuting CSOs $J_W$, $J_W'$, and $H$ satisfies both $\mathrm{Hom}(J_1, Id_U) \circ H = H \circ J_W$ and $\mathrm{Hom}(J_2, Id_U) \circ H = H \circ J_W'$, then $H$ respects the direct sums and the induced maps are c-linear.*

PROOF. The claims follow from Lemmas 1.89, 1.98, and 5.65. If the projections and inclusions induced on $W = W_c \oplus W_a$ by $J_W$, $J_W'$ are $P_i'$, $Q_i'$, then the induced maps $\mathrm{Hom}(Q_c, Id_U) \circ H \circ Q_c' : W_c \to \mathrm{Hom}(V_c, U)$ and $P_c'' \circ H \circ Q_c' : W_c \to (\mathrm{Hom}(V, U))_c$ are related by composition with the c-linear invertible map from the above Example:

$$\mathrm{Hom}(Q_c, Id_U) \circ H \circ Q_c' = (\mathrm{Hom}(Q_c, Id_U) \circ Q_c'') \circ (P_c'' \circ H \circ Q_c').$$

∎

EXERCISE 5.86. The results of the previous Lemma have analogues for a map $\mathrm{Hom}(V, U) \to W$. ∎

EXAMPLE 5.87. For $U$ and commuting CSOs $J_1$, $J_2$ on $V$ as in Example 5.83, suppose there are also commuting CSOs $J_1'$, $J_2'$ on $V'$. If $B : V \to V'$ respects the direct sums $V_c \oplus V_a \to V_c' \oplus V_a'$ (equivalently, $B \circ J_1 \circ J_2 = J_1' \circ J_2' \circ B$ by Lemma 1.126 and Lemma 5.60), then for $i = c, a$, there are induced maps $P_i' \circ B \circ Q_i : V_i \to V_i'$. By Lemma 1.92, for any map $A : W \to U$, the map $\mathrm{Hom}(A, B) : \mathrm{Hom}(U, V) \to \mathrm{Hom}(W, V')$ respects the direct sums

$$\mathrm{Hom}(U, V_c) \oplus \mathrm{Hom}(U, V_a) \to \mathrm{Hom}(W, V_c') \oplus \mathrm{Hom}(W, V_a')$$

from Example 5.83, and for $i = c, a$, the induced map

$$\mathrm{Hom}(Id_W, P_i') \circ \mathrm{Hom}(A, B) \circ \mathrm{Hom}(Id_U, Q_i)$$

is equal to $\mathrm{Hom}(A, P_i' \circ B \circ Q_i)$.

EXAMPLE 5.88. For $U$, $V$, $V'$, $A$ as in Example 5.87, if $B : V \to V'$ is c-linear with respect to both pairs $J_1$, $J_1'$ and $J_2$, $J_2'$, then $B$ satisfies the hypothesis from Example 5.87, and by Lemma 5.30, $\mathrm{Hom}(A, B)$ is also c-linear with respect to the pairs $\mathrm{Hom}(Id_U, J_1)$, $\mathrm{Hom}(Id_W, J_1')$ and $\mathrm{Hom}(Id_U, J_2)$, $\mathrm{Hom}(Id_W, J_2')$. Lemma 5.65 applies, so the induced maps from Example 5.87, $P_c' \circ B \circ Q_c : V_c \to V_c'$, and also $\mathrm{Hom}(A, P_c' \circ B \circ Q_c)$, are both c-linear.

EXERCISE 5.89. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the map $t_{UV}^W : \mathrm{Hom}(U, V) \to \mathrm{Hom}(\mathrm{Hom}(V, W), \mathrm{Hom}(U, W))$ respects the direct sums and the induced map

$$\mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}_c(\mathrm{Hom}(\mathbf{V}, W), \mathrm{Hom}(\mathbf{U}, W))$$

is c-linear.

HINT. Exercise 5.33, Exercise 5.34, and then Lemma 5.65 apply. ∎

EXAMPLE 5.90. For $W = \mathbb{R}$ in the previous Exercise, $t_{UV}^{\mathbb{R}} = t_{UV} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(V^*, U^*)$. $V^*$ has a CSO $J_V^*$ as in Example 5.5, and similarly $J_U^*$ is a CSO for $U^*$. $t_{UV}$ respects the direct sums

$$\mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \oplus \mathrm{Hom}_a(U, V) \to \mathrm{Hom}_c(V^*, U^*) \oplus \mathrm{Hom}_a(V^*, U^*)$$

and the induced map $\mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}_c(V^*, U^*)$, $A \mapsto t_{UV}(A) = \mathrm{Hom}(A, Id_{\mathbb{R}}) = A^*$, is c-linear.

EXERCISE 5.91. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the map $e_{UV}^W : \mathrm{Hom}(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W) \otimes \mathbf{U}, W)$ respects the direct sums and the induced map

$$\mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \to \mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W) \otimes_c \mathbf{U}, W)$$

is c-linear.

HINT. Exercise 5.38, Exercise 5.39, and then Lemma 5.85, apply. ∎

EXERCISE 5.92. Given $U$ and $V$, with $V$ finite-dimensional and $\mathbf{V} = (V, J_V)$, the image of $\eta_{VU} : U \to V \otimes \mathrm{Hom}(V, U)$ from Notation 4.44 is contained in the subspace of the target where the two induced CSOs agree: for any $u \in U$,

$$\eta_{VU}(u) \in \mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{V}, U).$$

HINT. The claim follows from Lemma 4.47. ∎

EXERCISE 5.93. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$ and $\mathbf{W} = (W, J_W)$, the map $T_{U,V;W} : \mathrm{Hom}(\mathbf{U}, \mathrm{Hom}(V, \mathbf{W})) \to \mathrm{Hom}(V, \mathrm{Hom}(\mathbf{U}, \mathbf{W}))$ respects the direct sums and the induced map

$$\mathrm{Hom}_c(\mathbf{U}, \mathrm{Hom}(V, \mathbf{W})) \to \mathrm{Hom}(V, \mathrm{Hom}_c(\mathbf{U}, \mathbf{W}))$$

is c-linear and invertible, where the inverse is induced by $T_{V,U;W}$.

HINT. The direct sum for the domain is as in Example 5.78, and for the target is as in Example 5.83. Exercise 5.43, Exercise 5.44, and then Lemma 5.65 apply. ∎

EXERCISE 5.94. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the map $T_{U,V;W} : \mathrm{Hom}(\mathbf{U}, \mathrm{Hom}(\mathbf{V}, W)) \to \mathrm{Hom}(\mathbf{V}, \mathrm{Hom}(\mathbf{U}, W))$ respects the direct sums and the induced map

$$\mathrm{Hom}_c(\mathbf{U}, \mathrm{Hom}(\mathbf{V}, W)) \to \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{U}, W))$$

is c-linear and invertible, where the inverse is induced by $T_{V,U;W}$.

HINT. The direct sums are as in Example 5.78. Exercise 5.44 and Lemma 5.65 apply. ∎

EXERCISE 5.95. Given $W$ and $\mathbf{V} = (V, J_V)$, the involution $T_{V;W}$ from Notation 4.3 is c-linear with respect to both induced CSOs as follows:

$$\begin{aligned}\mathrm{Hom}(\mathbf{V}, \mathrm{Hom}(V, W)) &\to \mathrm{Hom}(V, \mathrm{Hom}(\mathbf{V}, W))\\ \mathrm{Hom}(V, \mathrm{Hom}(\mathbf{V}, W)) &\to \mathrm{Hom}(\mathbf{V}, \mathrm{Hom}(V, W)).\end{aligned}$$

The induced map

$$\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W)) \to \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))$$

is a c-linear involution, and its fixed point subspace has a canonical CSO.

HINT. The c-linearity claims are the $U = V$ special case of Exercise 5.44, and the induced map is the special case from Exercise 5.94. Lemma 5.72 applies to the involution $T_{V;W}$ and the commuting CSOs on $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$. ∎

EXAMPLE 5.96. For $W$, $\mathbf{V} = (V, J_V)$, and $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ as in Exercise 5.95, Example 5.70 applies to the commuting involutions $T_{V;W}$ and

$$-\mathrm{Hom}(J_V, Id_{\mathrm{Hom}(V,W)}) \circ \mathrm{Hom}(Id_V, \mathrm{Hom}(J_V, Id_W)) = -\mathrm{Hom}(J_V, \mathrm{Hom}(J_V, Id_W)).$$

Recalling the direct sums from (4.6) and (5.11),

$$\begin{aligned}\mathrm{Hom}(V, \mathrm{Hom}(V, W)) &= \mathit{Sym}(V; W) \oplus \mathit{Alt}(V; W),\\ \mathrm{Hom}(V, \mathrm{Hom}(V, W)) &= \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W)) \oplus \mathrm{Hom}_a(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W)),\end{aligned}$$

the four distinguished subspaces from Example 5.70 can be described in terms of properties of their elements, $W$-valued bilinear forms $h$, and are denoted as follows:

$$\begin{aligned}\mathit{Sym}_c(\mathbf{V}, W) &= \mathit{Sym}(V; W) \cap \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))\\ &= \{h : (h(v_1))(v_2) = (h(v_2))(v_1) = -(h(J_V(v_1)))(J_V(v_2))\},\\ \mathit{Sym}_a(\mathbf{V}, W) &= \mathit{Sym}(V; W) \cap \mathrm{Hom}_a(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))\\ &= \{h : (h(v_1))(v_2) = (h(v_2))(v_1) = (h(J_V(v_1)))(J_V(v_2))\},\\ \mathit{Alt}_c(\mathbf{V}, W) &= \mathit{Alt}(V; W) \cap \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))\\ &= \{h : (h(v_1))(v_2) = -(h(v_2))(v_1) = -(h(J_V(v_1)))(J_V(v_2))\},\\ \mathit{Alt}_a(\mathbf{V}, W) &= \mathit{Alt}(V; W) \cap \mathrm{Hom}_a(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))\\ &= \{h : (h(v_1))(v_2) = -(h(v_2))(v_1) = (h(J_V(v_1)))(J_V(v_2))\}.\end{aligned}$$

The c-linear involution on $\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))$ induced by $T_{V;W}$ from Exercise 5.95 produces the direct sum

$$\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W)) = \mathit{Sym}_c(\mathbf{V}, W) \oplus \mathit{Alt}_c(\mathbf{V}, W),$$

and there is a canonical CSO on $\mathit{Sym}_c(\mathbf{V}, W)$.

EXERCISE 5.97. Given $V$ with CSOs $J_1$, $J_2$, if $J_1$ and $J_2$ commute and $J_2 \neq \pm J_1$, then the span of $\{Id_V, J_1, J_2, J_1 \circ J_2\}$ is a 4-dimensional subspace of $\mathrm{End}(V)$. This subspace is closed under composition, so it is a subrng of $\mathrm{End}(V)$.

HINT. The first claim follows from Exercise 5.22. Such a subspace is a commutative ring, and isomorphic to the associative algebra of bicomplex numbers of C. Segre. ∎

EXERCISE 5.98. Given $U$, $\mathbf{V} = (V, J_V)$, $\mathbf{W} = (W, J_W)$, let $P_c$ denote the projection $V \otimes W \twoheadrightarrow \mathbf{V} \otimes_c \mathbf{W}$ as in Example 5.74. For a map $B : U \to V$, the map $[B \otimes Id_W] : U \otimes W \to V \otimes W$ is c-linear with respect to $[Id_U \otimes J_W]$ and $[Id_V \otimes J_W]$, and $P_c$ is c-linear as in Example 5.61, so the composite $P_c \circ [B \otimes Id_W] : U \otimes \mathbf{W} \to \mathbf{V} \otimes_c \mathbf{W}$ is c-linear. If the image subspace $B(U)$ contains a $J_V$-complex line in $\mathbf{V}$, spanned by $B(u_1)$ and $J_V(B(u_1)) = B(u_2) \neq 0_V$, then for any non-zero $w \in W$,

$$v = u_1 \otimes w + u_2 \otimes (J_W(w))$$

and

$$[Id_U \otimes J_W](v) = u_1 \otimes (J_W(w)) - u_2 \otimes w$$

span a $[Id_U \otimes J_W]$-complex line in $\ker(P_c \circ [B \otimes Id_W]) \subseteq U \otimes \mathbf{W}$.

HINT. The property that $v \neq 0_{U \otimes W}$ follows from Lemma 5.11 and Claim 6.36. ∎

EXERCISE 5.99. Given $V = V_1 \oplus V_2$, suppose there is an invertible map $A : V_2 \to V_1$ and there is a CSO $J_1$ on $V_1$. The composite $A^{-1} \circ J_1 \circ A$ is a CSO on $V_2$ as in Lemma 5.46, and $V$ admits six CSOs: $\pm J_V$, $\pm J$, $\pm J'$,

$$J_V = Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2, \tag{5.14}$$

$$J = Q_1 \circ J_1 \circ P_1 + Q_2 \circ A^{-1} \circ J_1 \circ A \circ P_2, \tag{5.15}$$

$$J' = Q_1 \circ J_1 \circ P_1 - Q_2 \circ A^{-1} \circ J_1 \circ A \circ P_2.$$

$J_V$ is as in Example 5.8 and does not depend on $J_1$. $J$ and $J'$ are as in Example 5.69. The CSO $J$ commutes with both $J_V$ and $J'$; the direct sum produced by the involution $-J \circ J'$ is equivalent to the given direct sum $V = V_1 \oplus V_2$ as in Example 5.69, but in general is not equivalent to the direct sum produced by $-J \circ J_V$, which is denoted here by $V = V_c \oplus V_a$, with projections $(P_c, P_a)$ as in Example 5.59. The CSOs $J_V$ and $J'$ anticommute, so the involutions $-J \circ J'$ and $-J \circ J_V$ also anticommute. Theorem 1.137 applies: for any $\beta \neq 0$, the map

$$\beta \cdot P_c \circ Q_1 : V_1 \to V_c \tag{5.16}$$

is invertible, with inverse $\frac{2}{\beta} \cdot P_1 \circ Q_c : V_c \to V_1$. The map (5.16) is also c-linear with respect to $J_1$ and the induced CSO on $V_c$, $P_c \circ J \circ Q_c = P_c \circ J_V \circ Q_c$. Similarly by Theorem 1.137, $\beta \cdot P_a \circ Q_1 : V_1 \to V_a$ is invertible, and it is c-linear with respect to $J_1$ and the CSO

$$P_a \circ J \circ Q_a = -P_a \circ J_V \circ Q_a$$

from Example 5.61 (so it is a-linear with respect to $P_a \circ J_V \circ Q_a$). ∎

EXERCISE 5.100. For $V = V_1 \oplus V_2$, $A : V_2 \to V_1$, and CSOs $J_1$, $J_V$, $J$ as in Exercise 5.99, let $U = U_1 \oplus U_2$ be another direct sum with projections and inclusions $(P'_1, P'_2)$, $(Q'_1, Q'_2)$, an invertible map $A' : U_2 \to U_1$, and a CSO

$$J_U = Q'_2 \circ (A')^{-1} \circ P'_1 - Q'_1 \circ A' \circ P'_2 \tag{5.17}$$

as in Example 5.8 and (5.14). Note that unlike $V_1$, no complex structure on $U_1$ is assumed, so the situation with $J_U$ and $J_V$ is as in Lemma 5.48. Let $B : U_1 \to V_1$ be any map, and recall from Example 5.50 that there is a c-linear extension $B_c : (U, J_U) \to (V, J_V)$,

$$B_c = Q_1 \circ B \circ P'_1 + Q_2 \circ A^{-1} \circ B \circ A' \circ P'_2. \tag{5.18}$$

Suppose, as in Exercise 5.98, that the image subspace $B(U_1)$ contains a $J_1$-complex line in $V_1$, spanned by $B(u_1)$ and $J_1(B(u_1)) = B(u_2) \neq 0_{V_1}$. Let

$$v = Q_1'(u_1) + Q_2'((A')^{-1}(u_2)) \in U,$$

and

$$J_U(v) = Q_2'((A')^{-1}(u_1)) - Q_1'(u_2) \in U.$$

These two elements span a $J_U$-complex line in $\ker(P_c \circ B_c) \subseteq U$. Conversely, if $v$ is a non-zero element of $\ker(P_c \circ B_c)$, then $u_1 = P_1'(v)$ and $u_2 = A'(P_2'(v))$ are not both zero, and are related by $J_1(B(u_1)) = B(u_2)$.

HINT. The vector $v$ is non-zero ($P_1'(v) = u_1 \neq 0_{U_1}$).

$$\begin{aligned}
B_c(v) &= (Q_1 \circ B \circ P_1' + Q_2 \circ A^{-1} \circ B \circ A' \circ P_2')(Q_1'(u_1) + Q_2'((A')^{-1}(u_2))) \\
&= Q_1(B(u_1)) + Q_2(A^{-1}(B(u_2))) = ((Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B)(u_1), \\
P_c(B_c(v)) &= (P_c \circ Q_c \circ P_c)(B_c(v)) \\
&= (P_c \circ (\frac{1}{2} \cdot (Id_V - J_V \circ J)) \circ (Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B)(u_1)).
\end{aligned}$$

Using $J \circ Q_1 = Q_1 \circ J_1$ and $J \circ Q_2 = Q_2 \circ A^{-1} \circ J_1 \circ A$,

$$\begin{aligned}
& (Id_V - J_V \circ J) \circ (Q_1 + Q_2 \circ A^{-1} \circ J_1) \\
= \;& Q_1 + Q_2 \circ A^{-1} \circ J_1 - J_V \circ Q_1 \circ J_1 + J_V \circ Q_2 \circ A^{-1} \\
= \;& Q_1 + Q_2 \circ A^{-1} \circ J_1 - (Q_2 \circ A^{-1} \circ P_1 - Q_1 \circ A \circ P_2) \circ (Q_1 \circ J_1 - Q_2 \circ A^{-1}) \\
= \;& 0_{\mathrm{Hom}(V_1, V)}.
\end{aligned}$$

The other vector, $J_U(v)$, is also in the kernel, by the c-linearity of $B_c : U \to V$ and $P_c : V \to V_c$.

For the converse, if $v = (Q_1' \circ P_1' + Q_2' \circ P_2')(v) \neq 0_U$ then $P_1'(v)$ and $P_2'(v)$ are not both zero, and $P_c(B_c(v)) = 0_{V_c}$ is equivalent to $J_V(B_c(v)) = -J(B_c(v))$. We want to show that $J_1(B(P_1'(v))) = B(A'(P_2'(v)))$. Expanding using (5.14), (5.15), and (5.18) gives:

$$\begin{aligned}
J_V(B_c(v)) &= Q_2(A^{-1}(B(P_1'(v)))) - Q_1(B(A'(P_2'(v)))), \\
J(B_c(v)) &= Q_1(J_1(B(P_1'(v)))) + Q_2(A^{-1}(J_1(B(A'(P_2'(v)))))),
\end{aligned} \tag{5.19}$$

and then using the equality $J_V(B_c(v)) = -J(B_c(v))$, the claim follows from applying $P_1$ to both sides. ∎

EXERCISE 5.101. For $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, $A$, $A'$, $J_1$, $J_U$, $B$, and $B_c$ as in Exercise 5.100, suppose $(u_1, \ldots, u_{2\ell})$ is a list of elements of $U_1$ such that for each $\iota = 1, \ldots, \ell$, $J_1(B(u_{2\iota-1})) = B(u_{2\iota})$. For each $\iota$, denote an element $v_\iota \in U$ by:

$$v_\iota = Q_1'(u_{2\iota-1}) + Q_2'((A')^{-1}(u_{2\iota})).$$

If $B$ is one-to-one then the following are equivalent.

(1) $(v_1, J_U(v_1), \ldots, v_\ell, J_U(v_\ell))$ is an independent list in $U$.
(2) $(u_1, \ldots, u_{2\ell})$ is an independent list in $U_1$.

HINT. For (1) $\implies$ (2), by Exercise 5.51, $B_c$ is also one-to-one, so by Exercise 0.49,

$$(B_c(v_1), B_c(J_U(v_1)), \ldots, B_c(v_\ell), B_c(J_U(v_\ell)))$$

is an independent list in $V$. For each $\iota$,

$$\begin{aligned} B_c(v_\iota) &= Q_1(B(u_{2\iota-1})) + Q_2(A^{-1}(B(u_{2\iota}))) \\ &= ((Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B)(u_{2\iota-1}), \\ B_c(J_U(v_\iota)) &= Q_2(A^{-1}(B(u_{2\iota-1}))) - Q_1(B(u_{2\iota})) \\ &= ((Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B)(-u_{2\iota}), \end{aligned}$$

so $(Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B$ transforms $(u_1, -u_2, \ldots, u_{2\ell-1}, -u_{2\ell})$ to an independent list, which implies (2). Conversely, $P_1$ is a left inverse of $Q_1 + Q_2 \circ A^{-1} \circ J_1$, so $(Q_1 + Q_2 \circ A^{-1} \circ J_1) \circ B$ is one-to-one and transforms the independent list $(u_1, -u_2, \ldots, u_{2\ell-1}, -u_{2\ell})$ to an independent list. So $(v_1, J_U(v_1), \ldots, v_\ell, J_U(v_\ell))$ is transformed by $B_c$ into an independent list, which implies (1). ∎

REMARK 5.102. The constructions of Exercise 5.98 and Exercise 5.99 are two different generalizations of the notion of "complexification" of a complex vector space. In particular, the $V_c \oplus V_a$ direct sum and complex linear isomorphism $V_1 \to V_c$ from line (5.16) are analogous to the well-known construction often denoted by $V^{1,0} \oplus V^{0,1}$ in complex geometry. The idea from Exercise 5.98 and Exercise 5.100 that a real linear map $B$ into a complex vector space can be "complexified" to a c-linear map $B_c$, and then composed with a c-linear projection, so that the kernel detects whether the image of $B$ contains a complex line (or a $J$-invariant subspace, as in Exercise 5.101), has been used in geometry, for example, [W].

EXERCISE 5.103. For $V = V_1 \oplus V_2$, $U = U_1 \oplus U_2$, $A$, $A'$, $J_1$, $J_U$, $B$, and $B_c$ as in Exercise 5.100, suppose that additionally there is a CSO $\tilde{J}_1$ on $U_1$, and in analogy with (5.15), there is the CSO on $U$,

$$\tilde{J} = Q_1' \circ \tilde{J}_1 \circ P_1' + Q_2' \circ (A')^{-1} \circ \tilde{J}_1 \circ A' \circ P_2'. \tag{5.20}$$

If there is an element $u_1 \in U_1$ so that

$$B(\tilde{J}_1(u_1)) = J_1(B(u_1)) \in V_1, \tag{5.21}$$

then $B$ maps the $\tilde{J}_1$-invariant subspace spanned by $\{u_1, \tilde{J}_1(u_1)\}$ in $U_1$ to a $J_1$-invariant subspace in $V_1$, spanned by $\{B(u_1), J_1(B(u_1))\}$, as in Lemma 5.24. If $B(u_1) \neq 0_{V_1}$, then the result of Exercise 5.100 applies, with $u_2 = \tilde{J}_1(u_1)$: these two elements of $U$:

$$\begin{aligned} v &= Q_1'(u_1) + Q_2'((A')^{-1}(\tilde{J}_1(u_1))), \\ J_U(v) &= Q_2'((A')^{-1}(u_1)) - Q_1'(\tilde{J}_1(u_1)) \end{aligned}$$

span a $J_U$-complex line in $\ker(P_c \circ B_c) \subseteq U$. The direct sum produced by $-\tilde{J} \circ J_U$ is $U_c \oplus U_a$, with projections $(P_c', P_a')$, and $Q_a' \circ P_a' = \frac{1}{2} \cdot (Id_U + \tilde{J} \circ J_U)$ as in Example 5.59. A straightforward computation shows that $v = 2 \cdot (Q_a' \circ P_a')(Q_1'(u_1))$ and $J_U(v) = -2 \cdot (Q_a' \circ P_a')(Q_1'(\tilde{J}_1(u_1)))$. So, the $J_U$-complex line spanned by $\{v, J_U(v)\}$ is contained in $\ker(P_c \circ B_c \circ Q_a') \subseteq U_a$, and it is the image under the invertible map $P_a' \circ Q_1'$ of the $\tilde{J}_1$-complex line spanned by $\{u_1, \tilde{J}_1(u_1)\}$ in $U_1$.

The composite $P_c \circ B_c \circ Q_a' : U_a \to V_c$ is c-linear with respect to the canonically induced CSO on $V_c$, and the CSO $P_a' \circ J_U \circ Q_a' = -P_a' \circ \tilde{J} \circ Q_a'$ on $U_a$ as in Example 5.61. As in Exercise 5.99, $P_a' \circ Q_1' : U_1 \to U_a$ is c-linear with respect to $\tilde{J}_1$ and $P_a' \circ \tilde{J} \circ Q_a' = -P_a' \circ J_U \circ Q_a'$.

Conversely, if $v \in U_a$ is a non-zero element of $\ker(P_c \circ B_c \circ Q_a')$, then $u_1 = P_1'(Q_a'(v)) \in U_1$ is non-zero and satisfies (5.21).

HINT. To sketch a proof of the last claim, $P_1'\circ Q_a'$ is invertible by Theorem 1.137. We want to show $B(\tilde{J}_1(P_1'(Q_a'(v)))) = J_1(B(P_1'(Q_a'(v))))$. By the construction of $U = U_c \oplus U_a$, $Q_a'(v)$ satisfies $J_U(Q_a'(v)) = -\tilde{J}(Q_a'(v))$. Expanding the formulas from (5.17) and (5.20) gives:

$$\begin{aligned} J_U(Q_a'(v)) &= Q_2'((A')^{-1}(P_1'(Q_a'(v)))) - Q_1'(A'(P_2'(Q_a'(v)))), \\ \tilde{J}(Q_a'(v)) &= Q_1'(\tilde{J}_1(P_1'(Q_a'(v)))) + Q_2'((A')^{-1}(\tilde{J}_1(A'(P_2'(Q_a'(v)))))), \end{aligned}$$

Setting these to be opposite and then applying $P_1'$ to both sides gives the equation

$$A'(P_2'(Q_a'(v))) = \tilde{J}_1(P_1'(Q_a'(v))). \tag{5.22}$$

The hypothesis $(P_c \circ B_c \circ Q_a')(v) = 0_{V_c}$ is equivalent to

$$J_V(B_c(Q_a'(v))) = -J(B_c(Q_a'(v))).$$

Expanding as in (5.19),

$$\begin{aligned} J_V(B_c(Q_a'(v))) &= Q_2(A^{-1}(B(P_1'(Q_a'(v))))) - Q_1(B(A'(P_2'(Q_a'(v))))), \\ J(B_c(Q_a'(v))) &= Q_1(J_1(B(P_1'(Q_a'(v))))) + Q_2(A^{-1}(J_1(B(A'(P_2'(Q_a'(v))))))), \end{aligned}$$

setting these to be opposite and then applying $P_1$ to both sides gives

$$J_1(B(P_1'(Q_a'(v)))) = B(A'(P_2'(Q_a'(v)))),$$

so the claim follows from using (5.22). ∎

EXERCISE 5.104. For $V = V_1 \oplus V_2$ and $A$ as in Exercise 5.99, if $J_1$ and $\tilde{J}_1$ are two (not necessarily commuting) CSOs on the same vector space $V_1$, then Exercise 5.103 applies in the special case $B = Id_{V_1}$, $B_c = Id_V$. $J_1$ and $\tilde{J}_1$ define different complexifications, $V = V_c \oplus V_a$ produced by $-J \circ J_V$ as in Exercise 5.99, and $V = U_c \oplus U_a$ produced by $-\tilde{J} \circ J_V$ as in Exercise 5.103, respectively. If $u_1$ is a non-zero element of $V_1$ where $J_1(u_1) = \tilde{J}_1(u_1)$ (as in Example 5.25), then the $J_1$-complex line spanned by $\{u_1, J_1(u_1) = \tilde{J}_1(u_1)\}$ is mapped by $P_a' \circ Q_1$ to a $J_V$-complex line in the kernel of the map $P_c \circ Q_a' : U_a \to V_c$. ∎

REMARK 5.105. As in Remark 5.102, describing the subspace of vectors where a real linear map happens to respect the CSOs, in terms of the kernel of the composite of a complexified map and a projection, has been used in geometry. The special case from Example 5.25 and Exercise 5.104, describing a set on which two CSOs coincide, is considered by [HL]. The connection between the property in Exercise 5.100 (the image of $B$ contains a $J_1$-invariant subspace) and the property in Exercise 5.103 ($B$ is c-linear on a subspace) is considered in Exercise 5.54.

### 5.3.2. Three Commuting Complex Structure Operators.

EXAMPLE 5.106. Given $V$ and three commuting CSOs $J_1$, $J_2$, $J_3$, consider an ordered triple $(i_1, i_2, i_3)$ which is a permutation (no repeats) of the indices 1, 2, 3. For the first two indices, the two CSOs $J_{i_1}$, $J_{i_2}$ produce a direct sum $V = V_{c(i_1 i_2)} \oplus V_{a(i_1 i_2)}$ with projection $P_{c(i_1 i_2)} = \frac{1}{2} \cdot (Id_V - J_{i_1} \circ J_{i_2})$ as in Example 5.59. The ordering of the pair is irrelevant: $V_{c(i_1 i_2)} = V_{c(i_2 i_1)}$. The remaining CSO $J_{i_3}$ respects this direct sum by Lemma 5.60, and by Lemma 5.47 induces a CSO $P_{c(i_1 i_2)} \circ J_{i_3} \circ Q_{c(i_1 i_2)}$ on $V_{c(i_1 i_2)}$, which commutes with the CSO induced by $J_{i_1}$ and $J_{i_2}$. So, again as in Example 5.59, there is a direct sum $V_{c(i_1 i_2)} =$

$(V_{c(i_1i_2)})_c \oplus (V_{c(i_1i_2)})_a$ with projection $P_{c((i_1i_2)i_3)} : V_{c(i_1i_2)} \twoheadrightarrow (V_{c(i_1i_2)})_c$. Simplifying the composition of projections gives the formula

$$P_{c((i_1i_2)i_3)} \circ P_{c(i_1i_2)} = \frac{1}{4} \cdot (Id_V - J_1 \circ J_2 - J_2 \circ J_3 - J_1 \circ J_3).$$

Considering $(V_{c(i_1i_2)})_c$ as a subspace of $V$, the above formula shows that neither the composite map nor its image depends on the ordering of the three indices, and so $(V_{c(i_1i_2)})_c$ is equal to the subspace where all three CSOs coincide and it can be denoted $V_{c(123)}$. The composite of inclusions $Q_{c(i_1i_2)} \circ Q_{c((i_1i_2)i_3)}$ also does not depend on the ordering. The canonical CSO on $V_{c(123)}$ is:

$$P_{c((i_1i_2)i_3)} \circ P_{c(i_1i_2)} \circ J_i \circ Q_{c(i_1i_2)} \circ Q_{c((i_1i_2)i_3)}, \tag{5.23}$$

for any $i = 1$, 2, or 3.

Example 5.106 is also a special case of both Theorem 1.130 and Example 5.70: given three commuting CSOs, there are commuting involutions on $V$,

$$K_1 = -J_2 \circ J_3, \quad K_2 = -J_1 \circ J_3, \quad K_1 \circ K_2 = -J_1 \circ J_2. \tag{5.24}$$

The direct sums from Theorem 1.130 produced by these involutions are exactly $V = V_{c(i_1i_2)} \oplus V_{a(i_1i_2)}$, and each $V_{c(i_1i_2)}$ admits a canonically induced involution and direct sum. The conclusions of Theorem 1.130 are $V_{c(12)} \cap V_{c(23)} \cap V_{c(13)} = V_{c(123)}$ and $(V_{c(i_1i_2)})_a = V_{a(i_1i_3)} \cap V_{a(i_2i_3)}$. From (1.18), each projection $P_{c(i_1i_2)i_3}$ is equal to a map induced by $P_{c(i_1i_3)}$ and also to a map induced by $P_{c(i_2i_3)}$.

LEMMA 5.107. *For $V$ with three commuting CSOs $J_1$, $J_2$, $J_3$, $\mathbf{U} = (U, J_U)$, and a map $H : V \to U$, if $H$ is c-linear with respect to $(J_1, J_U)$, then $H \circ Q_{c(12)} : V_{c(12)} \to \mathbf{U}$ is c-linear with respect to $(P_{c(12)} \circ J_1 \circ Q_{c(12)}, J_U)$ and $H \circ Q_{c(13)} : V_{c(13)} \to \mathbf{U}$ is c-linear with respect to $(P_{c(13)} \circ J_1 \circ Q_{c(13)}, J_U)$. For any ordering $(i_1, i_2, i_3)$, $H \circ Q_{c(i_1i_2)} \circ Q_{c((i_1i_2)i_3)} : V_{c(123)} \to \mathbf{U}$ is c-linear.*

PROOF. The first claim follows from Lemma 5.63, for the induced CSO $P_{c(12)} \circ J_1 \circ Q_{c(12)} = P_{c(12)} \circ J_2 \circ Q_{c(12)}$ on $V_{c(12)}$. The second claim similarly follows from Lemma 5.63, and there are analogous claims if $H$ is instead assumed to be c-linear with respect to either $(J_2, J_U)$ or $(J_3, J_U)$. Lemma 5.63 then applies to $H \circ Q_{c(12)}$, and the two CSOs on $V_{c(12)}$ from Example 5.106, $P_{c(12)} \circ J_1 \circ Q_{c(12)}$ and $P_{c(12)} \circ J_3 \circ Q_{c(12)}$, so $(H \circ Q_{c(12)}) \circ Q_{c((12)3)}$ is c-linear with respect to $(P_{c((12)3)} \circ (P_{c(12)} \circ J_1 \circ Q_{c(12)}) \circ Q_{c((12)3)}, J_U)$. The last claim follows from the composites not depending on the ordering. ∎

LEMMA 5.108. *Given $V$ with commuting CSOs $J_V^1$, $J_V^2$, $J_V^3$, and $U$ with commuting CSOs $J_U^1$, $J_U^2$, $J_U^3$, if $H : U \to V$ satisfies $H \circ J_U^1 = J_V^1 \circ H$ and $H \circ J_U^2 = J_V^2 \circ H$ and $H \circ J_U^3 = J_V^3 \circ H$, then $H$ respects the corresponding direct sums from Example 5.106, and each induced map $P_{c(i_1i_2)}^V \circ H \circ Q_{c(i_1i_2)}^U$ is c-linear with respect to the CSOs induced by $J_U^{i_1}$, $J_V^{i_1}$ and also c-linear with respect to the CSOs induced by $J_U^{i_3}$, $J_V^{i_3}$. The induced map*

$$P_{c((i_1i_2)i_3)}^V \circ P_{c(i_1i_2)}^V \circ H \circ Q_{c(i_1i_2)}^U \circ Q_{c((i_1i_2)i_3)}^U : U_{c(123)} \to V_{c(123)}$$

*is c-linear and does not depend on the ordering $(i_1, i_2, i_3)$. If, also, $H$ is invertible, then the induced maps are invertible.*

PROOF. Theorem 1.131 applies, with commuting involutions on both $V$ and $U$ as in Example 5.106. In particular, $H$ respects the direct sums

$$U_{c(i_1i_2)} \oplus U_{a(i_1i_2)} \to V_{c(i_1i_2)} \oplus V_{a(i_1i_2)}.$$

The induced map $P^V_{c(i_1i_2)} \circ H \circ Q^U_{c(i_1i_2)}$ respects the direct sum

$$U_{c((i_1i_2)i_3)} \oplus (U_{c(i_1i_2)})_a \to V_{c((i_1i_2)i_3)} \oplus (V_{c(i_1i_2)})_a,$$

where $U_{c((i_1i_2)i_3)} = U_{c(123)}$ and $(U_{c(i_1i_2)})_a = U_{a(i_1i_3)} \cap U_{a(i_2i_3)}$ as in Example 5.106. The first claim of c-linearity follows from Lemma 5.65. The second c-linearity requires checking

$$P^V_{c(i_1i_2)} \circ H \circ Q^U_{c(i_1i_2)} \circ P^U_{c(i_1i_2)} \circ J_U^{i_3} \circ Q^U_{c(i_1i_2)} = P^V_{c(i_1i_2)} \circ J_V^{i_3} \circ Q^V_{c(i_1i_2)} \circ P^V_{c(i_1i_2)} \circ H \circ Q^U_{c(i_1i_2)},$$

and then the last claims follow from Lemma 5.65 applied again to the commuting CSOs induced on $U_{c(i_1i_2)}$, $V_{c(i_1i_2)}$. ∎

EXAMPLE 5.109. Given $U$ with commuting CSOs $J_1$, $J_2$, $J_3$, suppose $H$ is an involution on $U$ such that $H \circ J_1 = J_2 \circ H$ and $H \circ J_3 = J_3 \circ H$. By Lemma 5.72, $H$ respects the direct sum $U_{c(12)} \oplus U_{a(12)} \to U_{c(12)} \oplus U_{a(12)}$, and induces involutions on $U_{c(12)}$ and $U_{a(12)}$. Lemma 5.108 applies to the triple $(J_1, J_2, J_3)$ on the domain of $H$ and $(J_2, J_1, J_3)$ on the target. So, $H$ respects the direct sums:

$$\begin{aligned} U_{c(13)} \oplus U_{a(13)} &\to U_{c(23)} \oplus U_{a(23)} \\ U_{c(23)} \oplus U_{a(23)} &\to U_{c(13)} \oplus U_{a(13)}, \end{aligned} \tag{5.25}$$

and the induced maps $U_{c(13)} \to U_{c(23)} \to U_{c(13)}$ are c-linear and mutually inverses. The induced maps $U_{a(13)} \to U_{a(23)} \to U_{a(13)}$ are also mutually inverses. The subspace $U_{c(12)}$ admits commuting CSOs $P_{c(12)} \circ J_1 \circ Q_{c(12)} = P_{c(12)} \circ J_2 \circ Q_{c(12)}$ and $P_{c(12)} \circ J_3 \circ Q_{c(12)}$ as in Example 5.106, producing the direct sum $U_{c(123)} \oplus (U_{c(12)})_a$. The induced involution $P_{c(12)} \circ H \circ Q_{c(12)}$ on $U_{c(12)}$ commutes with both of these CSOs, respects this direct sum, and induces a c-linear involution on $U_{c(123)}$ by Lemma 5.71. The involutions on $U$ from (5.24) satisfy

$$(-J_2 \circ J_3) \circ H = H \circ (-J_1 \circ J_3), \tag{5.26}$$

so we have a special case of Example 1.132. Adapting the notation from Example 1.132, the involutions $H$ and $H \circ (-J_1 \circ J_2)$ produce direct sums $U = U_7 \oplus U_8$ and $U_9 \oplus U_{10}$. The involution $P_{c(12)} \circ H \circ Q_{c(12)}$ on $U_{c(12)}$ commutes with $P_{c(12)} \circ (-J_1 \circ J_3) \circ Q_{c(12)} = P_{c(12)} \circ (-J_2 \circ J_3) \circ Q_{c(12)}$, and their product $P_{c(12)} \circ (-J_1 \circ J_3 \circ H) \circ Q_{c(12)}$ is an involution with fixed point subspace $U_{11}$. The following commutative diagram is adapted from Example 1.132.

$$\begin{tikzcd}
 & & U \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
U_{c(23)} \arrow[dr] & U_{c(13)} \arrow[d] & U_{c(12)} \arrow[dl] \arrow[d] \arrow[dr] & U_7 \arrow[d] & U_9 \arrow[dl] \\
 & U_{c(123)} \arrow[dr] & U_{11} \arrow[d] & U_{c(12)} \cap U_7 \cap U_9 \arrow[dl] & \\
 & & U_{c(123)} \cap U_7 & &
\end{tikzcd}$$

The subspace $U_{c(123)} \cap U_7$ has a canonical CSO.

EXERCISE 5.110. For $J_1$, $J_2$, $J_3$, and $H$ as in Example 5.109, the sixteen operators

$$\{\pm Id_U, \pm H, \pm J_1 \circ J_2, \pm J_1 \circ J_3, \pm J_2 \circ J_3, \pm H \circ J_1 \circ J_2, \pm H \circ J_1 \circ J_3, \pm H \circ J_2 \circ J_3\}$$

form a group which is the image of a representation of $D_4 \times \mathbb{Z}_2$. Unlike the group from Exercise 1.139, there is no pair of anticommuting elements. ∎

EXAMPLE 5.111. For $\mathbf{U} = (U, J_U)$, $\mathbf{V} = (V, J_V)$, and $\mathbf{W} = (W, J_W)$, the space $U \otimes V \otimes W$ admits three commuting CSOs $[J_U \otimes Id_{V \otimes W}]$, $\dots$, $[Id_{U \otimes V} \otimes J_W]$. The subspaces $(\mathbf{U} \otimes_c \mathbf{V}) \otimes_c \mathbf{W}$ and $\mathbf{U} \otimes_c (\mathbf{V} \otimes_c \mathbf{W})$ are equal, as a special case of Example 5.106.

EXAMPLE 5.112. For $\mathbf{U} = (U, J_U)$ and commuting CSOs $J_1$, $J_2$ on $V$, $U \otimes V$ has three commuting CSOs: $[J_U \otimes Id_V]$, $[Id_U \otimes J_1]$, $[Id_U \otimes J_2]$. This is another special case of Example 5.106. The projection $P_{c(23)} : U \otimes V \twoheadrightarrow (U \otimes V)_{c(23)} = U \otimes V_c$ is as in Example 5.73, so $P_{c(23)} = [Id_U \otimes P_c]$, where $P_c : V \twoheadrightarrow V_c$ is as in Example 5.59. The subspace where all three CSOs agree is $(U \otimes V)_{c(123)} = \mathbf{U} \otimes_c V_c$.

EXAMPLE 5.113. Given $U$ and $V$, suppose $\mathbf{V} = (V, J_V)$ and there are commuting CSOs $J_U$, $J'_U$ on $U$. Then $J_1 = \operatorname{Hom}(J_U, Id_V)$, $J_2 = \operatorname{Hom}(J'_U, Id_V)$, $J_3 = \operatorname{Hom}(Id_U, J_V)$ are three commuting CSOs on $\operatorname{Hom}(U, V)$, and this is a special case of Example 5.106. There are three direct sums:

$$\operatorname{Hom}(U, V) = (\operatorname{Hom}(U, V))_{c(13)} \oplus (\operatorname{Hom}(U, V))_{a(13)},$$

where $(\operatorname{Hom}(U, V))_{c(13)} = \operatorname{Hom}_c((U, J_U), \mathbf{V})$ as in Example 5.78,

$$\operatorname{Hom}(U, V) = (\operatorname{Hom}(U, V))_{c(23)} \oplus (\operatorname{Hom}(U, V))_{a(23)},$$

where $(\operatorname{Hom}(U, V))_{c(23)} = \operatorname{Hom}_c((U, J'_U), \mathbf{V})$, and

$$\operatorname{Hom}(U, V) = (\operatorname{Hom}(U, V))_{c(12)} \oplus (\operatorname{Hom}(U, V))_{a(12)},$$

as in Example 5.84. Each $(\operatorname{Hom}(U, V))_{c(i_1 i_2)}$ admits a direct sum with projection onto

$$((\operatorname{Hom}(U, V))_{c(i_1 i_2)})_c = (\operatorname{Hom}(U, V))_{c(123)} = \{A : U \to V : A \circ J_U = A \circ J'_U = J_V \circ A\}.$$

The invertible map

$$P''_c \circ \operatorname{Hom}(P_c, Id_U) : \operatorname{Hom}(U_c, V) \to (\operatorname{Hom}(U, V))_{c(12)}$$

from Example 5.84 (where in this case, $P''_c = P_{c(12)}$) is c-linear with respect to $\operatorname{Hom}(Id_{U_c}, J_V)$ and $P_{c(12)} \circ J_3 \circ Q_{c(12)}$, and is also c-linear with respect to $\operatorname{Hom}(P_c \circ J_U \circ Q_c, Id_V)$ and $P_{c(12)} \circ J_1 \circ Q_{c(12)}$, so by Lemma 5.65, it respects the direct sums and induces an invertible, c-linear map $\operatorname{Hom}_c(U_c, \mathbf{V}) \to (\operatorname{Hom}(U, V))_{c(123)}$, as indicated in the following diagram.

$$\begin{tikzcd}
 & & \mathrm{Hom}(U,V) \arrow[dll, shift right, "\mathrm{Hom}(Q_c,Id_V)"'] \arrow[dl, "P_{c(12)}"'] \arrow[d, "P_{c(13)}"] \arrow[dr, "P_{c(23)}"] & \\
\mathrm{Hom}(U_c,V) \arrow[urr, shift right, "\mathrm{Hom}(P_c,Id_V)"'] \arrow[r, shift left] \arrow[d] & (\mathrm{Hom}(U,V))_{c(12)} \arrow[l, shift left] \arrow[dr] & (\mathrm{Hom}(U,V))_{c(13)} \arrow[d] & (\mathrm{Hom}(U,V))_{c(23)} \arrow[dl] \\
\mathrm{Hom}_c(U_c,\mathbf{V}) \arrow[rr, shift left] & & (\mathrm{Hom}(U,V))_{c(123)} \arrow[ll, shift left] &
\end{tikzcd}$$

LEMMA 5.114. *Given* $\mathbf{V} = (V, J_V)$ *and* $U$ *with commuting CSOs* $J_U^1$, $J_U^2$, *let* $W$ *be a space with three commuting CSOs* $J_W^1$, $J_W^2$, $J_W^3$. *If* $H : W \to \mathrm{Hom}(U,V)$ *satisfies* $\mathrm{Hom}(J_U^1, Id_V) \circ H = H \circ J_W^1$ *and* $\mathrm{Hom}(J_U^2, Id_V) \circ H = H \circ J_W^2$ *and* $\mathrm{Hom}(Id_U, J_V) \circ H = H \circ J_W^3$, *then* $H$ *respects the corresponding direct sums from Example 5.113 and the induced maps are c-linear. If also* $H$ *is invertible, then the induced maps are invertible.*

PROOF. That $H$ respects the direct sums produced by the three corresponding pairs of CSOs, and that the induced maps

$$P_{c(i_1i_2)} \circ H \circ Q'_{c(i_1i_2)} : W_{c(i_1i_2)} \to (\mathrm{Hom}(U,V))_{c(i_1i_2)}$$

(for example, the arrow labeled $a_3$ in the diagram below) are c-linear, follow from Lemma 5.108, which also showed the induced map $\tilde{a}_3 : W_{c(123)} \to (\mathrm{Hom}(U,V))_{c(123)}$ is c-linear, and invertible if $H$ is. In the diagram, $a_1$ and $\tilde{a}_1$ are the canonical invertible maps which appeared as horizontal arrows in the diagram from Example 5.113; the adjacent projection arrows are also copied from that diagram. Lemma 5.85 showed that $H$ also respects the direct sum $W_{c(12)} \oplus W_{a(12)} \to \mathrm{Hom}(U_c,V) \oplus \mathrm{Hom}(U_a,V)$ and that the induced map $\mathrm{Hom}(Q_c, Id_V) \circ H \circ Q'_{c(12)}$, denoted $a_2$ in the diagram below, is c-linear with respect to the CSO induced by $J_W^1$ and the CSO induced by $\mathrm{Hom}(J_U^1, Id_V)$. In fact, $a_2$ is also c-linear with respect to the other pair of CSOs, induced by $J_W^3$ and $\mathrm{Hom}(Id_U, J_V)$. By Lemma 5.65, $a_2$ respects the direct sums produced by the commuting CSOs and induces a c-linear map $\tilde{a}_2 : W_{c(123)} \to \mathrm{Hom}_c(U_c, \mathbf{V})$; it satisfies the identity $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$.

$$\begin{tikzcd}
 & \mathrm{Hom}(U,V) \arrow[ddl, "P_{c(12)}"'] \arrow[dd, "\mathrm{Hom}(Q_c,Id_V)"] & W \arrow[l, "H"'] \arrow[d, "P'_{c(12)}"] \\
 & & W_{c(12)} \arrow[dll, "a_3"'] \arrow[dl, "a_2"] \arrow[d, "P'_{c((12)3)}"] \\
(\mathrm{Hom}(U,V))_{c(12)} \arrow[r, "a_1"'] \arrow[d, "P_{c((12)3)}"] & \mathrm{Hom}(U_c,V) \arrow[d] & W_{c(123)} \arrow[dll, "\tilde{a}_3"'] \arrow[dl, "\tilde{a}_2"] \\
(\mathrm{Hom}(U,V))_{c(123)} \arrow[r, "\tilde{a}_1"'] & \mathrm{Hom}_c(U_c,V) &
\end{tikzcd}$$

∎

EXAMPLE 5.115. Given $U$ and $V$, suppose there are three commuting CSOs $J_U$, $J'_U$, $J''_U$ on $U$. Then $J_1 = \mathrm{Hom}(J_U, Id_V)$, $J_2 = \mathrm{Hom}(J'_U, Id_V)$, $J_3 = \mathrm{Hom}(J''_U, Id_V)$ are three commuting CSOs on $\mathrm{Hom}(U,V)$, and this is a special case of Example 5.106. There are three direct sums: $\mathrm{Hom}(U,V) = (\mathrm{Hom}(U,V))_{c(i_1i_2)} \oplus (\mathrm{Hom}(U,V))_{a(i_2i_2)}$, each of which is equivalent to a direct sum $\mathrm{Hom}(U_{c(i_1i_2)}, V) \oplus \mathrm{Hom}(U_{a(i_1i_2)}, V)$ as in Example 5.84, with projection $\mathrm{Hom}(Q_{c(i_2,i_2)}, Id_V) : \mathrm{Hom}(U,V) \to \mathrm{Hom}(U_{c(i_1i_2)}, V)$ and inclusion $\mathrm{Hom}(P_{c(i_2,i_2)}, Id_V)$. Each $(\mathrm{Hom}(U,V))_{c(i_1i_2)}$ admits a direct sum with projection $P'_{c((i_1i_2)i_3)}$ onto

$$((\mathrm{Hom}(U,V))_{c(i_1i_2)})_c = (\mathrm{Hom}(U,V))_{c(123)} = \{A\colon U \to V\colon A \circ J_U = A \circ J'_U = A \circ J''_U\}.$$

Each subspace $\mathrm{Hom}(U_{c(i_2i_2)}, V)$ has two commuting CSOs, and Example 5.84 applies again; there are equivalent direct sums:

$$\begin{aligned}
\mathrm{Hom}(U_{c(i_1i_2)}, V) &= (\mathrm{Hom}(U_{c(i_1i_2)}, V))_c \oplus (\mathrm{Hom}(U_{c(i_1i_2)}, V))_a, \\
\mathrm{Hom}(U_{c(i_1i_2)}, V) &= \mathrm{Hom}(U_{c(123)}, V) \oplus \mathrm{Hom}((U_{c(i_1i_2)})_a, V).
\end{aligned}$$

The following diagram shows the $(i_1i_2) = (12)$ case, the other two cases being similar.

$$\begin{tikzcd}
 & & \mathrm{Hom}(U,V) \arrow[dl, shift right, "\mathrm{Hom}(Q_{c(12)}, Id_V)"'] \arrow[d, "P'_{c(12)}"] \\
 & \mathrm{Hom}(U_{c(12)}, V) \arrow[ur, shift right] \arrow[r, "a_1"] \arrow[dl, shift right, "\mathrm{Hom}(Q_{c((12)3)}, Id_V)"'] \arrow[d, "P''_c"] & (\mathrm{Hom}(U,V))_{c(12)} \arrow[d, "P'_{c((12)3)}"] \\
\mathrm{Hom}(U_{c(123)}, V) \arrow[ur, shift right] \arrow[r, "a_2"] & (\mathrm{Hom}(U_{c(12)}, V))_c \arrow[r, "a_3"] & (\mathrm{Hom}(U,V))_{c(123)}
\end{tikzcd}$$

The horizontal arrows are

$$\begin{aligned}
a_1 &= P'_{c(12)} \circ \mathrm{Hom}(P_{c(12)}, Id_V) \\
a_2 &= P''_c \circ \mathrm{Hom}(P_{c((12)3)}, Id_V) \\
a_3 &= P'_{c((12)3)} \circ a_1 \circ Q''_c.
\end{aligned}$$

Both $a_1$ and $a_2$ are c-linear and invertible, canonically induced from the equivalent direct sums as in Example 5.84. The $a_1$ map is also c-linear with respect to the CSOs induced by $J''_U$, so the induced map $a_3$ is c-linear and invertible by Lemma 5.65. The c-linear invertible composite

$$a_3 \circ a_2 = P'_{c((12)3)} \circ P'_{c(12)} \circ \mathrm{Hom}(P_{c((12)3)} \circ P_{c(12)}, Id_V)$$

is canonical, not depending on the choice of $(i_1i_2)$, as in Example 5.106.

EXAMPLE 5.116. Given $U$ and $V$, suppose $\mathbf{U} = (U, J_U)$ and there are commuting CSOs $J_V$, $J'_V$ on $V$. Then $J_1 = \mathrm{Hom}(J_U, Id_V)$, $J_2 = \mathrm{Hom}(Id_U, J_V)$, $J_3 = \mathrm{Hom}(Id_U, J'_V)$ are three commuting CSOs on $\mathrm{Hom}(U,V)$, and this is a special case of Example 5.106. There are three direct sums:

$$\mathrm{Hom}(U,V) = (\mathrm{Hom}(U,V))_{c(12)} \oplus (\mathrm{Hom}(U,V))_{a(12)},$$

where $(\mathrm{Hom}(U,V))_{c(12)} = \mathrm{Hom}_c(\mathbf{U}, (V, J_V))$ as in Example 5.78,

$$\mathrm{Hom}(U,V) = (\mathrm{Hom}(U,V))_{c(13)} \oplus (\mathrm{Hom}(U,V))_{a(13)},$$

where $(\mathrm{Hom}(U,V))_{c(13)} = \mathrm{Hom}_c(\mathbf{U},(V,J'_V))$, and

$$\mathrm{Hom}(U,V) = (\mathrm{Hom}(U,V))_{c(23)} \oplus (\mathrm{Hom}(U,V))_{a(23)},$$

where $(\mathrm{Hom}(U,V))_{c(23)} = \mathrm{Hom}(U,V_c)$ as in Example 5.83. Each $(\mathrm{Hom}(U,V))_{c(i_1i_2)}$ admits a direct sum with projection onto

$$((\mathrm{Hom}(U,V))_{c(i_1i_2)})_c = (\mathrm{Hom}(U,V))_{c(123)} = \mathrm{Hom}_c(\mathbf{U},V_c),$$

as follows:

There are two ways to construct the projection

$$P^1 : (\mathrm{Hom}(U,V))_{c(23)} = \mathrm{Hom}(U,V_c) \twoheadrightarrow (\mathrm{Hom}(U,V))_{c(123)} = \mathrm{Hom}_c(\mathbf{U},V_c),$$

which will turn out to give the same map. Denote the projection $P'_c : V \twoheadrightarrow V_c$ as in Example 5.59 with corresponding inclusion $Q'_c$; then $P_{c(23)} = \mathrm{Hom}(Id_U, P'_c) : \mathrm{Hom}(U,V) \twoheadrightarrow \mathrm{Hom}(U,V_c)$ is the projection from (5.13) in Example 5.83, with corresponding inclusion $Q_{c(23)} = \mathrm{Hom}(Id_U, Q'_c)$.

The first construction of $P^1$ is to consider $\mathrm{Hom}(U,V_c)$ as a space with commuting CSOs $\mathrm{Hom}(J_U, Id_{V_c})$, $\mathrm{Hom}(Id_U, J_{V_c})$ and directly apply Example 5.78 to get a projection

$$\begin{aligned} P^1 &= \frac{1}{2}\cdot(Id_{\mathrm{Hom}(U,V_c)} - \mathrm{Hom}(J_U, Id_{V_c}) \circ \mathrm{Hom}(Id_U, J_{V_c})) \\ &= \frac{1}{2}\cdot(Id_{\mathrm{Hom}(U,V_c)} - \mathrm{Hom}(J_U, P'_c \circ J_V \circ Q'_c)). \end{aligned} \tag{5.27}$$

Second, consider the subspace $(\mathrm{Hom}(U,V))_{c(23)}$, with two induced CSOs that commute, as in Example 5.106:

$$(P_{c(23)} \circ \mathrm{Hom}(J_U, Id_V) \circ Q_{c(23)}), \quad (P_{c(23)} \circ \mathrm{Hom}(Id_U, J_V) \circ Q_{c(23)}).$$

Then, using $P_{c(23)} = \mathrm{Hom}(Id_U, P'_c)$ and $Q_{c(23)} = \mathrm{Hom}(Id_U, Q'_c)$, the projection $P_{c((23)1)}$ defined by these two CSOs as in Example 5.106 is the same as (5.27).

The projection $P_{c((12)3)} : (\mathrm{Hom}(U,V))_{c(12)} \twoheadrightarrow (\mathrm{Hom}(U,V))_{c(123)}$ can also be defined by two methods with the same result (and similarly for $P_{c((13)2)}$). The commuting induced CSOs:

$$(P_{c(12)} \circ \mathrm{Hom}(J_U, Id_V) \circ Q_{c(12)}), \quad (P_{c(12)} \circ \mathrm{Hom}(Id_U, J'_V) \circ Q_{c(12)}) \tag{5.28}$$

define $P_{c((12)3)}$ as in Example 5.106. The other way to define the projection is to consider the map $\mathrm{Hom}(Id_U, P'_c) : \mathrm{Hom}(U,V) \twoheadrightarrow \mathrm{Hom}(U,V_c)$, which is c-linear in two different ways: with respect to the pair $\mathrm{Hom}(J_U, Id_V)$, $\mathrm{Hom}(J_U, Id_{V_c})$ and also the pair $\mathrm{Hom}(Id_U, J_V)$, $\mathrm{Hom}(Id_U, J_{V_c})$, as in Lemma 5.80. The induced map $P^2 = P^1 \circ \mathrm{Hom}(Id_U, P'_c) \circ Q_{c(12)} : (\mathrm{Hom}(U,V))_{c(12)} \to \mathrm{Hom}_c(U,V_c)$ is c-linear; it can be denoted $\mathrm{Hom}_c(Id_U, P'_c)$ as in Notation 5.81. By the equality of the composite projections from Example 5.106,

$$P^2 = P^1 \circ \mathrm{Hom}(Id_U, P'_c) \circ Q_{c(12)} \tag{5.29}$$

$$= P_{c((23)1)} \circ P_{c(23)} \circ Q_{c(12)} \tag{5.30}$$

$$= P_{c((12)3)} \circ P_{c(12)} \circ Q_{c(12)} = P_{c((12)3)}.$$

The expression (5.30) is an example of the construction (1.18) from Theorem 1.130.

Similarly, since the composite inclusions are equal:

$$Q_{c(12)} \circ Q_{c((12)3)} = \mathrm{Hom}(Id_U, Q'_c) \circ Q_{c((23)1)}, \tag{5.31}$$

the inclusion $Q_{c((12)3)}$ is equal the induced map:

$$Q_{c((12)3)} = P_{c(12)} \circ \operatorname{Hom}(Id_U, Q'_c) \circ Q_{c((23)1)} = \operatorname{Hom}_c(Id_U, Q'_c).$$

EXERCISE 5.117. Given $\mathbf{V} = (V, J_V)$ and $\mathbf{W} = (W, J_W)$, $\operatorname{Hom}(\operatorname{Hom}(V, W), W)$ admits three commuting CSOs,

$$\begin{array}{rcl} J_1 & = & \operatorname{Hom}(\operatorname{Hom}(J_V, Id_W), Id_W), \\ J_2 & = & \operatorname{Hom}(\operatorname{Hom}(Id_V, J_W), Id_W), \\ J_3 & = & \operatorname{Hom}(Id_{\operatorname{Hom}(V,W)}, J_W). \end{array}$$

As in Example 5.113, there are three direct sums; $(\operatorname{Hom}(\operatorname{Hom}(V, W), W))_{c(23)}$ was considered in Exercise 5.37. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ d_{VW} : \mathbf{V} \to (\operatorname{Hom}(\operatorname{Hom}(V, W), W))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let $Q_c : \operatorname{Hom}_c(\mathbf{V}, \mathbf{W}) \hookrightarrow \operatorname{Hom}(V, W)$ denote the inclusion from Example 5.78. Then the image of

$$\operatorname{Hom}(Q_c, Id_W) \circ d_{VW} : V \to \operatorname{Hom}(\operatorname{Hom}_c(\mathbf{V}, \mathbf{W}), W)$$

is contained in $\operatorname{Hom}_c(\operatorname{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$, i.e., for any $v \in V$,

$$d_{VW}(v) \circ Q_c : \operatorname{Hom}_c(\mathbf{V}, \mathbf{W}) \to \mathbf{W} : H \mapsto (Q_c(H))(v) = H(v)$$

is a c-linear map. From the commutativity of the diagram from Example 5.113, considering $\operatorname{Hom}(Q_c, Id_W) \circ d_{VW}$ as a map $\mathbf{V} \to \operatorname{Hom}_c(\operatorname{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$, it is identical to the composite of the above map $P_{c((23)1)} \circ P_{c(23)} \circ d_{VW}$ with the canonical map $(\operatorname{Hom}(\operatorname{Hom}(V, W), W))_{c(123)} \to \operatorname{Hom}_c(\operatorname{Hom}_c(\mathbf{V}, \mathbf{W}), \mathbf{W})$, so it is c-linear. ∎

Recall from Definition 1.40 and Notation 1.41 the canonical map

$$n : V \otimes \operatorname{Hom}(U, W) \to \operatorname{Hom}(U, V \otimes W) : (n'(v \otimes E)) : u \mapsto v \otimes (E(u)).$$

THEOREM 5.118. *If* $\mathbf{U} = (U, J_U)$ *and* $\mathbf{V} = (V, J_V)$ *and* $\mathbf{W} = (W, J_W)$*, then* $n : V \otimes \operatorname{Hom}(U, W) \to \operatorname{Hom}(U, V \otimes W)$ *is c-linear with respect to corresponding pairs of the three commuting CSOs induced on each space, so it respects the direct sums and induces c-linear maps*

$$\begin{array}{rcl} n^1 : V \otimes \operatorname{Hom}_c(\mathbf{U}, \mathbf{W}) & \to & \operatorname{Hom}_c(\mathbf{U}, V \otimes \mathbf{W}) \\ n^2 : \mathbf{V} \otimes_c \operatorname{Hom}(\mathbf{U}, W) & \to & \operatorname{Hom}_c(\mathbf{U}, \mathbf{V} \otimes W) \\ n^3 : \mathbf{V} \otimes_c \operatorname{Hom}(U, \mathbf{W}) & \to & \operatorname{Hom}(U, \mathbf{V} \otimes_c \mathbf{W}) \\ \mathbf{n} : \mathbf{V} \otimes_c \operatorname{Hom}_c(\mathbf{U}, \mathbf{W}) & \to & \operatorname{Hom}_c(\mathbf{U}, \mathbf{V} \otimes_c \mathbf{W}), \end{array}$$

*which are invertible if* $n$ *is.*

PROOF. The c-linearity claims for $n$ follow from Lemma 1.42 (adjusted for variations in ordering; they are also straightforward to check directly), and then Lemma 5.108 applies. The direct sums for the domain $V \otimes \operatorname{Hom}(U, W)$ are as in Example 5.112. The projection onto the subspace $V \otimes \operatorname{Hom}_c(\mathbf{U}, \mathbf{W})$ is equal to $[Id_V \otimes P_H]$ as in Example 5.73, where $P_H$ is the projection $\operatorname{Hom}(U, W) \twoheadrightarrow \operatorname{Hom}_c(\mathbf{U}, \mathbf{W})$. The direct sums for the target space $\operatorname{Hom}(U, V \otimes W)$ are as in Example 5.116. The following diagram shows some of the canonical projections, including $P^2$ as in (5.29).

$$\begin{tikzcd}
V \otimes \mathrm{Hom}(U, W) \arrow[r, "n"] \arrow[d, "{[Id_V \otimes P_H]}"'] & \mathrm{Hom}(U, V \otimes W) \arrow[d] \\
V \otimes \mathrm{Hom}_c(\mathbf{U}, \mathbf{W}) \arrow[r, "n^1"] \arrow[d] & \mathrm{Hom}_c(\mathbf{U}, V \otimes \mathbf{W}) \arrow[d, "P^2 = \mathrm{Hom}_c(Id_U, P'_c)"] \\
\mathbf{V} \otimes_c \mathrm{Hom}_c(\mathbf{U}, \mathbf{W}) \arrow[r, "\mathbf{n}"] & \mathrm{Hom}_c(\mathbf{U}, \mathbf{V} \otimes_c \mathbf{W})
\end{tikzcd}$$

The invertibility also follows from Lemma 5.108; in particular, if $U$ or $V$ is finite-dimensional then these maps are invertible. ∎

EXAMPLE 5.119. Given $\mathbf{U} = (U, J_U)$, $\mathbf{V} = (V, J_V)$, $\mathbf{W} = (W, J_W)$, the canonical invertible map

$$q : \mathrm{Hom}(V, \mathrm{Hom}(U, W)) \to \mathrm{Hom}(V \otimes U, W)$$

from Definition 1.46 is c-linear with respect to the three corresponding pairs of induced CSOs, by Lemma 1.50. Lemma 5.114 applies, so that the following induced maps are c-linear and invertible:

$$\begin{array}{rcl}
\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{U}, W)) & \to & \mathrm{Hom}(\mathbf{V} \otimes_c \mathbf{U}, W) \\
\mathrm{Hom}(V, \mathrm{Hom}_c(\mathbf{U}, \mathbf{W})) & \to & \mathrm{Hom}_c(V \otimes \mathbf{U}, \mathbf{W}) \\
\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(U, \mathbf{W})) & \to & \mathrm{Hom}_c(\mathbf{V} \otimes U, \mathbf{W}) \\
\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}_c(\mathbf{U}, \mathbf{W})) & \to & \mathrm{Hom}_c(\mathbf{V} \otimes_c \mathbf{U}, \mathbf{W}).
\end{array}$$

The direct sums for the domain $\mathrm{Hom}(V, \mathrm{Hom}(U, W))$ are as in Example 5.116. The direct sums for the target space $\mathrm{Hom}(V \otimes U, W)$ are as in Example 5.113.

EXAMPLE 5.120. Given $\mathbf{V} = (V, J_V)$ and $\mathbf{W} = (W, J_W)$, $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ admits three commuting CSOs, and the involution $T_{V;W}$ is c-linear with respect to three corresponding pairs of commuting CSOs as in Exercise 5.43 and Exercise 5.44. $\mathrm{Hom}(V, \mathrm{Hom}(V, W))$ also admits the three involutions from (5.24), and the involution $T_{V;W}$ satisfies

$$(-\mathrm{Hom}(Id_V, \mathrm{Hom}(J_V, J_W))) \circ T_{V;W} = T_{V;W} \circ (-\mathrm{Hom}(J_V, \mathrm{Hom}(Id_V, J_W))),$$

as in (5.26), so this is a special case of Example 5.109. As in (5.25), $T_{V;W}$ induces c-linear invertible maps:

$$\begin{array}{rcl}
\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(V, \mathbf{W})) & \to & \mathrm{Hom}(V, \mathrm{Hom}_c(\mathbf{V}, \mathbf{W})) \\
\mathrm{Hom}(V, \mathrm{Hom}_c(\mathbf{V}, \mathbf{W})) & \to & \mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(V, \mathbf{W})).
\end{array}$$

The subspace $\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))$ admits commuting CSOs as in (5.28), and projects onto the subspace $\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}))$ as in Example 5.116. The involution on $\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}(\mathbf{V}, W))$ induced by $T_{V;W}$ from Exercise 5.95 induces a c-linear involution on $\mathrm{Hom}_c(\mathbf{V}, \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}))$, producing a direct sum denoted $Sym_c(\mathbf{V}; \mathbf{W}) \oplus Alt_c(\mathbf{V}; \mathbf{W})$. Combining the notation from Example 1.132, Example 5.96, and Example 5.109, the following commutative diagram shows some of the projections from Example 5.109.

$$\begin{tikzcd}[column sep=small]
 & \mathrm{Hom}(V,\mathrm{Hom}(V,W)) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{Hom}_c(\mathbf{V},\mathrm{Hom}(V,\mathbf{W})) \arrow[d] & \mathrm{Hom}_c(\mathbf{V},\mathrm{Hom}(\mathbf{V},W)) \arrow[dl, "P^2"'] \arrow[d] \arrow[dr] & Sym(V;W) \arrow[d] \\
\mathrm{Hom}_c(\mathbf{V},\mathrm{Hom}_c(\mathbf{V},\mathbf{W})) \arrow[dr] & U_{11} \arrow[d] & Sym_c(\mathbf{V};W) \arrow[dl] \\
 & Sym_c(\mathbf{V};\mathbf{W}) &
\end{tikzcd}$$

The projection $P^2 = P_{c((12)3)}$ labeled in the diagram, and the vertical arrow $P_{c((13)2)}$ on the left, are as in (5.29) from Example 5.116. The subspace $U_{11}$ is this fixed point subspace:

$$\{h : (h(v_1))(v_2) = -(h(J_V(v_1)))(J_V(v_2)) = -J_W((h(J_V(v_2)))(v_1))\}.$$

The conclusions from Example 5.109 are that

$$Sym_c(\mathbf{V};\mathbf{W}) = \mathrm{Hom}_c(\mathbf{V},\mathrm{Hom}_c(\mathbf{V},\mathbf{W})) \cap Sym(V;W),$$

and that $Sym_c(\mathbf{V};\mathbf{W})$ has a canonical CSO.

EXERCISE 5.121. Given $\mathbf{U} = (U, J_U)$ and $\mathbf{W} = (W, J_W)$, $\mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W)$ admits three commuting CSOs, as in Example 5.113, so there are three direct sums; $(\mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W))_{c(23)}$ was considered in Exercise 5.40. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W : \mathrm{Hom}(\mathbf{U}, V) \to (\mathrm{Hom}(\mathrm{Hom}(V,\mathbf{W}) \otimes \mathbf{U}, \mathbf{W}))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let

$$Q_c : \mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U} \hookrightarrow \mathrm{Hom}(V,W) \otimes U$$

denote the inclusion from Example 5.74. Then the image of

$$\mathrm{Hom}(Q_c, Id_W) \circ e_{UV}^W : \mathrm{Hom}(\mathbf{U}, V) \to \mathrm{Hom}(\mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U}, W)$$

is contained in $\mathrm{Hom}_c(\mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$, i.e., for any $A \in \mathrm{Hom}(U,V)$,

$$e_{UV}^W(A) \circ Q_c : \mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U} \to \mathbf{W} : B \otimes u \mapsto (e_{UV}^W(A))(Q_c(B \otimes u)) = B(A(u))$$

is a c-linear map. From the commutativity of the diagram from Example 5.113, considering $\mathrm{Hom}(Q_c, Id_W) \circ e_{UV}^W$ as a map $\mathrm{Hom}(\mathbf{U}, V) \to \mathrm{Hom}_c(\mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$, it is identical to the composite of the above map $P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W$ with the canonical map $(\mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W))_{c(123)} \to \mathrm{Hom}_c(\mathrm{Hom}(V,\mathbf{W}) \otimes_c \mathbf{U}, \mathbf{W})$, so it is c-linear. ∎

EXERCISE 5.122. Given $\mathbf{V} = (V, J_V)$ and $\mathbf{W} = (W, J_W)$, $\mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W)$ admits three commuting CSOs, as in Example 5.113, so there are three direct sums; $(\mathrm{Hom}(\mathrm{Hom}(V,W) \otimes U, W))_{c(23)}$ was considered in Exercises 5.40, 5.121. The composite

$$P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W : \mathrm{Hom}(U, \mathbf{V}) \to (\mathrm{Hom}(\mathrm{Hom}(\mathbf{V},\mathbf{W}) \otimes U, \mathbf{W}))_{c(123)}$$

is c-linear with respect to the canonical CSO. Let $Q_c : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U \to \mathrm{Hom}(V, W) \otimes U$ denote the inclusion from Examples 5.73 and 5.78. Then the image of

$$\mathrm{Hom}(Q_c, Id_W) \circ e_{UV}^W : \mathrm{Hom}(U, \mathbf{V}) \to \mathrm{Hom}(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, W)$$

is contained in $\mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$, i.e., for any $A \in \mathrm{Hom}(U, V)$,

$$e_{UV}^W(A) \circ Q_c : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U \to \mathbf{W} : B \otimes u \mapsto (e_{UV}^W(A))(Q_c(B \otimes u)) = B(A(u))$$

is a c-linear map. From the commutativity of the diagram from Example 5.113, considering $\mathrm{Hom}(Q_c, Id_W) \circ e_{UV}^W$ as a map $\mathrm{Hom}(U, \mathbf{V}) \to \mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$, it is identical to the composite of the above map $P_{c((23)1)} \circ P_{c(23)} \circ e_{UV}^W$ with the canonical map $(\mathrm{Hom}(\mathrm{Hom}(V, W) \otimes U, W))_{c(123)} \to \mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes U, \mathbf{W})$, so it is c-linear. ∎

Exercise 5.123. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$, $\mathbf{V} = (V, J_V)$, and $\mathbf{W} = (W, J_W)$, we consider the three commuting CSOs on $\mathrm{Hom}(V, W) \otimes U$, so Example 5.115 applies to $\mathrm{Hom}(\mathrm{Hom}(V, W) \otimes U, W)$. The lower square in the following commutative diagram is a specific case of a square from the diagram in Example 5.115, with the same labeling of arrows, and (12) referring to the CSOs induced by $J_U$ and $J_V$. The $e_1$, $e_2$ arrows are the maps induced by $e_{UV}^W$ on the two equivalent direct sums from Exercise 5.91.

$$\begin{tikzcd}
 & \mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \arrow[dl, "e_1"'] \arrow[d, "e_2"] \\
\mathrm{Hom}(\mathrm{Hom}(\mathbf{V}, W) \otimes_c \mathbf{U}, W) \arrow[r, "a_1"] \arrow[d, "\mathrm{Hom}(Q_{c((12)3)}, Id_W)"'] & (\mathrm{Hom}(\mathrm{Hom}(V, W) \otimes U, W))_{c(12)} \arrow[d, "P'_{c((12)3)}"] \\
\mathrm{Hom}(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U}, W) \arrow[r, "a_3 \circ a_2"] & (\mathrm{Hom}(\mathrm{Hom}(V, W) \otimes U, W))_{c(123)}
\end{tikzcd}$$

The composite $P'_{c((12)3)} \circ e_2$ is c-linear, so the composite $\mathrm{Hom}(Q_{c((12)3)}, Id_W) \circ e_1$ is also c-linear. As in Exercises 5.40, 5.121, 5.122, the image of $\mathrm{Hom}(Q_{c((12)3)}, Id_W) \circ e_1$ is contained in $\mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes \mathbf{U}, \mathbf{W})$, i.e., if $Q'_c$ denotes the inclusion of $\mathrm{Hom}_c(\mathbf{U}, \mathbf{V}) \hookrightarrow \mathrm{Hom}(U, V)$ and $Q_{c(12)} \circ Q_{c((12)3)}$ is the composite inclusion of $\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U} \hookrightarrow \mathrm{Hom}(V, W) \otimes U$, then for any $A \in \mathrm{Hom}_c(\mathbf{U}, \mathbf{V})$, the map

$$e_{UV}^W(Q'_c(A)) \circ Q_{c(12)} \circ Q_{c((12)3)} : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{U} \to \mathbf{W} \tag{5.32}$$

is c-linear. ∎

Example 5.124. For $\mathbf{V} = (V, J_V)$ and $\mathbf{W} = (W, J_W)$, the space $\mathrm{Hom}(V, W) \otimes V$ admits three commuting CSOs as in Example 5.112. The composite of the inclusions, $Q_{c(12)} \circ Q_{c((12)3)}$,

$$\mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} \xhookrightarrow[Q_{c((12)3)}]{} \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes V \xhookrightarrow[Q_{c(12)}]{} \mathrm{Hom}(V, W) \otimes V$$

does not depend on the ordering of the indices, but in the case of the above diagram, $Q_{c(12)} = [Q_c \otimes Id_V]$ for $Q_c : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \hookrightarrow \mathrm{Hom}(V, W)$, as in Example 5.73.

Define a c-linear evaluation map

$$Ev_{\mathbf{VW}}^c = Ev_{VW} \circ Q_{c(12)} \circ Q_{c((12)3)} : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} \to \mathbf{W}. \tag{5.33}$$

This is the restriction of the canonical evaluation from Definition 2.71; it is c-linear by Exercise 5.41 and Lemma 5.107. Considering the formula $Ev_{VW} = e^W_{VV}(Id_V)$ from Equation (2.16), this construction is also a special case of (5.32) from Exercise 5.123. The domain $\mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \otimes_c \mathbf{V}$ is spanned by elements of the form $P_{c((12)3)}(A \otimes v)$ for c-linear maps $A : V \to W$, on which $Ev^c_{\mathbf{VW}}$ acts as follows:

$$\begin{aligned}
& Ev^c_{\mathbf{VW}}(P_{c((12)3)}(A \otimes v)) \\
= \;& (Ev_{VW} \circ Q_{c(12)} \circ Q_{c((12)3)} \circ P_{c((12)3)})(A \otimes v) \\
= \;& (Ev_{VW} \circ [Q_c \otimes Id_V] \circ \frac{1}{2}(Id_{\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes V} - [J_{\mathrm{Hom}_c(\mathbf{V},\mathbf{W})} \otimes J_V]))(A \otimes v) \\
= \;& Ev_{VW}(\frac{1}{2}((Q_c(A)) \otimes v - (Q_c(A \circ J_V)) \otimes (J_V(v)))) \\
= \;& \frac{1}{2}(Q_c(A))(v) - \frac{1}{2}(Q_c(A \circ J_V))(J_V(v)) \\
(5.34) = \;& A(v) = Ev_{VW}(A \otimes v),
\end{aligned}$$

where in line (5.34), we just forget that $A = Q_c(A)$ and $A \circ J_V = Q_c(A \circ J_V)$ are c-linear.

LEMMA 5.125. *For any* $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, *and c-linear map* $B : \mathbf{U} \to \mathbf{W}$, *the following diagram is commutative.*

$$\begin{tikzcd}
\mathbf{U} \arrow[rr, "B"] & & \mathbf{W} \\
\mathrm{Hom}_c(\mathbf{V},\mathbf{U}) \otimes_c \mathbf{V} \arrow[u, "Ev^c_{\mathbf{VU}}"] \arrow[rr, "{[\mathrm{Hom}_c(Id_V,B)\otimes_c Id_V]}"'] & & \mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \otimes_c \mathbf{V} \arrow[u, "Ev^c_{\mathbf{VW}}"']
\end{tikzcd}$$

PROOF. This is a c-linear version of Lemma 2.73 (the case $G = Id_V$), which states the commutativity of the upper block of this diagram (for any $B$, not necessarily c-linear).

$$\begin{tikzcd}
U \arrow[rr, "B"] & & W \\
\mathrm{Hom}(V,U) \otimes V \arrow[u, "Ev_{VU}"] \arrow[rr, "{[\mathrm{Hom}(Id_V,B)\otimes Id_V]}"'] & & \mathrm{Hom}(V,W) \otimes V \arrow[u, "Ev_{VW}"'] \\
\mathrm{Hom}_c(\mathbf{V},\mathbf{U}) \otimes V \arrow[u, hook, "{[Q'_c \otimes Id_V]}"] \arrow[rr, "{[\mathrm{Hom}_c(Id_V,B)\otimes Id_V]}"'] & & \mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \otimes V \arrow[u, hook, "{[Q_c \otimes Id_V]}"'] \\
\mathrm{Hom}_c(\mathbf{V},\mathbf{U}) \otimes_c \mathbf{V} \arrow[u, hook, "Q'_{c((12)3)}"] \arrow[rr, "{[\mathrm{Hom}_c(Id_V,B)\otimes_c Id_V]}"'] & & \mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \otimes_c \mathbf{V} \arrow[u, hook, "Q_{c((12)3)}"']
\end{tikzcd}$$

The middle block is commutative by definition of $\mathrm{Hom}_c(Id_V, B)$ for the c-linear maps $Id_V$ and $B$ as in Notation 5.81, together with Lemma 1.36. The lower block is commutative by the construction from Notation 5.76. The upward composites in the left and right columns are $Ev^c_{\mathbf{VU}}$ and $Ev^c_{\mathbf{VW}}$ as in Example 5.124. ∎

### 5.3.3. Four Commuting Complex Structure Operators.

EXAMPLE 5.126. Given $V$ and four commuting CSOs $J_1$, $J_2$, $J_3$, $J_4$, the construction of Example 5.106 shows that for any ordered pair $(i_1, i_2)$ selected without repeats from the indices 1, 2, 3, 4, there is a direct sum with projection $P_{c(i_1i_2)} : V \twoheadrightarrow V_{c(i_1i_2)}$, and for a third distinct index, another direct sum with projection $P_{c((i_1i_2)i_3)} : V_{c(i_1i_2)} \twoheadrightarrow V_{c(i_1i_2i_3)}$. Repeating the process for the remaining, fourth index, the fourth commuting CSO produces a direct sum $V_{c(i_1i_2i_3)} = (V_{c(i_1i_2i_3)})_c \oplus (V_{c(i_1i_2i_3)})_a$, with projection $P_{c((i_1i_2i_3)i_4)} : V_{c(i_1i_2i_3)} \twoheadrightarrow (V_{c(i_1i_2i_3)})_c$. As in Example 5.106, the composite $P_{c((i_1i_2i_3)i_4)} \circ P_{c((i_1i_2)i_3)} \circ P_{c(i_1i_2)}$ equals

$$\frac{1}{8} \cdot (Id_V - J_1 \circ J_2 - J_2 \circ J_3 - J_1 \circ J_3 - J_1 \circ J_4 - J_2 \circ J_4 - J_3 \circ J_4 + J_1 \circ J_2 \circ J_3 \circ J_4),$$

which shows the image of the last projection does not depend on the ordering of the four indices, so the subspace where all four CSOs coincide can be denoted $V_{c(1234)}$.

EXAMPLE 5.127. Given $V$ and four commuting CSOs $J_1$, $J_2$, $J_3$, $J_4$, the space $V_{c(i_1i_2)}$ from Example 5.126 admits three commuting CSOs: $P_{c(i_1i_2)} \circ J_{i_1} \circ Q_{c(i_1i_2)}$, $P_{c(i_1i_2)} \circ J_{i_3} \circ Q_{c(i_1i_2)}$, $P_{c(i_1i_2)} \circ J_{i_4} \circ Q_{c(i_1i_2)}$. Example 5.126 considered pairing the first one with one of the other two to get two direct sums, but as in Example 5.106, there are three possible direct sums on $V_{c(i_1i_2)}$, the third coming from $P_{c(i_1i_2)} \circ J_{i_3} \circ Q_{c(i_1i_2)}$, $P_{c(i_1i_2)} \circ J_{i_4} \circ Q_{c(i_1i_2)}$ to get a subspace denoted $V_{c(i_1i_2)(i_3i_4)}$, where $J_{i_1} = J_{i_2}$ and $J_{i_3} = J_{i_4}$, equal to the subspace $V_{c(i_3i_4)(i_1i_2)}$. The composite projection $V \twoheadrightarrow V_{c(i_1i_2)(i_3i_4)}$ is given by the formula

$$\frac{1}{4} \cdot (Id_V - J_1 \circ J_2 - J_3 \circ J_4 + J_1 \circ J_2 \circ J_3 \circ J_4).$$

The subspace $V_{c(i_1i_2)(i_3i_4)}$ admits two commuting CSOs, the one induced by $J_{i_1}$ and $J_{i_2}$, and the other by $J_{i_3}$ and $J_{i_4}$, so there is a direct sum, and a projection onto the subspace $V_{c(1234)}$ from Example 5.126.

THEOREM 5.128. *Given $V$ and four commuting CSOs $J_1$, $J_2$, $J_3$, $J_4$, the following diagram is commutative, where the arrows are all the projections from direct sums produced by commuting CSOs described in Examples 5.126 and 5.127.*

$$\begin{tikzcd}
 & & & V \arrow[dl] \arrow[dr] \arrow[ddll] \arrow[ddrr] \arrow[dddlll] \arrow[dddrrr] & & & \\
 & & V_{c(14)} \arrow[ddddd] \arrow[dddddr] \arrow[dddddrr] & & V_{c(23)} \arrow[dddllll] \arrow[dddddl] \arrow[dddrr] & & \\
 & V_{c(13)} \arrow[ddl] \arrow[dddrrrr] \arrow[ddddrrr] & & & & V_{c(24)} \arrow[ddddlll] \arrow[ddd] \arrow[ddr] & \\
V_{c(12)} \arrow[d] \arrow[ddr] \arrow[dddrr] & & & & & & V_{c(34)} \arrow[d] \arrow[ddlllll] \arrow[dddll] \\
V_{c(123)} \arrow[dddrrr] & & & & & & V_{c(234)} \arrow[dddlll] \\
 & V_{c(12)(34)} \arrow[ddrr] & & & & V_{c(13)(24)} \arrow[ddll] & \\
 & & V_{c(124)} \arrow[dr] & V_{c(14)(23)} \arrow[d] & V_{c(134)} \arrow[dl] & & \\
 & & & V_{c(1234)} & & &
\end{tikzcd}$$

PROOF. Some sub-diagrams were already considered in Examples 5.106, 5.126, 5.127. Some remain to be checked, for example, the equality of the composite projections $V_{c(12)} \twoheadrightarrow V_{c(123)} \twoheadrightarrow V_{c(1234)}$ and $V_{c(12)} \twoheadrightarrow V_{c(12)(34)} \twoheadrightarrow V_{c(1234)}$ follows from considering the three CSOs on $V_{c(12)}$ as in Example 5.106. The corresponding composites of inclusions are also equal. ∎

LEMMA 5.129. *Given $V$ with commuting CSOs $J_V^1$, $J_V^2$, $J_V^3$, $J_V^4$, and $U$ with commuting CSOs $J_U^1$, $J_U^2$, $J_U^3$, $J_U^4$, if $H : U \to V$ satisfies $H \circ J_U^1 = J_V^1 \circ H$ and $H \circ J_U^2 = J_V^2 \circ H$ and $H \circ J_U^3 = J_V^3 \circ H$ and $H \circ J_U^4 = J_V^4 \circ H$, then $H$ respects the corresponding direct sums from Examples 5.126 and 5.127, and induces maps $U_{c(i_1 i_2)} \to V_{c(i_1 i_2)}$, $U_{c(i_1 i_2 i_3)} \to V_{c(i_1 i_2 i_3)}$, and $U_{c(i_1 i_2)(i_3 i_4)} \to V_{c(i_1 i_2)(i_3 i_4)}$, which are c-linear with respect to all pairs of CSOs induced by $J_U^i$, $J_V^i$, and invertible if $H$ is. The induced map $U_{c(1234)} \to V_{c(1234)}$ is c-linear, and invertible if $H$ is.*

Proof. All the claims follow from Lemma 5.65 and Lemma 5.108. As in Lemma 5.108, the map $U_{c(1234)} \to V_{c(1234)}$ is canonically induced, not depending on the ordering of the indices. ∎

Example 5.130. For $V$ with commuting CSOs $J_V$, $J'_V$, and $U$ with commuting CSOs $J_U$, $J'_U$, the space $U \otimes V$ has four commuting CSOs:

$$J_1 = [J_U \otimes Id_V],\ J_2 = [Id_U \otimes J_V],\ J_3 = [J'_U \otimes Id_V],\ J_4 = [Id_U \otimes J'_V].$$

Theorem 5.128 applies, to give a collection of subspaces of $U \otimes V$. From Example 5.73, $(U \otimes V)_{c(13)} = U_c \otimes V$ and $(U \otimes V)_{c(24)} = U \otimes V_c$. From Example 5.74, $(U \otimes V)_{c(12)} = U \otimes_c V$, and we (temporarily) denote a similar construction $(U \otimes V)_{c(34)} = U \otimes' V$. The subspaces $(U \otimes V)_{c(14)}$ and $(U \otimes V)_{c(23)}$ did not appear in previous Examples, and are omitted from the following commutative diagram, where the positions of the objects match the corresponding positions in the diagram from Theorem 5.128.

$$\begin{tikzcd}
 & & & U \otimes V \arrow[dll] \arrow[ddlll] \arrow[drr] \arrow[ddrrr] & & & \\
 & U_c \otimes V \arrow[ddl] \arrow[dddrrrr] \arrow[ddddrrr] & & & & U \otimes V_c \arrow[ddddlll] \arrow[ddd] \arrow[ddr] & \\
U \otimes_c V \arrow[d] \arrow[ddr] \arrow[dddrr] & & & & & & U \otimes' V \arrow[d] \arrow[ddlllll] \arrow[dddll] \\
U_c \otimes_c V \arrow[dddrrr] & & & & & & U \otimes' V_c \arrow[dddlll] \\
 & U \otimes'_c V \arrow[ddrr] & & & & U_c \otimes V_c \arrow[ddll] & \\
 & & U \otimes_c V_c \arrow[dr] & & U_c \otimes' V \arrow[dl] & & \\
 & & & U_c \otimes_c V_c & & &
\end{tikzcd}$$

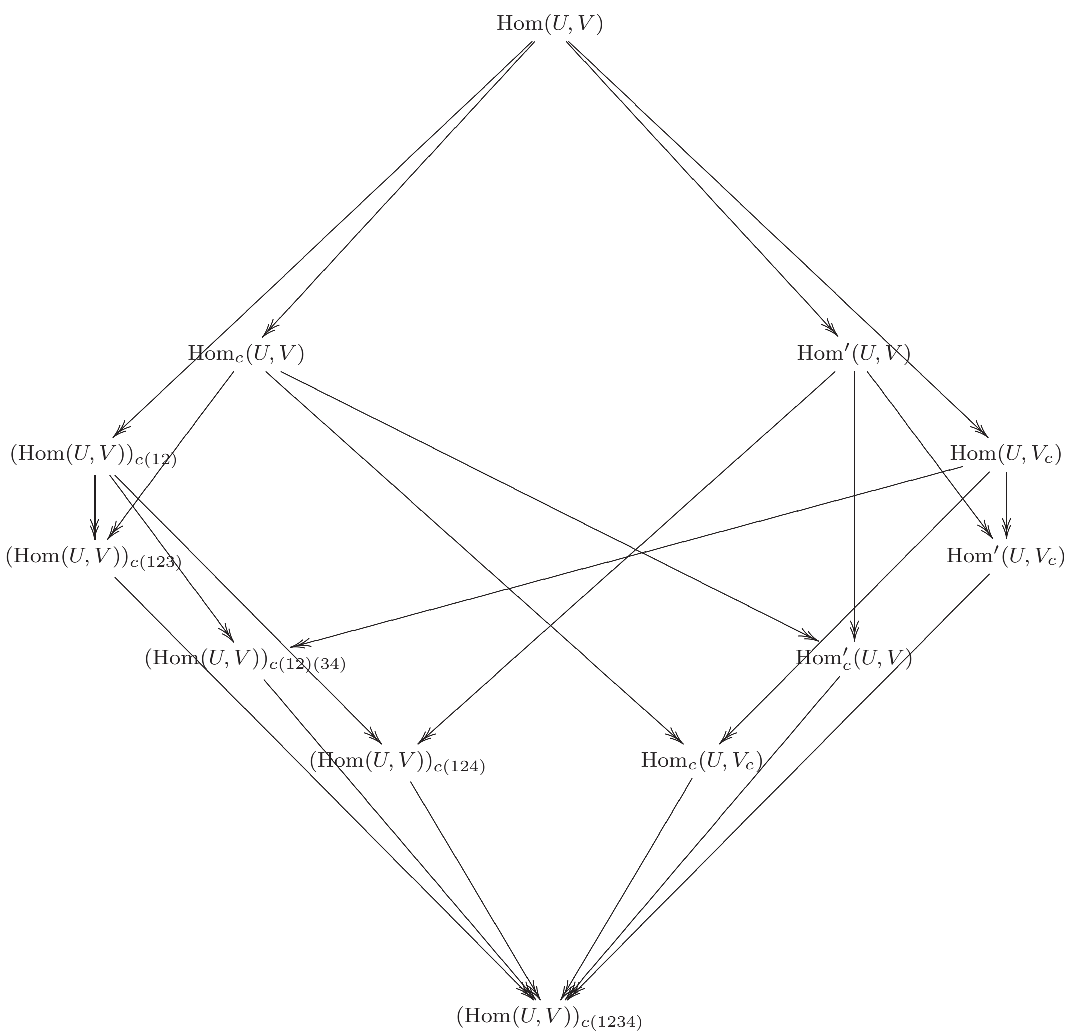
The subspaces $U_c \otimes_c V$, $U \otimes_c V_c$, $U_c \otimes' V$, and $U \otimes' V_c$ are as in Example 5.112. The set $U \otimes'_c V$ corresponds to $(U \otimes V)_{c(12)(34)}$, where $J_1 = J_2$ and $J_3 = J_4$; this notation resembles (5.10) from Example 5.70, with the commuting involutions $-J_1 \circ J_2$ and $-J_3 \circ J_4$.

EXAMPLE 5.131. For $V$ with commuting CSOs $J_V$, $J'_V$, and $U$ with commuting CSOs $J_U$, $J'_U$, the space $\mathrm{Hom}(U,V)$ has four commuting CSOs:

$$J_1 = \mathrm{Hom}(J_U, Id_V),\ \ J_2 = \mathrm{Hom}(J'_U, Id_V),$$

$$J_3 = \mathrm{Hom}(Id_U, J_V),\ \ J_4 = \mathrm{Hom}(Id_U, J'_V).$$

Theorem 5.128 applies, to give a collection of subspaces of $\mathrm{Hom}(U,V)$. The subspace $(\mathrm{Hom}(U,V))_{c(12)}$ was considered in Example 5.113. As in Example 5.78, denote $(\mathrm{Hom}(U,V))_{c(13)} = \mathrm{Hom}_c(U,V)$, and (temporarily) denote a similar construction $(\mathrm{Hom}(U,V))_{c(24)} = \mathrm{Hom}'(U,V)$. From Example 5.83, denote $(\mathrm{Hom}(U,V))_{c(34)} = \mathrm{Hom}(U,V_c)$. The subspaces $(\mathrm{Hom}(U,V))_{c(14)}$ and $(\mathrm{Hom}(U,V))_{c(23)}$ are omitted from the following commutative diagram, but otherwise the positions of the objects match the corresponding positions in the diagram from Theorem 5.128 and Example 5.130.



Adapting the notation from (5.10) in Example 5.70, $\mathrm{Hom}'_c(U,V)$ denotes the subspace

$$(\mathrm{Hom}(U,V))_{c(13)(24)} = \{A : U \to V : A \circ J_U = J_V \circ A \text{ and } A \circ J'_U = J'_V \circ A\}.$$

If we ignore $J'_U$, then the two projections onto $\mathrm{Hom}_c(U,V_c)$ in the above diagram are as in Example 5.116. Similarly ignoring $J_U$, the two projections onto $\mathrm{Hom}'(U,V_c)$ are also as in Example 5.116.

EXAMPLE 5.132. The space $(\mathrm{Hom}(U,V))_{c(12)}$ from Example 5.131 is related to $\mathrm{Hom}(U_c,V)$ as in Example 5.113, by an invertible map $P_{c(12)} \circ \mathrm{Hom}(P_c,Id_V)$. Both $(\mathrm{Hom}(U,V))_{c(12)}$ and $\mathrm{Hom}(U_c,V)$ admit three induced CSOs, and $\mathrm{Hom}(U_c,V)$ admits three direct sums as in Example 5.116. The map $P_{c(12)} \circ \mathrm{Hom}(P_c,Id_V)$ is c-linear with respect to the three corresponding pairs of CSOs, so by Lemma 5.108, it respects the direct sums and induces c-linear invertible maps as indicated by the unlabeled horizontal arrows in the following diagram. The left part is copied from the diagram in Example 5.131, and the top triangle and top square appeared already in the diagram for Example 5.113.

$$\begin{tikzcd}
 & & & \mathrm{Hom}(U,V) \arrow[dlll, shift left, "P_{c(12)}"] \arrow[d, shift left, "\mathrm{Hom}(Q_c,Id_V)"] & & \\
(\mathrm{Hom}(U,V))_{c(12)} \arrow[urrr, hook, shift left, "Q_{c(12)}"] \arrow[rrr, shift right, "\mathrm{Hom}(Q_c,Id_V)\circ Q_{c(12)}"'] \arrow[d] \arrow[ddr] \arrow[dddrr] & & & \mathrm{Hom}(U_c,V) \arrow[lll, shift right, "P_{c(12)}\circ\mathrm{Hom}(P_c,Id_V)"'] \arrow[u, shift left, "\mathrm{Hom}(P_c,Id_V)"] \arrow[d] \arrow[ddr] \arrow[dddrr] & & \\
(\mathrm{Hom}(U,V))_{c(123)} \arrow[rrr, shift right] \arrow[dddrrr] & & & \mathrm{Hom}_c(U_c,V) \arrow[lll, shift right] \arrow[dddrr, "P^2"] & & \\
 & (\mathrm{Hom}(U,V))_{c(12)(34)} \arrow[rrr, shift right] \arrow[ddrr] & & & \mathrm{Hom}(U_c,V_c) \arrow[lll, shift right] \arrow[ddr, "P^1"] & \\
 & & (\mathrm{Hom}(U,V))_{c(124)} \arrow[rrr, shift right] \arrow[dr] & & & \mathrm{Hom}'(U_c,V) \arrow[lll, shift right] \arrow[d] \\
 & & & (\mathrm{Hom}(U,V))_{c(1234)} \arrow[rr, shift right] & & \mathrm{Hom}_c(U_c,V_c) \arrow[ll, shift right]
\end{tikzcd}$$

The projections $P^1$, $P^2$ are labeled to match (5.27), (5.29) from Example 5.116, and the lower right vertical arrow is also from (5.29).

THEOREM 5.133. *For $V$ with commuting CSOs $J_V$, $J'_V$, and $U$ with commuting CSOs $J_U$, $J'_U$, let $W$ be a space with four commuting CSOs $J^1_W$, $J^2_W$, $J^3_W$ $J^4_W$. If $H : W \to \mathrm{Hom}(U, V)$ satisfies $\mathrm{Hom}(J_U, Id_V) \circ H = H \circ J^1_W$ and $\mathrm{Hom}(J'_U, Id_V) \circ H = H \circ J^2_W$ and $\mathrm{Hom}(Id_U, J_V) \circ H = H \circ J^3_W$ and $\mathrm{Hom}(Id_U, J'_V) \circ H = H \circ J^4_W$, then $H$ respects the corresponding direct sums from Examples 5.131 and 5.132, and the induced maps are c-linear. If also $H$ is invertible, then the induced maps are invertible.*

PROOF. The claims for the induced maps

$$\begin{aligned} W_{c(i_1i_2)} &\to (\mathrm{Hom}(U, V))_{c(i_1i_2)} \\ W_{c(i_1i_2i_3)} &\to (\mathrm{Hom}(U, V))_{c(i_1i_2i_3)} \\ W_{c(i_1i_2)(i_3i_4)} &\to (\mathrm{Hom}(U, V))_{c(i_1i_2)(i_3i_4)} \\ \tilde{a}_3 : W_{c(1234)} &\to (\mathrm{Hom}(U, V))_{c(1234)} \end{aligned}$$

follow from Lemma 5.129. The target spaces are as in the diagram from Example 5.131, for example, $H$ induces a map $W_{c(13)(24)} \to \mathrm{Hom}'_c(U, V)$, labeled $a_3$ in the diagram below, and $a_3$ induces $\tilde{a}_3$. The claims for the induced maps

$$a_2 = \mathrm{Hom}(Q_c, Id_V) \circ H \circ Q'_{c(12)} : W_{c(12)} \to \mathrm{Hom}(U_c, V),$$

and $W_{c(123)} \to \mathrm{Hom}_c(U_c, (V, J_V))$ ($= \mathrm{Hom}_c(U_c, V)$ in the diagram from Example 5.132), and $W_{c(124)} \to \mathrm{Hom}_c(U_c, (V, J'_V)) = \mathrm{Hom}'(U_c, V)$ follow from Lemma 5.114. The map $a_2$ is c-linear with respect to the pair of CSOs induced by $J^3_W$ and $\mathrm{Hom}(Id_U, J_V)$, as mentioned in the Proof of Lemma 5.114, and is also c-linear with respect to the pair of CSOs induced by $J^4_W$ and $\mathrm{Hom}(Id_U, J'_V)$, so it satisfies the hypotheses of Lemma 5.108, and respects the corresponding direct sums in the diagram from Example 5.132. The maps induced by $a_2$ are c-linear:

$$\begin{aligned} W_{c(12)(34)} &\to \mathrm{Hom}(U_c, V_c) \\ \tilde{a}_2 : W_{c(1234)} &\to \mathrm{Hom}_c(U_c, V_c). \end{aligned}$$

In the following diagram, the left half is copied from the diagram from Example 5.132, where $a_1 = \mathrm{Hom}(Q_c, Id_V) \circ Q_{c(12)}$ induces $\tilde{a}_1$, and they are both c-linear and invertible. The space $\mathrm{Hom}'_c(U, V)$ and projections $P_{c(13)(24)}$, $P_3$ are copied from Example 5.131, and the right half of the diagram is part of the diagram from

Theorem 5.128.

$$\begin{tikzcd}
 & & \mathrm{Hom}(U,V) \arrow[dll, "P_{c(12)}"'] \arrow[d, "\mathrm{Hom}(Q_c,Id_V)"] \arrow[ddr, "P_{c(13)(24)}"] & & & W \arrow[lll, "H"'] \arrow[dll, "P'_{c(12)}"'] \arrow[dd, "P'_{c(13)(24)}"] \\
(\mathrm{Hom}(U,V))_{c(12)} \arrow[rr, "a_1"'] \arrow[dddr, "P_1"] & & \mathrm{Hom}(U_c,V) \arrow[ddd, "P_2"] & W_{c(12)} \arrow[l, "a_2"'] \arrow[ddrr, "P'_1"] & & \\
 & & & \mathrm{Hom}'_c(U,V) \arrow[ddll, "P_3"'] & & W_{c(13)(24)} \arrow[ll, "a_3"'] \arrow[d, "P'_2"] \\
 & & & & & W_{c(1234)} \arrow[dllll, "\tilde{a}_3"'] \arrow[dlll, "\tilde{a}_2"] \\
 & (\mathrm{Hom}(U,V))_{c(1234)} \arrow[r, "\tilde{a}_1"'] & \mathrm{Hom}_c(U_c,V_c) & & &
\end{tikzcd}$$

Finally, we remark that $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$; an analogous property was observed in the Proof of Lemma 5.114. The identity can be checked directly, using the c-linearity of $H$. The map $\tilde{a}_2$ acts on $w \in W_{c(1234)}$ as: $\tilde{a}_2 : w \mapsto P'_c \circ H(w) \circ Q_c$, where $P'_c$ is the projection $V \twoheadrightarrow V_c$. ∎

Exercise 5.134. Given $U$, $V$, $W$, with $\mathbf{U} = (U, J_U)$, $\mathbf{V} = (V, J_V)$, and $\mathbf{W} = (W, J)$, for any $A \in \mathrm{Hom}_c(\mathbf{U},\mathbf{V})$, the map $t_{UV}^W(A) : \mathrm{Hom}(V,W) \to \mathrm{Hom}(U,W)$ (or, more precisely, $t_{UV}^W(Q'_c(A))$, where $Q'_c$ is the inclusion of $\mathrm{Hom}_c(\mathbf{U},\mathbf{V})$ in $\mathrm{Hom}(U,V)$) is c-linear with respect to both pairs $\mathrm{Hom}(Id_V, J_W)$, $\mathrm{Hom}(Id_U, J_W)$ and $\mathrm{Hom}(J_U, Id_W)$, $\mathrm{Hom}(J_V, Id_W)$, so $t_{UV}^W(A)$ respects the direct sums and induces a c-linear map $\mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \to \mathrm{Hom}_c(\mathbf{U},\mathbf{W})$. The resulting map, denoted

$$t_{\mathbf{UV}}^{\mathbf{W}} : \mathrm{Hom}_c(\mathbf{U},\mathbf{V}) \to \mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V},\mathbf{W}), \mathrm{Hom}_c(\mathbf{U},\mathbf{W})),$$

is c-linear.

Hint. Lemma 5.65, Exercise 5.89, and Exercise 5.35 apply. The last claim can be checked directly. However, by following the diagrams from Examples 5.131 and 5.132, a little more can be obtained. Let $Q_i$ and $P_i$ denote the data for the direct sum $\mathrm{Hom}(V,W) = \mathrm{Hom}_c(\mathbf{V},\mathbf{W}) \oplus \mathrm{Hom}_a(V,W)$, so that

$$\begin{aligned} \mathrm{Hom}(Q_c, Id_{\mathrm{Hom}(U,W)}) &\quad : \\ \mathrm{Hom}(\mathrm{Hom}(V,W), \mathrm{Hom}(U,W)) &\quad \to \quad \mathrm{Hom}(\mathrm{Hom}_c(\mathbf{V},\mathbf{W}), \mathrm{Hom}(U,W)) \end{aligned}$$

is as in Example 5.132. Then

$$\mathrm{Hom}(Q_c, Id_{\mathrm{Hom}(U,W)}) \circ t_{UV}^W : \mathrm{Hom}(U,V) \to \mathrm{Hom}(\mathrm{Hom}_c(\mathbf{V},\mathbf{W}), \mathrm{Hom}(U,W))$$

is c-linear with respect to $\mathrm{Hom}(J_U, Id_V)$ and the CSO induced by

$$J^3 = \mathrm{Hom}(Id_{\mathrm{Hom}(V,W)}, \mathrm{Hom}(J_U, Id_W)),$$

and is also c-linear with respect to $\mathrm{Hom}(Id_U, J_V)$ and the CSO induced by

$$J^1 = \mathrm{Hom}(\mathrm{Hom}(J_V, Id_W), Id_{\mathrm{Hom}(U,W)}).$$

So by Lemma 5.65, it induces a c-linear map

$$\mathrm{Hom}_c(\mathbf{U},\mathbf{V}) \to \mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V},\mathbf{W}), \mathrm{Hom}(\mathbf{U}, W)).$$

As claimed above, the image of this induced map is contained in the subspace $\mathrm{Hom}_c(\mathrm{Hom}_c(\mathbf{V},\mathbf{W}),\mathrm{Hom}_c(\mathbf{U},\mathbf{W}))$. For $A\in\mathrm{Hom}_c(\mathbf{U},\mathbf{V})$ and $K\in\mathrm{Hom}_c(\mathbf{V},\mathbf{W})$,

$$(t_{UV}^W(Q_c'(A)))\circ Q_c : K\mapsto (Q_c(K))\circ(Q_c'(A)) = K\circ A\in\mathrm{Hom}_c(\mathbf{U},\mathbf{W}).$$

∎

EXAMPLE 5.135. Given $U_1$, $U_2$, $V_1$, $V_2$, and $\mathbf{U_1}=(U_1,J_{U_1})$, the canonical map (Definition 1.33)

$$j:\mathrm{Hom}(\mathbf{U_1},V_1)\otimes\mathrm{Hom}(U_2,V_2)\to\mathrm{Hom}(\mathbf{U_1}\otimes U_2,V_1\otimes V_2)$$

is c-linear with respect to the induced CSOs, by Lemma 1.37. A similar statement holds if any one of the four spaces has a CSO.

If every one of the above four spaces has a CSO, $\mathbf{U_1}=(U_1,J_{U_1})$, $\mathbf{U_2}=(U_2,J_{U_2})$, $\mathbf{V_1}=(V_1,J_{V_1})$, $\mathbf{V_2}=(V_2,J_{V_2})$, then $\mathrm{Hom}(U_1,V_1)\otimes\mathrm{Hom}(U_2,V_2)$ admits four commuting CSOs:

$$\begin{aligned}
J_1' &= [\mathrm{Hom}(J_{U_1},Id_{V_1})\otimes Id_{\mathrm{Hom}(U_2,V_2)}]\\
J_2' &= [Id_{\mathrm{Hom}(U_1,V_1)}\otimes\mathrm{Hom}(J_{U_2},Id_{V_2})]\\
J_3' &= [\mathrm{Hom}(Id_{U_1},J_{V_1})\otimes Id_{\mathrm{Hom}(U_2,V_2)}]\\
J_4' &= [Id_{\mathrm{Hom}(U_1,V_1)}\otimes\mathrm{Hom}(Id_{U_2},J_{V_2})],
\end{aligned}$$

and $\mathrm{Hom}(U_1\otimes U_2,V_1\otimes V_2)$ also admits four commuting CSOs:

$$\begin{aligned}
J_1 &= \mathrm{Hom}([J_{U_1}\otimes Id_{U_2}],Id_{V_1\otimes V_2})\\
J_2 &= \mathrm{Hom}([Id_{U_1}\otimes J_{U_2}],Id_{V_1\otimes V_2})\\
J_3 &= \mathrm{Hom}(Id_{U_1\otimes U_2},[J_{V_1}\otimes Id_{V_2}])\\
J_4 &= \mathrm{Hom}(Id_{U_1\otimes U_2},[Id_{V_1}\otimes J_{V_2}]).
\end{aligned}$$

Since $j$ is c-linear with respect to each pair $J_i'$, $J_i$, Theorem 5.133 applies, with $H=j$ and $W=\mathrm{Hom}(U_1,V_1)\otimes\mathrm{Hom}(U_2,V_2)$, so $j$ induces maps on corresponding subspaces, which are c-linear with respect to (possibly several pairs of) corresponding CSOs, and which are invertible if $j$ is. From the diagrams in Theorem 5.128 and Examples 5.130, 5.131, some induced maps from Lemma 5.129 are evident:

$$\begin{aligned}
\mathrm{Hom}(\mathbf{U_1},V_1)\otimes_c\mathrm{Hom}(U_2,\mathbf{V_2}) &\to \mathrm{Hom}_c(\mathbf{U_1}\otimes U_2,V_1\otimes\mathbf{V_2})\\
\mathrm{Hom}(U_1,\mathbf{V_1})\otimes_c\mathrm{Hom}(\mathbf{U_2},V_2) &\to \mathrm{Hom}_c(U_1\otimes\mathbf{U_2},\mathbf{V_1}\otimes V_2)\\
\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes\mathrm{Hom}(U_2,V_2) &\to \mathrm{Hom}_c(\mathbf{U_1}\otimes U_2,\mathbf{V_1}\otimes V_2)\\
\mathrm{Hom}(U_1,V_1)\otimes\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) &\to \mathrm{Hom}_c(U_1\otimes\mathbf{U_2},V_1\otimes\mathbf{V_2})\\
\mathrm{Hom}(U_1,\mathbf{V_1})\otimes_c\mathrm{Hom}(U_2,\mathbf{V_2}) &\to \mathrm{Hom}(U_1\otimes U_2,\mathbf{V_1}\otimes_c\mathbf{V_2})\\
\mathrm{Hom}(U_1,\mathbf{V_1})\otimes_c\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) &\to \mathrm{Hom}_c(U_1\otimes\mathbf{U_2},\mathbf{V_1}\otimes_c\mathbf{V_2})\\
a_3:\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) &\to \mathrm{Hom}_c'(U_1\otimes U_2,V_1\otimes V_2) \qquad (5.35)\\
\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes_c\mathrm{Hom}(U_2,\mathbf{V_2}) &\to \mathrm{Hom}_c(\mathbf{U_1}\otimes U_2,\mathbf{V_1}\otimes_c\mathbf{V_2})\\
\tilde a_3:\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes_c\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) &\to (\mathrm{Hom}(U_1\otimes U_2,V_1\otimes V_2))_{c(1234)}.
\end{aligned}$$

For example, the seventh map, labeled $a_3$ as in the diagram from Theorem 5.133, is c-linear with respect to both corresponding pairs of induced CSOs, and for c-linear maps $A$ and $B$, takes $A\otimes B$ to the map $j(A\otimes B):U_1\otimes U_2\to V_1\otimes V_2$, which is c-linear with respect to the pair $[J_{U_1}\otimes Id_{U_2}]$, $[J_{V_1}\otimes Id_{V_2}]$, and also c-linear with respect to the pair $[Id_{U_1}\otimes J_{U_2}]$, $[Id_{V_1}\otimes J_{V_2}]$.

Also, if some but not all of the four spaces have CSOs, then there may still be some induced maps, for example, the first one in the above list makes sense if only $U_1$ and $V_2$ have CSOs.

Let $Q'_{c(12)}$, $P'_{c(12)}$, and $Q_{c(12)}$, $P_{c(12)}$ denote the inclusions and projections for the direct sums produced by $J'_1$, $J'_2$, and $J_1$, $J_2$, respectively, as appearing in the diagram from the Proof of Theorem 5.133. By Lemma 5.85, the map $j$ also respects the direct sum

$$\mathrm{Hom}(\mathbf{U_1} \otimes_c \mathbf{U_2}, V_1 \otimes V_2) \oplus \mathrm{Hom}(U_1 \otimes_a U_2, V_1 \otimes V_2),$$

and induces a c-linear map, labeled

$$a_2 = \mathrm{Hom}(Q_c, Id_{V_1 \otimes V_2}) \circ j \circ Q'_{c(12)},$$

as in the following diagram, a copy of two blocks of the diagram from Theorem 5.133.

$$\begin{tikzcd}
\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2) \arrow[d, "\mathrm{Hom}(Q_c, Id_{V_1 \otimes V_2})"'] & \mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2) \arrow[l, "j"'] \arrow[d, two heads, "P'_{c(12)}"] \\
\mathrm{Hom}(\mathbf{U_1} \otimes_c \mathbf{U_2}, V_1 \otimes V_2) \arrow[d, two heads, "P_2"'] & \mathrm{Hom}(\mathbf{U_1}, V_1) \otimes_c \mathrm{Hom}(\mathbf{U_2}, V_2) \arrow[l, "a_2"'] \arrow[d, two heads, "P'_1"] \\
\mathrm{Hom}_c(\mathbf{U_1} \otimes_c \mathbf{U_2}, \mathbf{V_1} \otimes_c \mathbf{V_2}) & \mathrm{Hom}_c(\mathbf{U_1}, \mathbf{V_1}) \otimes_c \mathrm{Hom}_c(\mathbf{U_2}, \mathbf{V_2}) \arrow[l, "\tilde{a}_2"']
\end{tikzcd}$$

The map $a_2$ is equal to the composite of the induced map

$$P_{c(12)} \circ j \circ Q'_{c(12)} : \mathrm{Hom}(\mathbf{U_1}, V_1) \otimes_c \mathrm{Hom}(\mathbf{U_2}, V_2) \to (\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(12)}$$

with the invertible c-linear map from Example 5.132, labeled $a_1$ in Theorem 5.133,

$$\mathrm{Hom}(Q_c, Id_{V_1 \otimes V_2}) \circ Q_{c(12)} : (\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(12)} \to \mathrm{Hom}(\mathbf{U_1} \otimes_c \mathbf{U_2}, V_1 \otimes V_2).$$

Also, $a_2$ is c-linear with respect to all three corresponding pairs of induced CSOs, so as in Example 5.131 and Theorem 5.133, it respects the corresponding direct sums, to induce c-linear maps:

$$\begin{aligned}
\mathrm{Hom}_c(\mathbf{U_1}, \mathbf{V_1}) \otimes_c \mathrm{Hom}(\mathbf{U_2}, V_2) &\to \mathrm{Hom}_c(\mathbf{U_1} \otimes_c \mathbf{U_2}, \mathbf{V_1} \otimes V_2) \\
\mathrm{Hom}(U_1, V_1) \otimes'_c \mathrm{Hom}(U_2, V_2) &\to \mathrm{Hom}(\mathbf{U_1} \otimes_c \mathbf{U_2}, \mathbf{V_1} \otimes_c \mathbf{V_2}) \\
\mathrm{Hom}(\mathbf{U_1}, V_1) \otimes_c \mathrm{Hom}_c(\mathbf{U_2}, \mathbf{V_2}) &\to \mathrm{Hom}_c(\mathbf{U_1} \otimes_c \mathbf{U_2}, V_1 \otimes \mathbf{V_2}) \\
\tilde{a}_2 : \mathrm{Hom}_c(\mathbf{U_1}, \mathbf{V_1}) \otimes_c \mathrm{Hom}_c(\mathbf{U_2}, \mathbf{V_2}) &\to \mathrm{Hom}_c(\mathbf{U_1} \otimes_c \mathbf{U_2}, \mathbf{V_1} \otimes_c \mathbf{V_2}).
\end{aligned}$$

As remarked in the Proof of Theorem 5.133, let $\tilde{a}_1$ denote the invertible map induced by $a_1$,

$$(\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(1234)} \to \mathrm{Hom}_c(\mathbf{U_1} \otimes_c \mathbf{U_2}, \mathbf{V_1} \otimes_c \mathbf{V_2}),$$

so that then $\tilde{a}_2 = \tilde{a}_1 \circ \tilde{a}_3$. The map $\tilde{a}_2$ is invertible if $j$ is; it acts on $w \in \mathrm{Hom}_c(\mathbf{U_1}, \mathbf{V_1}) \otimes_c \mathrm{Hom}_c(\mathbf{U_2}, \mathbf{V_2}) \subseteq \mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2)$ as: $\tilde{a}_2 : w \mapsto P'_c \circ j(w) \circ Q_c$, where $P'_c$ is the projection $V_1 \otimes V_2 \twoheadrightarrow \mathbf{V_1} \otimes_c \mathbf{V_2}$.

Example 5.136. For $\mathbf{U_1}$, $\mathbf{U_2}$, $\mathbf{V_1}$, $\mathbf{V_2}$ as in Example 5.135, suppose $A : \mathbf{U_1} \to \mathbf{V_1}$ and $B : \mathbf{U_2} \to \mathbf{V_2}$ are c-linear. Then

$$A \otimes B \in \mathrm{Hom}_c(\mathbf{U_1}, \mathbf{V_1}) \otimes \mathrm{Hom}_c(\mathbf{U_2}, \mathbf{V_2}) \subseteq \mathrm{Hom}(U_1, V_1) \otimes \mathrm{Hom}(U_2, V_2),$$

so $A \otimes B$ is in the domain of $a_3 = j|_{\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2})}$ as in (5.35) and the following diagram, a copy of two blocks of the diagram from Theorem 5.133.

$$\begin{tikzcd}
\mathrm{Hom}'_c(U_1 \otimes U_2, V_1 \otimes V_2) \arrow[dd, "P_3"'] & \mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1}) \otimes \mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) \arrow[l, "a_3"'] \arrow[d, "P'_2"] \\
 & \mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1}) \otimes_c \mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2}) \arrow[dl, "\tilde{a}_3"'] \arrow[d, "\tilde{a}_2"] \\
(\mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2))_{c(1234)} \arrow[r, "\tilde{a}_1"'] & \mathrm{Hom}_c(\mathbf{U}_1 \otimes_c \mathbf{U}_2, \mathbf{V}_1 \otimes_c \mathbf{V}_2)
\end{tikzcd}$$

Starting with $A \otimes B$, these outputs are equal by the commutativity of the diagram from Theorem 5.133:

$$\begin{aligned}
& (\tilde{a}_2 \circ P'_2)(A \otimes B) = \tilde{a}_2(\frac{1}{2}(A \otimes B - (J_{V_1} \circ A) \otimes (B \circ J_{U_2}))) \\
(5.36) \quad = \quad & P'_c \circ (\frac{1}{2}([A \otimes B] - [(J_{V_1} \circ A) \otimes (B \circ J_{U_2})])) \circ Q_c, \\
& (\tilde{a}_1 \circ P_3 \circ a_3)(A \otimes B) = \tilde{a}_1(P_3([A \otimes B])) \\
= \quad & P'_c \circ (\frac{1}{2}([A \otimes B] - [J_{V_1} \otimes Id_{V_2}] \circ [A \otimes B] \circ [Id_{U_1} \otimes J_{U_2}])) \circ Q_c.
\end{aligned}$$

As in Lemma 5.80, this is the restriction of the map $\frac{1}{2}([A \otimes B] - [(J_{V_1} \circ A) \otimes (B \circ J_{U_2})]) \in \mathrm{Hom}(U_1 \otimes U_2, V_1 \otimes V_2)$ to the subspace $\mathbf{U}_1 \otimes_c \mathbf{U}_2$ in the domain and $\mathbf{V}_1 \otimes_c \mathbf{V}_2$ in the target.

For $u_1 \in U_1$, $u_2 \in U_2$, these elements of $V_1 \otimes V_2$ are equal:

$$\begin{aligned}
& ([A \otimes B] \circ Q_c)(P_c(u_1 \otimes u_2)) \\
= \quad & [A \otimes B](\frac{1}{2}(u_1 \otimes u_2 - (J_{U_1}(u_1)) \otimes (J_{U_2}(u_2)))) \\
= \quad & \frac{1}{2}((A(u_1)) \otimes (B(u_2)) - (A(J_{U_1}(u_1))) \otimes (B(J_{U_2}(u_2)))), \\
& (-[(J_{V_1} \circ A) \otimes (B \circ J_{U_2})] \circ Q_c)(P_c(u_1 \otimes u_2)) \\
= \quad & -[(J_{V_1} \circ A) \otimes (B \circ J_{U_2})](\frac{1}{2}(u_1 \otimes u_2 - (J_{U_1}(u_1)) \otimes (J_{U_2}(u_2)))) \\
= \quad & \frac{1}{2}(-(J_{V_1}(A(u_1))) \otimes (B(J_{U_2}(u_2))) + (J_{V_1}(A(J_{U_1}(u_1)))) \otimes (B(J_{U_2}(J_{U_2}(u_2))))).
\end{aligned}$$

Because $\mathbf{U_1} \otimes_c \mathbf{U_2}$ is spanned by elements of the form $P_c(u_1 \otimes u_2)$, these maps $\mathbf{U_1} \otimes_c \mathbf{U_2} \to \mathbf{V_1} \otimes_c \mathbf{V_2}$ are equal:

$$P'_c \circ (\frac{1}{2}([A \otimes B] - [(J_{V_1} \circ A) \otimes (B \circ J_{U_2})])) \circ Q_c = P'_c \circ [A \otimes B] \circ Q_c.$$

The above LHS is as in (5.36), and the RHS is from Notation 5.76, giving the equality:

$$(\tilde{a}_2 \circ P'_2)(A \otimes B) = [A \otimes_c B] = (\mathrm{Hom}(Q_c, P'_c) \circ j|_{\mathrm{Hom}_c(\mathbf{U_1},\mathbf{V_1})\otimes\mathrm{Hom}_c(\mathbf{U_2},\mathbf{V_2})})(A \otimes B).$$

There would be no ambiguity in denoting $\mathbf{j} = \tilde{a}_2$.

LEMMA 5.137. *For any* $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$, *the following diagram is commutative.*

$$\begin{tikzcd}
\mathbf{U}_c \otimes_c \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} \arrow[r, "{[Id_U \otimes_c Ev^c_{\mathbf{VW}}]}"] \arrow[d, "{[\mathbf{n} \otimes_c Id_V]}"'] & \mathbf{U} \otimes_c \mathbf{W} \\
\mathrm{Hom}_c(\mathbf{V}, \mathbf{U} \otimes_c \mathbf{W}) \otimes_c \mathbf{V} \arrow[ur, "Ev^c_{\mathbf{V}, \mathbf{U} \otimes_c \mathbf{W}}"'] &
\end{tikzcd}$$

PROOF. This is a c-linear version of Lemma 2.78, so all these objects are subspaces of spaces from Lemma 2.78; the following argument keeps track of all of the inclusions. The spaces $U \otimes \mathrm{Hom}(V, W) \otimes V$ and $\mathrm{Hom}(V, U \otimes W) \otimes V$ each admit four commuting CSOs, and the map $[n \otimes Id_V]$ is c-linear with respect to four corresponding pairs as in Lemma 5.129, by Lemma 5.29 and Theorem 5.118. Some of the projections and induced maps from Lemma 5.129 are as in the following diagram.

$$\begin{tikzcd}
U \otimes \mathrm{Hom}(V, W) \otimes V \arrow[r, "{[n \otimes Id_V]}"] \arrow[d, "P_{c(23)}"'] & \mathrm{Hom}(V, U \otimes W) \otimes V \arrow[d, "P'_{c(23)}"] \\
U \otimes \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes V \arrow[r, "{[n^1 \otimes Id_V]}"] \arrow[d, "P_{c((23)1)}"'] & \mathrm{Hom}_c(\mathbf{V}, U \otimes \mathbf{W}) \otimes V \arrow[d, "P'_{c((23)1)}"] \\
\mathbf{U} \otimes_c \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes V \arrow[r, "{[\mathbf{n} \otimes Id_V]}"] \arrow[d, "P_{c((231)4)}"'] & \mathrm{Hom}_c(\mathbf{V}, \mathbf{U} \otimes_c \mathbf{W}) \otimes V \arrow[d, "P'_{c((231)4)}"] \\
\mathbf{U} \otimes_c \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} \arrow[r, "{[\mathbf{n} \otimes_c Id_V]}"] & \mathrm{Hom}_c(\mathbf{V}, \mathbf{U} \otimes_c \mathbf{W}) \otimes_c \mathbf{V}
\end{tikzcd}$$

In the upper block, $[n^1 \otimes Id_V]$ is induced by $[n \otimes Id_V]$ as in Lemma 1.91 and Theorem 5.118, where $n$ induces $n^1$. The middle block is similarly related to the lower block from Theorem 5.118. In the lowest block, $[\mathbf{n} \otimes_c Id_V]$ is induced as in

Example 5.75, and $[\mathbf{n}\otimes_c Id_V]$ also appears in the following diagram.

$$\begin{tikzcd}
U\otimes\mathrm{Hom}(V,W)\otimes V \arrow[ddrr, "{[Id_U\otimes Ev_{VW}]}"] & & \\
U\otimes\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes V \arrow[u, hook, "Q_{c(23)}"] & & \\
U\otimes\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes_c \mathbf{V} \arrow[u, hook, "Q_{c((23)4)}"] \arrow[rr, "{[Id_U\otimes Ev^c_{\mathbf{VW}}]}"] & & U\otimes W \arrow[dl, "P'_c"'] \\
\mathbf{U}\otimes_c\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes_c \mathbf{V} \arrow[u, hook, "Q_{c((234)1)}"] \arrow[r, "{[Id_U\otimes_c Ev^c_{\mathbf{VW}}]}"] \arrow[d, "{[\mathbf{n}\otimes_c Id_V]}"'] & \mathbf{U}\otimes_c\mathbf{W} & \\
\mathrm{Hom}_c(\mathbf{V},\mathbf{U}\otimes_c\mathbf{W})\otimes_c \mathbf{V} \arrow[ur, "Ev^c_{\mathbf{V},\mathbf{U}\otimes_c\mathbf{W}}"'] \arrow[d, hook, "Q'_{c((132)4)}"'] & & \\
\mathrm{Hom}_c(\mathbf{V},\mathbf{U}\otimes_c\mathbf{W})\otimes V \arrow[d, hook, "Q'_{c((13)2)}"'] & & \\
\mathrm{Hom}(V,\mathbf{U}\otimes_c\mathbf{W})\otimes V \arrow[uuur, "Ev_{V,\mathbf{U}\otimes_c\mathbf{W}}"] & & \\
\mathrm{Hom}(V,U\otimes W)\otimes V \arrow[u, "P'_{c(13)}"] \arrow[uuuuurr, "Ev_{V,U\otimes W}"'] & &
\end{tikzcd}$$

In the upper block, $Q_{c(23)}$ is the inclusion corresponding to the projection $P_{c(23)}$ from the previous diagram, and as in Example 5.73, $Q_{c(23)}=[Id_U\otimes[Q_c\otimes Id_V]]$ for $Q_c:\mathrm{Hom}(\mathbf{V},\mathbf{W})\hookrightarrow\mathrm{Hom}(V,W)$. The inclusion $Q_{c((23)4)}$ does not correspond to any projection from the previous diagram, but again as in Example 5.73, it is equal to $[Id_U\otimes\tilde{Q}]$, for the inclusion $\tilde{Q}:\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes_c\mathbf{V}\hookrightarrow\mathrm{Hom}_c(\mathbf{V},\mathbf{W})\otimes V$. The upper block is then commutative by Lemma 1.36 and adapting formula (5.33) from Example 5.124, $Ev^c_{\mathbf{VW}}=Ev_{VW}\circ[Q_c\otimes Id_V]\circ\tilde{Q}$. The second block is commutative by Notation 5.76 for the induced map $[Id_U\otimes_c Ev^c_{\mathbf{VW}}]$. The commutativity of the center block is the claim of the Lemma. The fourth block is commutative, it is an adaptation of formula (5.33) defining $Ev^c_{\mathbf{V},\mathbf{U}\otimes_c\mathbf{W}}$. The inclusion $Q'_{c((132)4)}$ is equal to the inclusion with the first three indices re-ordered, $Q'_{c((231)4)}$, corresponding to the projection in the first diagram. In the lowest block, $P'_{c(13)}$ does not appear in the first diagram; it is equal to $[\mathrm{Hom}(Id_V,P'_c)\otimes Id_V]$ as in Example 5.73 and Example 5.83. So, the lowest block is commutative by Lemma 2.73. The claim follows, using Lemma 2.78 and equalities of composite inclusions from Theorem

5.128.

$$\begin{aligned}
& Ev^{c}_{\mathbf{V},\mathbf{U}\otimes_c\mathbf{W}} \circ [\mathbf{n}\otimes_c Id_V] \\
= \ & Ev_{V,\mathbf{U}\otimes_c\mathbf{W}} \circ Q'_{c((13)2)} \circ Q'_{c((132)4)} \circ [\mathbf{n}\otimes_c Id_V] \\
= \ & Ev_{V,\mathbf{U}\otimes_c\mathbf{W}} \circ P'_{c(13)} \circ Q'_{c(13)} \circ Q'_{c((13)2)} \circ Q'_{c((132)4)} \circ [\mathbf{n}\otimes_c Id_V] \\
= \ & P'_c \circ Ev_{V,U\otimes W} \circ Q'_{c(23)} \circ Q'_{c((23)1)} \circ Q'_{c((231)4)} \circ [\mathbf{n}\otimes_c Id_V] \\
= \ & P'_c \circ Ev_{V,U\otimes W} \circ [n\otimes Id_V] \circ Q_{c(23)} \circ Q_{c((23)1)} \circ Q_{c((231)4)} \\
= \ & P'_c \circ [Id_U \otimes Ev_{VW}] \circ Q_{c(23)} \circ Q_{c((23)4)} \circ Q_{c((234)1)} \\
= \ & [Id_U \otimes_c Ev^{c}_{\mathbf{VW}}].
\end{aligned}$$

∎

THEOREM 5.138. *For any $\mathbf{V}=(V,J_V)$, $\mathbf{U}$, $\mathbf{W}$, and c-linear $F:\mathbf{V}\otimes_c\mathbf{U}\to\mathbf{V}\otimes_c\mathbf{W}$, if $V$ is finite-dimensional then the $\mathbf{n}$ maps in the following diagram are invertible:*

$$\begin{tikzcd}
\mathbf{V}\otimes_c\mathbf{U} \arrow[rr, "F"] & & \mathbf{V}\otimes_c\mathbf{W} \\
\mathbf{V}\otimes_c \mathrm{Hom}_c(\mathbf{V},\mathbf{U})\otimes_c\mathbf{V} \arrow[u, "{[Id_V\otimes_c Ev^c_{\mathbf{VU}}]}"] \arrow[d, "{[\mathbf{n}_2\otimes_c Id_V]}"'] & & \mathbf{V}\otimes_c \mathrm{Hom}(\mathbf{V},\mathbf{W})\otimes_c\mathbf{V} \arrow[u, "{[Id_V\otimes_c Ev^c_{\mathbf{VW}}]}"'] \arrow[d, "{[\mathbf{n}'_2\otimes_c Id_V]}"] \\
\mathrm{Hom}(\mathbf{V},\mathbf{V}\otimes_c\mathbf{U})\otimes_c\mathbf{V} \arrow[rr, "{[\mathrm{Hom}_c(Id_V,F)\otimes_c Id_V]}"'] & & \mathrm{Hom}(\mathbf{V},\mathbf{V}\otimes_c\mathbf{W})\otimes_c\mathbf{V}
\end{tikzcd}$$

*and the diagram is commutative, in the sense that*

$$\begin{aligned}
& F\circ[Id_V\otimes_c Ev^c_{\mathbf{VU}}]\circ[\mathbf{n}_2\otimes_c Id_V]^{-1} \\
= \ & [Id_V\otimes_c Ev^c_{\mathbf{VW}}]\circ[\mathbf{n}'_2\otimes_c Id_V]^{-1}\circ[\mathrm{Hom}_c(Id_V,F)\otimes_c Id_V].
\end{aligned}$$

PROOF. This is a c-linear version of Theorem 2.79, and the Proof is analogous. The maps $\mathbf{n}_2$ and $\mathbf{n}'_2$ are special cases of the $\mathbf{n}$ map from Lemma 5.137; they are invertible by Lemma 1.44 and Theorem 5.118, with $[\mathbf{n}_2\otimes_c Id_V]^{-1}=[\mathbf{n}_2^{-1}\otimes_c Id_V]$.

By Lemma 5.137, the upward composite on the left is equal to $Ev^c_{\mathbf{V},\mathbf{V}\otimes_c\mathbf{U}}$, and similarly the upward composite on the right is equal to $Ev^c_{\mathbf{V},\mathbf{V}\otimes_c\mathbf{W}}$. The claim then follows from Lemma 5.125. ∎

REMARK 5.139. The results in this Section on the c-linear evaluation map from Example 5.124: Lemma 5.125, Lemma 5.137, and Theorem 5.138, give some details omitted from [$\mathbf{C}_2$] §4.

## 5.4. Real trace with complex vector values

In this Section we develop the notion of vector valued trace of $\mathbb{R}$-linear maps, where the value spaces have complex structure operators. The approach will be to refer to Chapter 2, while avoiding scalar multiplication.

Theorem 5.118 on the c-linearity of $n$ maps generalizes in a straightforward way to the various orderings of $n$ maps from Notation 1.41, as in the following Corollary. Recall from Theorem 2.74 the special case

$$n':\mathrm{Hom}(V,W)\otimes V\to\mathrm{Hom}(V,V\otimes W): A\otimes v\mapsto(u\mapsto v\otimes(A(u))),$$

and that if $V$ is finite-dimensional then $n'$ is invertible.

COROLLARY 5.140. *If* $\mathbf{V} = (V, J_V)$ *and* $\mathbf{W} = (W, J_W)$*, then* $n' : \mathrm{Hom}(V, W) \otimes V \to \mathrm{Hom}(V, V \otimes W)$ *is c-linear with respect to corresponding pairs of the three commuting CSOs induced on each space, so it respects the direct sums and induces maps*

$$\begin{aligned} n_1' : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes V &\to \mathrm{Hom}_c(\mathbf{V}, V \otimes \mathbf{W}) \\ n_2' : \mathrm{Hom}(\mathbf{V}, W) \otimes_c \mathbf{V} &\to \mathrm{Hom}_c(\mathbf{V}, \mathbf{V} \otimes W) \\ n_3' : \mathrm{Hom}(V, \mathbf{W}) \otimes_c \mathbf{V} &\to \mathrm{Hom}(V, \mathbf{V} \otimes_c \mathbf{W}) \\ \mathbf{n}' : \mathrm{Hom}_c(\mathbf{V}, \mathbf{W}) \otimes_c \mathbf{V} &\to \mathrm{Hom}_c(\mathbf{V}, \mathbf{V} \otimes_c \mathbf{W}). \end{aligned}$$

*If* $n'$ *is invertible then so are these maps.* ∎

THEOREM 5.141. *If* $V$ *is finite-dimensional and* $\mathbf{W} = (W, J_W)$*, then the map*

$$Tr_{V;W} = Ev_{VW} \circ (n')^{-1} : \mathrm{Hom}(V, V \otimes \mathbf{W}) \to \mathbf{W}$$

*is c-linear.*

PROOF. The map $n'$ is from Corollary 5.140: it is c-linear with respect to $[\mathrm{Hom}(Id_V, J_W) \otimes Id_V]$ and $\mathrm{Hom}(Id_V, [Id_V \otimes J_W])$ (without assuming any CSO on $V$). The canonical evaluation $Ev_{VW} : A \otimes v \mapsto A(v)$ from Definition 2.71 is c-linear $\mathrm{Hom}(V, \mathbf{W}) \otimes V \to \mathbf{W}$ as in Exercise 5.41, and the equality $Tr_{V;W} \circ n' = Ev_{VW}$ is from Theorem 2.74. The result could also be proved by applying Corollary 2.59 (or Corollary 2.75) with $B = J_W$. ∎

THEOREM 5.142. *If* $V$ *is finite-dimensional and* $W$ *admits commuting CSOs* $J_1$*,* $J_2$*, then the map* $Tr_{V;W}$ *respects the direct sums*

$$\mathrm{Hom}(V, V \otimes W_c) \oplus \mathrm{Hom}(V, V \otimes W_a) \to W_c \oplus W_a,$$

*and the induced c-linear map* $\mathrm{Hom}(V, V \otimes W_c) \to W_c$ *is equal to* $Tr_{V;W_c}$*.*

PROOF. Lemma 2.61 applies. The direct sums on $V \otimes W$ and $\mathrm{Hom}(V, V \otimes W)$ are as in Example 5.73 and Example 5.83, with canonical projections as indicated in the diagram. $Tr_{V;W}$ is c-linear with respect to both corresponding pairs of CSOs by Theorem 5.141, and the c-linearity of the induced map follows from Lemma 5.65.

$$\begin{tikzcd} \mathrm{Hom}(V, V \otimes W) \arrow[r, "Tr_{V;W}"] \arrow[d, "\mathrm{Hom}(Id_V, [Id_V \otimes P_c])"'] & W \arrow[d, "P_c"] \\ \mathrm{Hom}(V, V \otimes W_c) \arrow[r, "Tr_{V;W_c}"] & W_c \end{tikzcd}$$

∎

THEOREM 5.143. *For finite-dimensional* $V$*, and* $U$*,* $W$ *with CSOs* $J_U$*,* $J_W$*, the generalized trace*

$$Tr_{V;U,W} : \mathrm{Hom}(V \otimes U, V \otimes W) \to \mathrm{Hom}(U, W)$$

*is c-linear with respect to both pairs of corresponding commuting CSOs, and respects the direct sums, inducing a c-linear map, denoted*

$$Tr_{V;\mathbf{U},\mathbf{W}} : \mathrm{Hom}_c(V \otimes \mathbf{U}, V \otimes \mathbf{W}) \to \mathrm{Hom}_c(\mathbf{U}, \mathbf{W}).$$

Proof. The c-linearity claims follow from Theorem 2.30, and then Lemma 5.65 applies. That is enough for the Proof, but as in Theorem 2.52 and Theorem 2.53, the generalized trace is related to some different vector valued traces.

First, following the construction of Theorem 2.52, consider this diagram,

$$\begin{tikzcd}
\mathrm{Hom}(V\otimes U, V\otimes W) \arrow[dd, two heads] \arrow[dr, shift left, "q^{-1}"] \arrow[drrr, "Tr_{V;U,W}"] & & & \\
& M_{11} \arrow[ul, shift left, "q"] \arrow[r, shift left, "\mathrm{Hom}(Id_V, n^{-1})"] \arrow[d, two heads] & M_{12} \arrow[l, shift left, "\mathrm{Hom}(Id_V, n)"] \arrow[r, "Tr_{V;\mathrm{Hom}(U,W)}"'] \arrow[d, two heads] & \mathrm{Hom}(U,W) \arrow[d, two heads, "P_H"] \\
\mathrm{Hom}_c(V\otimes \mathbf{U}, V\otimes \mathbf{W}) \arrow[r, shift left] & M_{21} \arrow[l, shift left, "q_1"] \arrow[r, shift left] & M_{22} \arrow[l, shift left] \arrow[r, "Tr_{V;\mathrm{Hom}_c(\mathbf{U},\mathbf{W})}"'] & \mathrm{Hom}_c(\mathbf{U},\mathbf{W})
\end{tikzcd}$$

where

$$\begin{aligned}
M_{11} &= \mathrm{Hom}(V, \mathrm{Hom}(U, V\otimes W)) \\
M_{12} &= \mathrm{Hom}(V, V\otimes \mathrm{Hom}(U, W)) \\
M_{21} &= \mathrm{Hom}(V, \mathrm{Hom}_c(\mathbf{U}, V\otimes \mathbf{W})) \\
M_{22} &= \mathrm{Hom}(V, V\otimes \mathrm{Hom}_c(\mathbf{U}, \mathbf{W})).
\end{aligned}$$

All the vertical arrows are the canonical projections of the direct sums produced by the commuting CSOs induced by $J_U$ and $J_W$. The right square is commutative by Lemma 2.61; this is an example of Theorem 5.142, where the projection $M_{12} \twoheadrightarrow M_{22}$ is equal to $\mathrm{Hom}(Id_V, [Id_V \otimes P_H])$. The map $n : V \otimes \mathrm{Hom}(U, W) \to \mathrm{Hom}(U, V \otimes W)$ from Theorem 5.118 is invertible and c-linear with respect to the commuting corresponding pairs of CSOs induced by $J_U$ and $J_W$. Example 5.88 applies to $\mathrm{Hom}(Id_V, n)$ and the middle square in the diagram: the induced map (lower middle arrow) is invertible, c-linear, and equal to $\mathrm{Hom}(Id_V, n^1)$, where $n^1$ is the induced map from Theorem 5.118. The map $q$ is as in Theorem 2.52, which asserts the commutativity of the diagram's top triangle. By Example 5.119, the map $q$ similarly induces an invertible c-linear map, $q_1$. We can conclude, for $K : V \to V \otimes \mathrm{Hom}_c(\mathbf{U}, \mathbf{W})$,

$$Tr_{V;\mathrm{Hom}_c(\mathbf{U},\mathbf{W})}(K) = Tr_{V;\mathbf{U},\mathbf{W}}(q_1(n^1 \circ K)).$$

The next diagram has the same left and right columns as the previous one. The upper triangle is commutative by Theorem 2.53, where the map from (2.10) is temporarily relabeled $\tilde{q}$.

$$\begin{tikzcd}
\mathrm{Hom}(V\otimes U, V\otimes W) \arrow[dd, two heads] \arrow[dr, shift left, "\tilde{q}^{-1}"] \arrow[drr, "Tr_{V;U,W}"] & & \\
& \mathrm{Hom}(U, \mathrm{Hom}(V, V\otimes W)) \arrow[ul, shift left, "\tilde{q}"] \arrow[r, "\mathrm{Hom}(Id_U, Tr_{V;W})"'] \arrow[d, two heads] & \mathrm{Hom}(U,W) \arrow[d, two heads, "P_H"] \\
\mathrm{Hom}_c(V\otimes \mathbf{U}, V\otimes \mathbf{W}) \arrow[r, shift left] & \mathrm{Hom}_c(\mathbf{U}, \mathrm{Hom}(V, V\otimes \mathbf{W})) \arrow[l, shift left, "q_2"] \arrow[r, "\mathrm{Hom}_c(Id_U, Tr_{V,W})"] & \mathrm{Hom}_c(\mathbf{U},\mathbf{W})
\end{tikzcd}$$

The map $\tilde{q}$ is c-linear with respect to the pairs of induced CSOs and by Example 5.119 again, induces an invertible c-linear map, $q_2$, as indicated in the diagram. In the right square, $Tr_{V;W}$ is c-linear as in Theorem 5.141, and $\mathrm{Hom}(Id_U, Tr_{V;W})$ is c-linear with respect to the pairs of CSOs as in Lemma 5.80, inducing a c-linear map $\mathrm{Hom}_c(Id_U, Tr_{V;W})$ as in Notation 5.81. The conclusion is an analogue of Equation (2.11) from Theorem 2.53,

$$Tr_{V;\mathbf{U},\mathbf{W}} = \mathrm{Hom}_c(Id_U, Tr_{V;W}) \circ q_2^{-1},$$

so that for a c-linear map $F : V \otimes \mathbf{U} \to V \otimes \mathbf{W}$,

$$\begin{aligned} Tr_{V;\mathbf{U},\mathbf{W}}(F) &= Tr_{V;W} \circ (q_2^{-1}(F)) \\ &= Ev_{VW} \circ (n')^{-1} \circ (q_2^{-1}(F)). \end{aligned} \tag{5.37}$$

Line (5.37) gives an analogue of Corollary 2.76, using the c-linear maps from the Proof of Theorem 5.141. ∎

CHAPTER 6

# Appendices

## 6.1. Appendix: Functions and Binary Operations

DEFINITION 6.1. Given sets $S$ and $T$, the product set $S \times T$ is the set of ordered pairs $\{(\sigma, \tau) : \sigma \in S, \tau \in T\}$.

DEFINITION 6.2. Given sets $S$ and $T$, suppose there is a subset $G \subseteq S \times T$ with the following properties.

- If $(s_1, t_1) \in G$ and $(s_2, t_2) \in G$ and $s_1 = s_2$ then $t_1 = t_2$.
- For each $s \in S$, there is an element $(s, t) \in G$.

Then for each $s \in S$, there is exactly one element $\alpha(s) \in T$ so that $(s, \alpha(s)) \in G$. This defines a function $\alpha$, with domain $S$, target $T$, and graph $G$, which is denoted (as in Notation 0.41) by the arrow notation $\alpha : S \rightsquigarrow T$.

DEFINITION 6.3. Given a set $S$, a binary operation on $S$ is any function from $S \times S$ to $S$. The notation $(S, *)$ denotes a set $S$, together with $*$, a binary operation on $S$. For $x, y \in S$, and a binary operation $*$, the element $*((x, y))$ will be abbreviated $x * y$.

DEFINITION 6.4. A binary operation $*$ on $S$ is associative means: for all $x, y, z \in S$, $(x * y) * z = x * (y * z)$. The binary operation $*$ is commutative means: for all $x, y \in S$, $x * y = y * x$.

DEFINITION 6.5. Given $(S, *)$, any element $e \in S$ such that $e * x = x * e = x$ for all $x \in S$ is called an identity element.

EXERCISE 6.6. Given $(S, *)$, suppose there is an identity element $e \in S$. Then, the identity element is unique. $\blacksquare$

EXERCISE 6.7. Given $(S, *)$, with an identity element $e$, if for all $x, y, z \in S$, $x * (y * z) = (x * z) * y$, then $*$ is commutative and associative. $\blacksquare$

EXERCISE 6.8. Give an example of a set and an operation $*$ where $x * (y * z) = (x * z) * y$ holds but $*$ is not associative. $\blacksquare$

DEFINITION 6.9. Given $(S, *)$, and an identity element $e \in S$, and $x, y \in S$, $y$ is a $*$-inverse for $x$ means that $x * y = y * x = e$.

Note that $*$-inverse cannot be defined without an identity element, so in any statement asserting the existence of a $*$-inverse, it is assumed that there exists an identity element for the operation $*$.

EXERCISE 6.10. Given $(S, *)$, let $e$ be an identity element. Then $e$ has a $*$-inverse, and this inverse is unique. $\blacksquare$

EXERCISE 6.11. Given $(S, *)$, and $x \in S$, if $*$ is associative, and there exist $y \in S$, $z \in S$ such that $y * x = e$ and $x * z = e$, then $y = z$ and $y$ is a $*$-inverse for $x$. In particular, any $*$-inverse for $x$ is unique. $\blacksquare$

NOTATION 6.12. Usually it is more convenient to call a $*$-inverse just an "inverse," and if an element $x$ has a unique inverse, then it can be denoted $x^{-1}$. There may be some other abbreviations for certain operations; customarily a $+$-inverse of $x$ is denoted $-x$.

EXERCISE 6.13. Given $(S, *)$, and $x, y \in S$, if $*$ is associative, and $x$ and $y$ both have $*$-inverses, then $x * y$ has a unique $*$-inverse, $y^{-1} * x^{-1}$. $\blacksquare$

EXERCISE 6.14. Given $(S, *)$, and $x \in S$, if $*$ is associative, and there exists a $*$-inverse for $x$, and $x * x = x$, then $x = e$. $\blacksquare$

EXAMPLE 6.15. Given a set $S$, let $\mathcal{F}$ denote the set of functions $\{\alpha : S \rightsquigarrow S\}$. Composition of functions is an example of a binary operation on $\mathcal{F}$, denoted $\circ$, so that the function $\alpha \circ \beta$ is defined by the formula depending on $x \in S$:

$$(\alpha \circ \beta)(x) = \alpha(\beta(x)). \tag{6.1}$$

The operation $\circ$ is associative and has an identity element denoted $Id_S \in \mathcal{F}$, which is the function with graph $G = \{(x, x) : x \in S\}$, so that $Id_S(x) = x$ for all $x \in S$.

NOTATION 6.16. The same symbol $\circ$ is used for composites of functions between other sets, although this is no longer an example of a binary operation as in Definition 6.3. For any sets $S$, $T$, $U$, and any functions $\beta : S \rightsquigarrow T$ and $\alpha : T \rightsquigarrow U$, there is a composite function $\alpha \circ \beta : S \rightsquigarrow U$, defined as in (6.1). An associative property holds: for $\gamma : R \rightsquigarrow S$, $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$.

EXERCISE 6.17. Given $S \neq \emptyset$, and a function $\alpha : S \rightsquigarrow T$, the following are equivalent.

(1) For all $s_1$, $s_2 \in S$, if $s_1 \neq s_2$, then $\alpha(s_1) \neq \alpha(s_2)$ ($\alpha$ has the one-to-one property).
(2) For any set $C$ and any functions $\gamma : C \rightsquigarrow S$, $\delta : C \rightsquigarrow S$, if $\alpha \circ \gamma : C \rightsquigarrow T$ and $\alpha \circ \delta : C \rightsquigarrow T$ are the same function, then $\gamma = \delta$ ($\alpha$ has the left cancellable property).
(3) There is a function $\beta : T \rightsquigarrow S$ so that $\beta \circ \alpha : S \rightsquigarrow S$ is equal to the identity function $Id_S : S \rightsquigarrow S$ ($\alpha$ has a left inverse).

$\blacksquare$

EXERCISE 6.18. Given a function $\alpha : S \rightsquigarrow T$, the following are equivalent.

(1) For all $t \in T$, there is some $s \in S$ so that $\alpha(s) = t$ ($\alpha$ has the onto property).
(2) For any set $C$ and any functions $\gamma : T \rightsquigarrow C$, $\delta : T \rightsquigarrow C$, if $\gamma \circ \alpha : S \rightsquigarrow C$ and $\delta \circ \alpha : S \rightsquigarrow C$ are the same function, then $\gamma = \delta$ ($\alpha$ has the right cancellable property).
(3) There is a function $\beta : T \rightsquigarrow S$ so that $\alpha \circ \beta : T \rightsquigarrow T$ is equal to the identity function $Id_T : T \rightsquigarrow T$ ($\alpha$ has a right inverse).

HINT. The (1) $\Longrightarrow$ (3) step requires the Axiom of Choice. $\blacksquare$

EXERCISE 6.19. If $\beta : T \rightsquigarrow S$ is a left inverse of $\alpha$ and $\gamma : T \rightsquigarrow S$ is a right inverse of $\alpha$ then $\beta = \gamma$.

HINT. This is analogous to Exercise 6.11. ∎

EXERCISE 6.20. Given $S \neq \emptyset$ and a function $\alpha : S \rightsquigarrow T$, the following are equivalent.

(1) $\alpha$ is both one-to-one and onto.
(2) $\alpha$ has a left inverse $\gamma : T \rightsquigarrow S$ and a right inverse $\beta : T \rightsquigarrow S$.
(3) There exists a function $\delta : T \rightsquigarrow S$ so that $\alpha \circ \delta = Id_T$ and $\delta \circ \alpha = Id_S$.

HINT. The equivalence (1) $\iff$ (2) uses Exercise 6.17 and Exercise 6.18, although with the one-to-one assumption in (1), the Axiom of Choice is no longer required to construct the right inverse in (2). (3) $\implies$ (2) is trivial, and the converse uses Exercise 6.19 to get $\delta = \beta = \gamma$. ∎

DEFINITION 6.21. A function $\alpha : S \rightsquigarrow T$ is <u>invertible</u> means that $\alpha$ satisfies any of the equivalent properties (1), (2), or (3) from Exercise 6.20.

NOTATION 6.22. A function (such as $\delta = \beta = \gamma$ as in Exercise 6.19), that is both a left inverse and a right inverse of the function $\alpha : S \rightsquigarrow T$ is an <u>inverse</u> of $\alpha$. If an inverse of $\alpha$ exists, then it is unique by Exercise 6.19, it can be denoted $\alpha^{-1} : T \rightsquigarrow U$, and $\alpha^{-1}$ is also invertible, with inverse $\alpha$.

### 6.2. Appendix: Quotient spaces

DEFINITION 6.23. Given any vector space $(V, +_V, \cdot_V)$ and a subspace $W$, for any element $v \in V$ the following subset of $V$ is called a coset of $W$:

$$v + W = \{v +_V w : w \in W\}.$$

DEFINITION 6.24. Given a subspace $W$ of $V$ as in Definition 6.23, the set of cosets of $W$ is a vector space with the following operations:

$$\begin{aligned}(v_1 + W) + (v_2 + W) &= (v_1 +_V v_2) + W, \\ \rho \cdot (v + W) &= (\rho \cdot_V v) + W,\end{aligned}$$

and zero element $0_V + W = W$. This vector space is the quotient space, denoted $V/W$.

EXERCISE 6.25. The function $\boldsymbol{\pi} : V \to V/W : v \mapsto v + W$ is linear and right cancellable. If $S$ is a subset of $V$ such that $V = \operatorname{span}(S)$, then $V/W = \operatorname{span}(\boldsymbol{\pi}(S))$. ■

EXERCISE 6.26. Given a subspace $W$ of $V$ as in Definition 6.23, and any set $S$, if $B : V \rightsquigarrow S$ is constant on each coset $v + W$ then there exists a unique function $b : V/W \rightsquigarrow S$ such that $b \circ \boldsymbol{\pi} = B$.

HINT. For $v + W \in V/W$, choose any element $x = v + w \in v + W$. Define $b(v + W) = B(x)$; this does not depend on the choice of $x$ by hypothesis, and $v \in v + W$ (because $0_V \in W$) so $b(v + W) = B(x) = B(v)$. Then for any $v \in V$, $(b \circ \boldsymbol{\pi})(v) = b(\boldsymbol{\pi}(v)) = b(v + W) = B(x) = B(v)$ as claimed. For uniqueness, use the right cancellable property of $\boldsymbol{\pi}$. ■

EXERCISE 6.27. Given a subspace $W$ of $V$ as in Definition 6.23, and another vector space $U$, if $B : V \to U$ is linear and satisfies $B(w) = 0_U$ for all $w \in W$, then there exists a unique function $b : V/W \to U$ such that $b \circ \boldsymbol{\pi} = B$, and $b$ is linear.

HINT. For any $v \in V$, if $x \in v + W$ then $x = v + w$ for some $w \in W$ and $B(x) = B(v + w) = B(v) + B(w) = B(v) + 0_U = B(v)$. So, $B$ is constant on the coset $v + W$, and the previous Exercise applies to show there is a unique $b$ with $b \circ \boldsymbol{\pi} = B$. The linearity of $b$ easily follows from the linearity of $B$. ■

## 6.3. Appendix: Construction of the tensor product

As mentioned in Section 1.2, we elaborate on the existence of a tensor product of two vector spaces. The notation and methods in this Appendix are specific to this construction and not widely used in the Chapters. We start with a set of functions $\mathcal{F}(S, W)$ only because it comes with a convenient vector space structure.

EXAMPLE 6.28. For any set $S \neq \emptyset$ and any vector space $W$, the set of functions

$$\mathcal{F}(S, W) = \{f : S \rightsquigarrow W\}$$

is a vector space, with the usual operations of pointwise addition of functions and scalar multiplication of functions, and zero element given by the constant function $f(x) \equiv 0_W$.

NOTATION 6.29. For a set $S \neq \emptyset$ and any field $\mathbb{K}$, for each $x \in S$ there is an element $\boldsymbol{\delta}(x) \in \mathcal{F}(S, \mathbb{K})$ defined by:

$$\begin{aligned} \boldsymbol{\delta}(x) : y &\mapsto 1 \text{ for } y = x, \\ \boldsymbol{\delta}(x) : y &\mapsto 0 \text{ for } y \neq x. \end{aligned} \tag{6.2}$$

The span of the set of such functions is denoted

$$\mathcal{F}_0(S, \mathbb{K}) = \operatorname{span}\left(\{\boldsymbol{\delta}(x) : x \in S\}\right),$$

so $\mathcal{F}_0(S, \mathbb{K})$ is a subspace of $\mathcal{F}(S, \mathbb{K})$. The notation (6.2) defines a function

$$\boldsymbol{\delta} : S \rightsquigarrow \mathcal{F}_0(S, \mathbb{K}) : x \mapsto \boldsymbol{\delta}(x).$$

EXERCISE 6.30. For $S$ as above and any function $f : S \rightsquigarrow \mathbb{K}$, the following are equivalent.

(1) $f \in \mathcal{F}_0(S, \mathbb{K})$.
(2) The function $f$ is uniquely expressible as a finite sum of functions with coefficients $\alpha_\nu \in \mathbb{K}$ and $x_\nu \in S$: $f = \sum_{\nu=1}^{N} \alpha_\nu \cdot \boldsymbol{\delta}(x_\nu)$.
(3) $f(x) = 0$ for all but finitely many $x \in S$.

■

EXERCISE 6.31. Given $S$, $\mathbb{K}$, and $\boldsymbol{\delta}$ as in Notation 6.29, and any vector space $W$, If $g : S \rightsquigarrow W$ is any function, then there exists a unique linear map $\bar{g} : \mathcal{F}_0(S, \mathbb{K}) \to W$ such that $\bar{g} \circ \boldsymbol{\delta} = g : S \rightsquigarrow W$.

HINT. As in Exercise 6.30, the general element $f$ of $\mathcal{F}_0(S, \mathbb{K})$ has a unique expression of the form $f = \sum_{\nu=1}^{N} \alpha_\nu \cdot \boldsymbol{\delta}(x_\nu)$; define $\bar{g}$ on such an expression by using the same coefficients $\alpha_\nu$ and elements $x_\nu$:

$$\bar{g}(f) = \sum_{\nu=1}^{N} \alpha_\nu \cdot g(x_\nu).$$

Then for each $x \in S$, the composite function satisfies the claim:

$$(\bar{g} \circ \boldsymbol{\delta})(x) = \bar{g}(1 \cdot \boldsymbol{\delta}(x)) = 1 \cdot g(x) = g(x).$$

The uniqueness and $\mathbb{K}$-linearity of $\bar{g}$ are easily checked. ■

Definition 6.32. Given vector spaces $U$ and $V$, define the following subsets of $\mathcal{F}_0(U \times V, \mathbb{K})$:

$$\begin{aligned} R_1 &= \{\boldsymbol{\delta}((u_1 + u_2, v)) - \boldsymbol{\delta}((u_1, v)) - \boldsymbol{\delta}((u_2, v)) : u_1, u_2 \in U, v \in V\} \\ R_2 &= \{\boldsymbol{\delta}((u, v_1 + v_2)) - \boldsymbol{\delta}((u, v_1)) - \boldsymbol{\delta}((u, v_2)) : u \in U, v_1, v_2 \in V\} \\ R_3 &= \{\boldsymbol{\delta}((\rho \cdot u, v)) - \rho \cdot \boldsymbol{\delta}((u, v)) : \rho \in \mathbb{K}, u \in U, v \in V\} \\ R_4 &= \{\boldsymbol{\delta}((u, \rho \cdot v)) - \rho \cdot \boldsymbol{\delta}((u, v)) : \rho \in \mathbb{K}, u \in U, v \in V\} \\ R &= \operatorname{span}(R_1 \cup R_2 \cup R_3 \cup R_4). \end{aligned}$$

The tensor product space of $U$ and $V$ is defined to be the quotient space:

$$U \otimes V = \mathcal{F}_0(U \times V, \mathbb{K})/R.$$

Let $\boldsymbol{\pi} : \mathcal{F}_0(U \times V, \mathbb{K}) \to U \otimes V$ denote the quotient map as in Exercise 6.25.

Definition 6.33. Define a function $\boldsymbol{\tau} : U \times V \rightsquigarrow U \otimes V$ by:

$$\boldsymbol{\tau} = \boldsymbol{\pi} \circ \boldsymbol{\delta} : (u, v) \mapsto \boldsymbol{\pi}(\boldsymbol{\delta}((u, v))) = \boldsymbol{\delta}((u, v)) + R.$$

The output $\boldsymbol{\tau}((u, v))$ is abbreviated $u \otimes v \in U \otimes V$.

Exercise 6.34. $\boldsymbol{\tau} : U \times V \rightsquigarrow U \otimes V$ is a bilinear function.

Hint. Definition 1.23 is easily checked. ∎

Theorem 6.35. *For any bilinear function $A : U \times V \rightsquigarrow W$, there exists a unique linear map $a : U \otimes V \to W$ such that $A = a \circ \boldsymbol{\tau}$.*

Proof. By Exercise 6.31, there exists a unique linear map $\bar{A} : \mathcal{F}_0(U \times V, \mathbb{K}) \to W$ such that $\bar{A} \circ \boldsymbol{\delta} = A$. The linear map $\bar{A}$ has value $0_W$ on every element of the subspace $R$; it is enough to check that $\bar{A}(r) = 0_W$ for $r$ in each of the four subsets $R_1, \dots, R_4$ from Definition 6.32, for example, for $r \in R_1$,

$$\begin{aligned} \bar{A}(r) &= \bar{A}(\boldsymbol{\delta}((u_1 + u_2, v)) - \boldsymbol{\delta}((u_1, v)) - \boldsymbol{\delta}((u_2, v))) \\ &= A((u_1 + u_2, v)) - A((u_1, v)) - A((u_2, v)) \\ &= 0_W. \end{aligned}$$

The other $\bar{A}(r)$ values follow similarly from the bilinear property of $A$. Exercise 6.27 applies to $\bar{A}$, to give a unique linear map

$$a : \mathcal{F}_0(U \times V, \mathbb{K})/R \to W : f + R \mapsto \bar{A}(f)$$

such that $a \circ \boldsymbol{\pi} = \bar{A}$. The conclusion is that

$$a \circ \boldsymbol{\tau} = a \circ (\boldsymbol{\pi} \circ \boldsymbol{\delta}) = (a \circ \boldsymbol{\pi}) \circ \boldsymbol{\delta} = \bar{A} \circ \boldsymbol{\delta} = A \implies a(u \otimes v) = A(u, v).$$

For the uniqueness, suppose there is some $a'$ with $a \circ \boldsymbol{\pi} \circ \boldsymbol{\delta} = A = a' \circ \boldsymbol{\pi} \circ \boldsymbol{\delta}$. The set $\{\boldsymbol{\delta}((u, v)) : u \in U, v \in V\}$ spans $\mathcal{F}_0(U \times V, \mathbb{K})$ as in Notation 6.29, and the image under $\boldsymbol{\pi}$ of this set, $\{u \otimes v : u \in U, v \in V\}$, spans $U \otimes V = \mathcal{F}_0(U \times V, \mathbb{K})/R$ by Exercise 6.25. So $a$ and $a'$ agree on a spanning set of $U \otimes V$ and must be equal. ∎

Claim 6.36. *If $(u_1, u_2)$ and $(v_1, v_2)$ are linearly independent lists of elements in $U$ and $V$, then $(u_1 \otimes v_1, u_2 \otimes v_1, u_1 \otimes v_2, u_2 \otimes v_2)$ is a linearly independent list of elements of $U \otimes V$.* ∎

## 6.4. Appendix: Comments on [$\mathbf{C}_2$]

**6.4.1. Errata.** The following typo appears in the published paper, [$\mathbf{C}_2$].

On page 535, line 3, the symbol should be $\vec{x}_{q'} \mapsto$ instead of $\vec{x}_{q'} =$.

**6.4.2. Updates.** The topic of defining a "trace without duals" (as in [$\mathbf{C}_2$] §3) is briefly considered by [**S**] §1.7.

Some details omitted from [$\mathbf{C}_2$] are presented here in Chapter 2 (see Remark 2.111) and Section 5.3 (see Remark 5.139).

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