Notes on Differential Equations and Differential Inequalities

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1 Real autonomous ODE

The following Lemma gives conditions for the existence of a solution of a differential equation which is bounded on the domain $\mathbb{R}$.

Lemma 1.1. *Given real numbers $a < b$, if $f : (a, b) \to \mathbb{R}$ is a continuous, nonvanishing function, and there are some constants $C_1 > 0$, $C_2 > 0$, $\delta_1 \in (0, b-a)$, $\delta_2 \in (0, b-a)$ so that $|f(t)| \leq C_1(t-a)$ for $a < t < a+\delta_1$ and $|f(t)| \leq C_2(b-t)$ for $b - \delta_2 < t < b$, then there exists a one-to-one, onto function $g : \mathbb{R} \to (a, b)$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$.*

Proof. $\frac{1}{f(x)}$ is continuous on (a, b), so the function

$$G(t) = \int_{\frac{a+b}{2}}^{t} \frac{1}{f(x)} dx \tag{1}$$

is differentiable on (a, b) with a nonvanishing, nonzero derivative, $\frac{d}{dt}G(t) = \frac{1}{f(t)}$. Because f and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on (a, b). Suppose $f(t) > 0$, so G is increasing; the $f(t) < 0$ case is similar.

For $t \in (b - \delta_2, b)$,

$$G(t) = \int_{\frac{a+b}{2}}^{b-\delta_2} \frac{1}{f(x)} dx + \int_{b-\delta_2}^{t} \frac{1}{f(x)} dx \geq \int_{\frac{a+b}{2}}^{b-\delta_2} \frac{1}{f(x)} dx + \int_{b-\delta_2}^{t} \frac{1}{C_2(b-x)} dx,$$

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which is unbounded. Similarly, G is also unbounded at the other endpoint, so $G : (a, b) \to \mathbb{R}$ is onto and invertible. Let T be any constant, and define

$$g(t) = G^{-1}(t + T), \tag{2}$$

so $g : \mathbb{R} \to (a, b)$ is onto and increasing, and (by the Inverse Function Theorem, [C]), $y = g(t)$ is differentiable with

$$\frac{dy}{dt} = \frac{1}{G'(G^{-1}(t+T))} = \frac{1}{\frac{1}{f(y)}} = f(y).$$

∎

Such solutions with domain $\mathbb{R}$ are unique up to translation.

Lemma 1.2. *Given an open (possibly infinite) interval I, if $f : I \to \mathbb{R}$ is a continuous, nonvanishing function, and $g_1 : \mathbb{R} \to I$ and $g_2 : \mathbb{R} \to I$ are solutions of the equation $\frac{dy}{dt} = f(y)$, then there exists a constant T so that $g_2(t) = g_1(t + T)$.*

Proof. Because $g_1'(x) = f(g_1(x))$ is continuous and nonzero, the Inverse Function Theorem applies. For $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}\left(g_1^{-1}(g_2(t)) - t\right) &= \frac{1}{g_1'(g_1^{-1}(g_2(t)))} g_2'(t) - 1 \qquad (3) \\ &= \frac{1}{f(g_1(g_1^{-1}(g_2(t))))} f(g_2(t)) - 1 \equiv 0. \end{aligned}$$

∎

There is also a local uniqueness theorem for solutions on an interval, with one initial condition.

Lemma 1.3. *Given open (possibly infinite) intervals I_0, I_1, I_2, if $f : I_0 \to \mathbb{R}$ is a continuous, nonvanishing function, and $g_1 : I_1 \to I_0$ and $g_2 : I_2 \to I_0$ are solutions of the equation $\frac{dy}{dt} = f(y)$, and there is a point $c \in I_1 \cap I_2$ such that $g_1(c) = g_2(c)$, then there exists $\delta > 0$ so that $g_1(t) = g_2(t)$ for all $t \in (c - \delta, c + \delta)$.*

Proof. Because $g_1'(x) = f(g_1(x))$ is continuous and nonzero, the Inverse Function Theorem applies: there exists some $\delta_1 > 0$ so that g_1 is one-to-one on $(c - \delta_1, c + \delta_1)$. Suppose $f > 0$, so g_1 is increasing; the $f < 0$ case is similar. Let $\varepsilon = \min\{g_1(c + \frac{1}{2}\delta_1) - g_1(c), g_1(c) - g_1(c - \frac{1}{2}\delta_1)\} > 0$. Because g_2 is continuous, there is some $\delta_2 > 0$ corresponding to ε, so that for all $t \in (c - \delta_2, c + \delta_2)$, $|g_2(t) - g_2(c)| = |g_2(t) - g_1(c)| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$, then $(c - \delta, c + \delta) \subseteq I_1 \cap I_2$, where both g_1 and g_2 are defined. Also, for any $t \in (c - \delta, c + \delta)$,

$$g_1(c - \frac{1}{2}\delta_1) \leq g_1(c) - \varepsilon < g_2(t) < g_1(c) + \varepsilon \leq g_1(c + \frac{1}{2}\delta_1),$$

and by the Intermediate Value Theorem, there is some $x \in (c - \frac{1}{2}\delta_1, c + \frac{1}{2}\delta_1)$ so that $g_1(x) = g_2(t)$; this shows that $x = g_1^{-1}(g_2(t))$, so $g_2(t)$ is in the domain of g_1^{-1}. As in (3), for $c - \delta < t < c + \delta$,

$$\frac{d}{dt}(g_1^{-1}(g_2(t))) \equiv 1 \implies g_1^{-1}(g_2(t)) = t + T$$

for some constant T. Evaluating $g_2(t) = g_1(t + T)$ at $t = c$ gives $g_2(c) = g_1(c + T)$, and $g_2(c) = g_1(c)$ by hypothesis, so $c + T = c$ because g_1 is one-to-one. It follows that $T = 0$ and $g_1(t) = g_2(t)$ for all $t \in (c - \delta, c + \delta)$. ∎

Lemma 1.4. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous, nonvanishing function, then there exist some open interval I and a one-to-one, onto function $g : I \to \mathbb{R}$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$.*

Proof. $\frac{1}{f(x)}$ is continuous on $\mathbb{R}$, so the function

$$G(t) = \int_0^t \frac{1}{f(x)} dx$$

is differentiable on $\mathbb{R}$ with a nonvanishing, nonzero derivative, $\frac{d}{dt}G(t) = \frac{1}{f(t)}$. Because f and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $\mathbb{R}$. Its image is some open interval I, so $G : \mathbb{R} \to I$ is invertible.

Let T be any constant, and define $g(t) = G^{-1}(t + T)$ on the interval $I - T = \{x \in \mathbb{R} : x + T \in I\}$, so $g : I - T \to \mathbb{R}$ is invertible, and therefore not bounded. g is a solution of the ODE as in Lemma 1.1. ∎

Lemma 1.5. *Given $b \in \mathbb{R}$, if $f : (-\infty, b) \to \mathbb{R}$ is a continuous, nonvanishing function, and there are some constants $C_3 > 0$, $\delta_3 \in (0, 1)$ so that $|f(t)| \leq C_3(b - t)$ for $b - \delta_3 < t < b$, then there exist an open interval I and a one-to-one, onto function $g : I \to (-\infty, b)$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$.*

Proof. $\frac{1}{f(x)}$ is continuous on $(-\infty, b)$, so the function

$$G(t) = \int_{b-1}^t \frac{1}{f(x)} dx$$

is differentiable on $(-\infty, b)$ with a nonvanishing, nonzero derivative, $\frac{d}{dt}G(t) = \frac{1}{f(t)}$. Because f and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $(-\infty, b)$. Suppose $f(t) > 0$, so G is increasing; the $f(t) < 0$ case is similar.

For $t \in (b - \delta_3, b)$,

$$G(t) = \int_{b-1}^{b-\delta_3} \frac{1}{f(x)} dx + \int_{b-\delta_3}^t \frac{1}{f(x)} dx \geq \int_{b-1}^{b-\delta_3} \frac{1}{f(x)} dx + \int_{b-\delta_3}^t \frac{1}{C_3(b - x)} dx,$$

which is unbounded. So, the image of G is either $I = (L, \infty)$ or $I = \mathbb{R}$.

Let T be any constant, and define $g(t) = G^{-1}(t + T)$ on the interval $I - T = \{x \in \mathbb{R} : x + T \in I\}$, so $g : I - T \to (-\infty, b)$ is invertible, and therefore not bounded. g is a solution of the ODE as in Lemma 1.1. ∎

Theorem 1.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real analytic function. The following are equivalent:*

1. *there exists a non-constant, bounded, real analytic function $g : \mathbb{R} \to \mathbb{R}$ so that $y = g(t)$ is a solution of the equation $\frac{dy}{dt} = f(y)$;*

2. *there are at least two distinct points a_0, b_0 where $f(a_0) = f(b_0) = 0$.*

Proof. At any zero of f, say c_0, the real analyticity implies there is some constant C_0 so that $|f(t)| \leq C_0|x - c_0|$ for x near c_0.

To show *2.* $\implies$ *1.*, the property that f is real analytic on $\mathbb{R}$ implies that a_0 is an isolated zero, and because a_0 is not the only zero of f, there is some open interval (a, b) with either $a = a_0$ or $b = a_0$, satisfying the hypotheses of Lemma 1.1. The conclusion is that there exists a non-constant solution $g : \mathbb{R} \to (a, b)$, which by construction, (1) and (2), is real analytic.

To show *1.* $\implies$ *2.*, suppose, toward a contradiction, that there exists a solution g_1 as claimed, and that f has fewer than two zeros.

Case 1. If f is nonvanishing on $\mathbb{R}$ then Lemma 1.4 applies and there is some nonempty open interval I and some onto function $g_2 : I \to \mathbb{R}$ which is a solution of $\frac{dy}{dt} = f(y)$. By the construction of Lemma 1.4, g_2 is real analytic. On the interval I, $g_2 - g_1$ is continuous, and because g_1 is bounded and g_2 is onto, $g_2 - g_1$ attains some negative value and some positive value, so by the Intermediate Value Theorem, there is some c so that $g_1(c) = g_2(c)$. By Lemma 1.3, $g_1(t) \equiv g_2(t)$ on some interval $(c - \delta, c + \delta)$, and because both functions are real analytic, $g_1(t) \equiv g_2(t)$ on I. The contradiction is that g_2 is unbounded on I while g_1 is bounded.

Case 2. If f has exactly one zero, $b \in \mathbb{R}$, then Lemma 1.5 applies to f on $(-\infty, b)$: there is some interval I_2 and some one-to-one, onto solution $g_2 : I_2 \to (-\infty, b)$. By an analogous existence result for f on (b, ∞), there is some interval I_3 and some one-to-one, onto solution $g_3 : I_3 \to (b, \infty)$. Because g_1 is non-constant, there is some x_0 where $g(x_0) \neq b$.

If $g_1(x_0) < b$, then there is some $T \in I_2$ with $g_2(T) = g_1(x_0)$, and the function $g_4(t) = g_2(t + T - x_0)$ is, by construction, a real analytic solution of $\frac{dy}{dt} = f(y)$ on an open interval $I_2 - (T - x_0)$ satisfying $g_4(x_0) = g_1(x_0)$. As in Case 1., the uniqueness from Lemma 1.3 shows that $g_4 = g_1$ on $I - (T - x_0)$, contradicting the boundedness of g_1.

The $g_1(x_0) > b$ case is similar, using g_3. ∎

2 Ordinary differential inequalities

2.1 Linear differential inequalities

Lemma 2.1. *If $u(t)$ is continuous on $[a, b]$ and $u(a) > 0$, then either $u(t) > 0$ for all $t \in [a, b]$, or there is some $t_1 \in (a, b]$ so that $u(t) > 0$ on $[a, t_1)$ and $u(t_1) = 0$.*

Proof. Consider the set $S = \{x \in [a, b] : u(t) > 0 \text{ for all } t \in [a, x)\}$; it is non-empty (by continuity of u and $u(a) > 0$), and bounded above, so it has a least upper bound $t_1 \in (a, b]$. If there were some t_0 with $a < t_0 < t_1$ and $u(t_0) \leq 0$, then there would be no $x \in S$ with $x > t_0$, and t_0 would be an upper bound, contradicting the least property of t_1. So, $u(t) > 0$ for all $t \in [a, t_1)$. Case 1: If $u(t_1) = 0$, then $u > 0$ on $[a, t_1)$ as claimed. Case 2: If $u(t_1) < 0$ then by continuity, there is some t_2 with $a < t_2 < t_1$ and $u(t_2) < 0$, contradicting the above property that $u(t) > 0$ for all $t \in [a, t_1)$. Case 3: If $t_1 = b$ and $u(t_1) > 0$, then $u > 0$ on $[a, b]$ as claimed. Case 4: If $t_1 < b$ and $u(t_1) > 0$, then by continuity, $u(t)$ would be positive on some interval $[a, t_1 + \delta)$, contradicting the property that t_1 is an upper bound for S. Only Cases 1 and 3 do not lead to a contradiction. ∎

Lemma 2.2. *Suppose $a(t)$ is a real function on $[0, 1)$ such that $a(t) \geq 0$ and a is bounded on every subinterval $[0, x] \subseteq [0, 1)$. If y is continuous on $[0, 1]$ with $y(0) \geq 0$, $\lim_{t \to 0^+} y'(t) \geq 0$, and $y''(t) \geq a(t)y(t)$ for $0 < t < 1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$ and $y'(t) \geq 0$ for all $0 < t < 1$.*

Proof. Case 1: $y(0) > 0$ and $\lim_{t \to 0^+} y'(t) > 0$. In this case, we can show that $y > 0$ on $[0, 1]$ and $y' > 0$ on $(0, 1)$. By Lemma 2.1 applied to y on $[0, 1]$, either $y > 0$ on $[0, 1]$, or there is some $t_1 \in (0, 1]$ so that $y(t) > 0$ for all $t \in [0, t_1)$ and $y(t_1) = 0$. In the latter case, y attains some positive maximum value on $[0, t_1]$. If $y(0)$ is the maximum, then by the Mean Value Theorem, for any $0 < \delta < t_1$, there is some t_2 with $0 < t_2 < \delta$ and $y'(t_2) = \frac{y(\delta) - y(0)}{\delta} \leq 0$, which contradicts $\lim_{t \to 0^+} y'(t) > 0$. If the maximum is at an interior point t_3 with $0 < t_3 < t_1$, then $y'(t_3) = 0$. From $\lim_{t \to 0^+} y'(t) > 0$, there is some t_4 with $0 < t_4 < t_3$ and $y'(t_4) > 0$. Applying the Mean Value Theorem to y' on $[t_4, t_3]$, there is some t_5 with $t_4 < t_5 < t_3$ and $y''(t_5) = \frac{y'(t_3) - y'(t_4)}{t_3 - t_4} < 0$. This contradicts $y''(t_5) \geq a(t_5)y(t_5) \geq 0$. We can conclude that $y(t_1)$ must be the maximum value, and $y(t_1) > 0$, which contradicts $y(t_1) = 0$. This shows $y(t) > 0$ for all $t \in [0, 1]$.

For any $t_7 \in (0, 1)$, there is some t_8 with $0 < t_8 < t_7$ and $y'(t_8) > 0$. By the Mean Value Theorem, there is some t_9 with $\frac{y'(t_7)-y'(t_8)}{t_7-t_8} = y''(t_9) \geq a(t_9)y(t_9) \geq 0$. It follows that $y'(t_7) \geq y'(t_8) > 0$.

Case 1 did not use the boundedness of a, just $a \geq 0$.

Case 2: $y(0) \geq 0$ and $\lim_{t \to 0^+} y'(t) \geq 0$. Suppose, toward a contradiction, that there is some t_0 with $0 < t_0 < 1$ and $y(t_0) < 0$. For $0 \leq t \leq t_0$, there is a bound $A > 0$ with $0 \leq a(t) \leq A$. For $t \in [0, t_0]$, define $u(t) = y(t) - \frac{1}{2}y(t_0)e^{\sqrt{A}(t-t_0)}$. Then, by construction, $u(0) > 0$ and $u(t_0) < 0$. For $0 < t < t_0$,

$$\begin{aligned}
u'(t) &= y'(t) - \frac{1}{2}y(t_0)\sqrt{A}e^{\sqrt{A}(t-t_0)} \\
\implies \lim_{t \to 0^+} u'(t) &= \left(\lim_{t \to 0^+} y'(t) \right) - \frac{1}{2}y(t_0)\sqrt{A} > 0, \\
u''(t) &= y''(t) - \frac{1}{2}y(t_0)Ae^{\sqrt{A}(t-t_0)} \\
&\geq a(t)y(t) - \frac{1}{2}y(t_0)a(t)e^{\sqrt{A}(t-t_0)} \\
&\quad + \frac{1}{2}y(t_0)a(t)e^{\sqrt{A}(t-t_0)} - \frac{1}{2}y(t_0)Ae^{\sqrt{A}(t-t_0)} \\
&= a(t)u(t) + \frac{1}{2}y(t_0)(a(t) - A)e^{\sqrt{A}(t-t_0)} \\
&\geq a(t)u(t).
\end{aligned}$$

Let $w(t) = u(t_0 t)$, just horizontally re-scaling u to the domain $[0, 1]$ so that $w(0) > 0$, $w(1) < 0$, $\lim_{t \to 0^+} w'(t) > 0$, and $w''(t) \geq t_0^2 a(t_0 t)w(t)$, so Case 1 applies to w, contradicting $w(1) < 0$. The conclusion is that $y(t) \geq 0$ on $[0, 1)$, and on $[0, 1]$ by continuity.

To establish the inequality $y' \geq 0$, for any t_1 with $0 < t_1 < 1$ and any $\epsilon > 0$, from $\lim_{t \to 0^+} y'(t) \geq 0$, there is some t_2 with $0 < t_2 < t_1$ and $y'(t_2) > -\epsilon$. By the Mean Value Theorem, there is some t_3 with $t_2 < t_3 < t_1$ and $\frac{y'(t_1)-y'(t_2)}{t_1-t_2} = y''(t_3) \geq a(t_3)y(t_3) \geq 0$. It follows that $y'(t_1) \geq y'(t_2) > -\epsilon$. ∎

Here's a higher order generalization, using only the Mean Value Theorem, not the maximum value.

Lemma 2.3. *Let $k \geq 2$ be an integer. Suppose $a(t)$ is a real function on $[0,1)$ such that $a(t) \geq 0$ and a is bounded on every subinterval $[0,x] \subseteq [0,1)$. If y is continuous on $[0,1]$ with $y(0) \geq 0$, and $\lim_{t\to 0^+} y^{(j)} \geq 0$ for $j = 1,\ldots,k-1$, and $y^{(k)}(t) \geq a(t)y(t)$ for $0 < t < 1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$ and $y^{(j)}(t) \geq 0$ for all $0 < t < 1$, $j = 1,\ldots,k$.*

Proof. Case 1: $y(0) > 0$ and $\lim_{t\to 0^+} y^{(j)}(t) > 0$. In this case, we can show that $y > 0$ on $[0,1]$ and $y^{(j)} > 0$ on $(0,1)$ for $j = 1,\ldots,k-1$.

By Lemma 2.1 applied to y on $[0,1]$, either $y > 0$ on $[0,1]$, or there is some $t_1 \in (0,1]$ so that $y(t) > 0$ for all $t \in [0,t_1)$ and $y(t_1) = 0$.

In the latter case, $y(t_1) = 0 < y(0)$, so by the Mean Value Theorem for y on $[0,t_1]$, there is some t_2 with $0 < t_2 < t_1$ and $y'(t_2) < 0$. Then, the MVT applies to y' on $[t_3,t_2]$ for some $t_3 > 0$ where $y'(t_3) > 0$, using $\lim_{t\to 0^+} y'(t) > 0$, so there is some $t_4 > 0$ where $y''(t_4) = \frac{y'(t_2)-y'(t_3)}{t_2-t_3} < 0$. Repeatedly applying this MVT argument to $y^{(j)}$ until $j = k$, gives some t_N with $0 < t_N < t_1$, $y^{(k)}(t_N) < 0$, contradicting $y^{(k)}(t_N) \geq a(t_N)y(t_N) \geq 0$.

So, the only case not leading to a contradiction is that $y(t) > 0$ on $[0,1]$.

The above MVT argument also shows that all $y^{(j)}$ are positive on $(0,1)$ for $j = 1,\ldots,k-1$, since any point t_n with $0 < t_n < t_1 = 1$ and $y^{(j)}(t_n) \leq 0$ leads to another point t_m with $0 < t_m < t_n$ and $y^{(j+1)}(t_m) < 0$, eventually contradicting $y^{(k)}(t_N) \geq a(t_N)y(t_N) \geq 0$.

Case 1 did not use the boundedness of a, just $a \geq 0$.

Case 2: $y(0) \geq 0$ and $\lim_{t\to 0^+} y^{(j)} \geq 0$, $j = 1,\ldots,k-1$. Suppose, toward a contradiction, that there is some t_0 with $0 < t_0 < 1$ and $y(t_0) < 0$. For $0 \leq t \leq t_0$, there is a bound $A > 0$ with $0 \leq a(t) \leq A$. For $t \in [0,t_0]$, define $u(t) = y(t) - \frac{1}{2}y(t_0)e^{A^{1/k}(t-t_0)}$. Then, by construction, $u(0) > 0$ and

$u(t_0) < 0$. For $0 < t < t_0$, $1 \leq j \leq k-1$,

$$\begin{aligned}
u^{(j)}(t) &= y^{(j)}(t) - \frac{1}{2}y(t_0)A^{j/k}e^{A^{1/k}(t-t_0)} \\
\implies \lim_{t\to 0^+} u^{(j)}(t) &= \left(\lim_{t\to 0^+} y^{(j)}(t)\right) - \frac{1}{2}y(t_0)A^{j/k} > 0, \\
u^{(k)}(t) &= y^{(k)}(t) - \frac{1}{2}y(t_0)Ae^{A^{1/k}(t-t_0)} \\
&\geq a(t)y(t) - \frac{1}{2}y(t_0)a(t)e^{A^{1/k}(t-t_0)} \\
&\quad + \frac{1}{2}y(t_0)a(t)e^{A^{1/k}(t-t_0)} - \frac{1}{2}y(t_0)Ae^{A^{1/k}(t-t_0)} \\
&= a(t)u(t) + \frac{1}{2}y(t_0)(a(t)-A)e^{A^{1/k}(t-t_0)} \\
&\geq a(t)u(t).
\end{aligned}$$

Let $w(t) = u(t_0 t)$, just horizontally re-scaling u to the domain $[0,1]$ so that $w(0) > 0$, $w(1) < 0$, $\lim_{t\to 0^+} w^{(j)}(t) > 0$, and $w^{(k)}(t) \geq t_0^k a(t_0 t)w(t)$, so Case 1 applies to w, contradicting $w(1) < 0$. The conclusion is that $y(t) \geq 0$ on $[0,1)$, and on $[0,1]$ by continuity.

To establish the inequalities $y^{(j)} \geq 0$, start with $j = k-1$. Then, for any t_1 with $0 < t_1 < 1$ and any $\epsilon > 0$, from $\lim_{t\to 0^+} y^{(k-1)}(t) \geq 0$, there is some t_2 with $0 < t_2 < t_1$ and $y'(t_2) > -\epsilon$. By the MVT, there is some t_3 with $t_2 < t_3 < t_1$ and $\frac{y^{(k-1)}(t_1) - y^{(k-1)}(t_2)}{t_1 - t_2} = y^{(k)}(t_3) \geq a(t_3)y(t_3) \geq 0$. It follows that $y^{(k-1)}(t_1) \geq y^{(k-1)}(t_2) > -\epsilon$. A similar argument applies for j decreasing from $k-1$ to 1. ∎

Lemma 2.4. *If the left-side limit* $\lim_{t\to b^-} f(t) = -\infty$, *then there is no interval* $(b-\delta, b)$ *on which* $f'(t)$ *is bounded below.*

Proof. (See [C].) ∎

Lemma 2.5. *Suppose $a(t)$ is a real function on $[0,1)$ such that a is bounded above on every subinterval $[0,x] \subseteq [0,1)$ and bounded below on every subinterval $[x_1,x_2] \subseteq (0,1)$. If y is continuous on $[0,1]$ with $y(0) \geq 0$ and $y'(t) \geq a(t)y(t)$ for $0 < t < 1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$.*

Proof. Case 1: $y(0) > 0$.

By Lemma 2.1 applied to y on $[0,1]$, either $y > 0$ on $[0,1]$, or there is some $t_1 \in (0,1]$ so that $y(t) > 0$ for all $t \in [0,t_1)$ and $y(t_1) = 0$. If $t_1 = 1$, then $y \geq 0$ as claimed.

So, suppose toward a contradiction that $t_1 < 1$. On the interval $[\frac{1}{2}t_1, t_1]$, a is bounded below: there is some $K < 0$ so that $K \leq a(t)$. Define the function $f(t) = \ln(y(t))$ for t in the interval $(\frac{1}{2}t_1, t_1)$. f has left-side limit $\lim_{t \to t_1^-} f(t) = -\infty$. For all t in $(\frac{1}{2}t_1, t_1)$, the derivative is bounded below: $f'(t) = \frac{1}{y(t)}y'(t) \geq \frac{1}{y(t)}a(t)y(t) = a(t) \geq K$, but this contradicts Lemma 2.4.

Case 2: $y(0) = 0$.

Suppose toward a contradiction that there is some $p \in [0,1]$ with $y(p) < 0$. $p \neq 0$ by assumption, and if $p = 1$, then by continuity of y, there is some nearby point $p - \delta_1/2$ with $y(p - \delta_1/2) < 0$. So by re-labeling if necessary, we can assume $0 < p < 1$. On the interval $[0,p]$, a is bounded above: there is some $A > 0$ so that $a(t) \leq A$.

Define $g(t) = -y(p-pt)$ on the domain $0 \leq t \leq 1$, so that $g(0) = -y(p) > 0$, $g(1) = -y(0) = 0$, and g is continuous on $[0,1]$. The derivative satisfies

$$g'(t) = py'(p-pt) \geq pa(p-pt)y(p-pt) = -pa(p-pt)g(t),$$

and the coefficient $-pa(p-pt)$ is bounded below by $-pA$. Case 1 applies to g, so $g(t) > 0$ on $[0,1)$ and $g(1) = 0$. Define the function $f(t) = \ln(g(t))$ for t in the interval $(0,1)$. f has left-side limit $\lim_{t \to 1^-} f(t) = -\infty$. For all t in $(0,1)$, the derivative is bounded below: $f'(t) = \frac{1}{g(t)}g'(t) \geq \frac{1}{g(t)}(-pa(p-pt))g(t) = -pa(p-pt) \geq -pA$, but this contradicts Lemma 2.4. ∎

Lemma 2.6. *Suppose $a(t)$ is a bounded real function on $[0, X]$. Then there is some δ with $0 < \delta \leq X$, with the property that if y is continuous on $[0, X]$ with $y(0) \geq 0$, $\lim_{t\to 0^+} y'(t) \geq 0$, and $y''(t) \geq a(t)y(t)$ for $0 < t < X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.*

Proof. Step 1. Pick any x in $(0, X]$, so that by hypothesis, there are some $A > 0$ and $K < 0$ so that $K \leq a(t) \leq A$ for $t \in [0, x]$. Let $\delta = \min\{x, \frac{1}{\sqrt{A}}, \frac{1}{\sqrt{-K}}\} > 0$. (Remark: depending on a, it may be possible to choose x that optimizes δ.) To show that this is a δ as claimed by the Lemma, suppose toward a contradiction that there is some c with $0 \leq c \leq \delta$ and $y(c) < 0$. $c > 0$ by hypothesis.

Step 2. Let $y(b)$ be the minimum value of y on $[0, c]$, so $y(b) \leq y(c) < 0$ and $0 < b \leq c \leq \delta$. The MVT applies to y on $[0, b]$: there is some t_0 with $0 < t_0 < b$ and $y'(t_0) = \frac{y(b)-y(0)}{b-0}$. The MVT applies to y' (extended to $y'(0) = \lim_{t\to 0^+} y(t) \geq 0$) on $[0, t_0]$: there is some t_1 with $0 < t_1 < t_0$ and

$$y''(t_1) = \frac{y'(t_0) - y'(0)}{t_0 - 0} = \frac{\frac{y(b)-y(0)}{b} - y'(0)}{t_0}.$$

By hypothesis,

$$a(t_1)y(t_1) \leq y''(t_1) = \frac{y(b) - y(0) - by'(0)}{bt_0} \leq \frac{y(b)}{bt_0} < 0.$$

If $a(t_1) > 0$ and $y(t_1) < 0$, then $y(t_1) \leq \frac{y(b)}{bt_0 a(t_1)} < \frac{y(b)}{\delta^2 A} \leq y(b)$, contradicting the minimum property of $y(b)$. So, $a(t_1) < 0$ and $y(t_1) > 0$.

Step 3. Lemma 2.1 applies to y on the interval $[t_1, b]$, so there is some t_3 with $t_1 < t_3 < b$, $y(t_3) = 0$, and $y(t) > 0$ for all $t \in [t_1, t_3)$. A left-side version of Lemma 2.1 applies to y on the interval $[0, t_1]$; there are two cases:

Case 1. There is some t_2 with $0 \leq t_2 < t_1$, $y(t_2) = 0$, and $y(t) > 0$ for all $t \in (t_2, t_1]$.

Case 2. $y(t) > 0$ for all $t \in [0, t_1]$. In this case denote $t_2 = 0$.

In either case, there is some interval $[t_2, t_3]$ where $0 \leq t_2 < t_1 < t_3 < b$, $y(t_3) = 0$, $y(t_2) \geq 0$, and $y(t) > 0$ for all $t \in (t_2, t_3)$. Let $y(t_4)$ be the maximum value of y on $[t_2, t_3]$. In Case 1, $t_2 < t_4 < t_3$, so the maximum occurs at an interior point and $y'(t_4) = 0$. In Case 2, t_4 is either an interior point of $[t_2, t_3]$, or the maximum occurs at the endpoint $t_4 = t_2 = 0$, where there is a right-side derivative $y'(0) \geq 0$ as in Step 2. In either case, $y'(t_4) \geq 0$.

Step 4. The MVT applies to y on $[t_4, t_3]$: there is some t_5 with $t_4 < t_5 < t_3$ and $y'(t_5) = \frac{y(t_3)-y(t_4)}{t_3-t_4}$. The MVT applies to y' on $[t_4, t_5]$: there is some t_6 with $t_4 < t_6 < t_5$ and

$$y''(t_6) = \frac{y'(t_5) - y'(t_4)}{t_5 - t_4} = \frac{\frac{y(t_3)-y(t_4)}{t_3-t_4} - y'(t_4)}{t_5 - t_4} < \frac{-y(t_4)}{b^2}.$$

Using the lower bound for a and the property $y(t_6) > 0$,

$$\begin{aligned} Ky(t_6) &\leq a(t_6)y(t_6) \leq y''(t_6) < \frac{-y(t_4)}{b^2} \\ \implies y(t_6) &> \frac{-y(t_4)}{b^2 K} \geq \frac{-y(t_4)}{\delta^2 K} \geq y(t_4), \end{aligned}$$

contradicting the maximum property of $y(t_4)$. ∎

Theorem 2.7. *Suppose $a(t)$ and $b(t)$ are real functions on $[0, 1)$, and there is a point X such that $0 < X < 1$ and a, b, and b' are bounded on $(0, X]$. Then there is some δ with $0 < \delta \leq X$, with the property that if y is continuous on $[0, 1)$ with $y(0) \geq 0$, $\lim_{t\to 0^+} y'(t) \geq 0$, and $y''(t) \geq a(t)y(t) + b(t)y'(t)$ for $0 < t < X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.*

Proof. By hypothesis, there are some $A > 0$ and $K < 0$ so that $K \leq a(t) \leq A$ for $t \in [0, X]$, and there are some $B > 0$ and $L < 0$ so that $L \leq b(t) \leq B$ for $t \in [0, X]$. Let $x = \min\{X, \frac{1}{\sqrt{2A}}, \frac{1}{4B}\} > 0$.

Recall the elementary calculus fact that if b is continuous and bounded on $(0, x)$, then b is (Riemann) integrable on $[0, x]$. Let $p(t) = \int_0^t -\frac{1}{2}b(x)dx$, so p is continuous on $[0, x]$ and for $0 < t < x$, $p'(t) = -\frac{1}{2}b(t)$.

Let

$$f(t) = e^{p(t)} \left[y(t) - Ky(0)t^2 - y(0)\right],$$

so f is continuous on $[0, x]$ and $f(0) = 0$. For $0 < t < x$,

$$\begin{aligned} f'(t) &= e^{p(t)} \left[y'(t) - 2Ky(0)t\right] \\ &\quad + e^{p(t)}(-\frac{1}{2}b(t)) \left[y(t) - Ky(0)t^2 - y(0)\right], \end{aligned}$$

and using $\lim_{t\to 0^+} (y(t) - y(0)) = 0$ and the boundedness of b, the limit exists:

$$\lim_{t\to 0^+} f'(t) = \lim_{t\to 0^+} y'(t) \geq 0.$$

For $0 < t < x$,

$$\begin{aligned}
f''(t) &= e^{p(t)}\left[y''(t) - 2Ky(0)\right] + e^{p(t)}(-\frac{1}{2}b(t))\left[y'(t) - 2Ky(0)t\right] \\
&\quad + e^{p(t)}(-\frac{1}{2}b(t))^2\left[y(t) - Ky(0)t^2 - y(0)\right] \\
&\quad + e^{p(t)}(-\frac{1}{2}b'(t))\left[y(t) - Ky(0)t^2 - y(0)\right] \\
&\quad + e^{p(t)}(-\frac{1}{2}b(t))\left[y'(t) - 2Ky(0)t\right] \\
&\geq e^{p(t)}\left[a(t)y(t) + b(t)y'(t) - 2Ky(0)\right] - e^{p(t)}b(t)\left[y'(t) - 2Ky(0)t\right] \\
&\quad + e^{p(t)}(-\frac{1}{2}b(t))^2\left[y(t) - Ky(0)t^2 - y(0)\right] \\
&\quad + e^{p(t)}(-\frac{1}{2}b'(t))\left[y(t) - Ky(0)t^2 - y(0)\right] \\
&= e^{p(t)}\left[y(t) - Ky(0)t^2 - y(0)\right]\left(a(t) + \frac{1}{4}(b(t))^2 - \frac{1}{2}b'(t)\right) \qquad (4) \\
&\quad + e^{p(t)}y(0)\left(K(a(t)t^2 + 2b(t)t - 2) + a(t)\right). \qquad (5)
\end{aligned}$$

In the last step, the y' terms cancel by construction. Term (4) is equal to $\tilde{a}(t)f(t)$, where $\tilde{a}(t) = \left(a(t) + \frac{1}{4}(b(t))^2 - \frac{1}{2}b'(t)\right)$ is bounded by hypothesis. The upper bounds $a(t) \leq A$ and $b(t) \leq B$ and the initial choice of x imply, for $0 < t < x$,

$$\begin{aligned}
a(t)t^2 + 2b(t)t - 2 &\leq At^2 + 2Bt - 2 \\
&\leq A\left(\frac{1}{2A}\right) + 2B\left(\frac{1}{4B}\right) - 2 = -1 \\
\implies K(a(t)t^2 + 2b(t)t - 2) + a(t) &\geq -K + a(t) \geq 0,
\end{aligned}$$

so the entire term (5) is non-negative, and for $0 < t < x$, $f''(t) \geq \tilde{a}(t)f(t)$. Lemma 2.6 applies to f, so there is some δ_1 depending on a, b, b', X, but not on y, with $f \geq 0$ on $[0, \delta_1]$. The factor $[y(t) - Ky(0)t^2 - y(0)]$ is non-negative on the same interval, where

$$y(t) - Ky(0)t^2 - y(0) \geq 0 \implies y(t) \geq y(0)(1 + Kt^2),$$

so $y(t) \geq 0$ for $0 \leq t \leq \delta = \min\{\delta_1, \frac{1}{\sqrt{-K}}\}$. ∎

2.2 A nonlinear differential inequality

Theorem 2.8. *Given a function f that satisfies $f''f - (f')^2 \geq 0$ on (a, b), at every critical point c with $f(c) \neq 0$, there is either a positive local min. or a negative local max.*

Proof. Suppose c is a critical point, meaning $f'(c) = 0$. Suppose also that $f(c) \neq 0$, so that the function $g(x) = f'(x)/f(x)$ is defined on a neighborhood $N = (c - \delta, c + \delta) \subseteq (a, b)$. By the quotient rule,

$$g'(x) = \frac{f(x)f''(x) - (f'(x))^2}{(f(x))^2}$$

which is ≥ 0 on N by hypothesis. It follows that $g(x)$ is weakly increasing on N. For $c < x < c + \delta$, $f'(x)/f(x) = g(x) \geq g(c) = 0$, and for $c - \delta < x < c$, $f'(x)/f(x) = g(x) \leq g(c) = 0$.

If $f(c) > 0$, then $f(x) > 0$ on N so $f'(x) \geq 0$ on the right and $f'(x) \leq 0$ on the left. $f(c)$ is a local min. by the first derivative test.

If $f(c) < 0$, then $f(x) < 0$ on N so $f'(x) \leq 0$ on the right and $f'(x) \geq 0$ on the left. $f(c)$ is a local max. ∎

Note that $\mathcal{C}^2$ is not used, just the existence of f''. Constant functions trivially satisfy both the hypotheses and conclusions.

Lemma 2.9. *If $p(x)$ satisfies $p''(x) \geq 0$ on (a, b) then for any $c \in (a, b)$, p satisfies $p(x) \geq p(c) + p'(c)(x - c)$ for all $x \in (a, b)$.* ∎

Proof. (See [C].) ∎

Theorem 2.10. *Given a function f that satisfies $f''f - (f')^2 \geq 0$ on (a, b), if there is a point c in (a, b) with $f(c) > 0$, then f satisfies*

$$f(x) \geq f(c) \cdot \exp(\frac{f'(c)}{f(c)}(x - c))$$

for all $x \in (a, b)$.

Proof. By continuity, there is some neighborhood $(s, t) \subseteq (a, b)$ so that $s < c < t$ and $f(x) > 0$ on (s, t). Suppose $f(z) = 0$ for some $z \in (c, b)$. Then, the set $\{t : f(x) > 0 \text{ on } (c, t)\}$ is non-empty and has sup $= T \leq z < b$. By construction and using continuity again, $f(T) = 0$ and $f(x) > 0$ on (s, T).

Consider $h(x) = \ln(f(x))$, which is well-defined on (s, T). $h' = f'/f = g$, from Theorem 2.8, so $h'' = g' \geq 0$ on (s, T). By Lemma 2.9, $h(x) \geq h(c) + h'(c) \cdot (x - c)$ on (s, T):

$$\begin{aligned} \ln(f(x)) &\geq \ln(f(c)) + \frac{f'(c)}{f(c)} \cdot (x - c) \\ f(x) &\geq f(c) \cdot \exp(\frac{f'(c)}{f(c)} \cdot (x - c)) \end{aligned}$$

for all x in (s, T). This implies $\lim_{x \to T^-} f(x) = f(T) > 0$, which contradicts the construction of T. We can conclude that f is never zero on (c, b), and always positive there, so the inequality holds on (s, b). The inequality on the other side of c follows from an analogous inf argument. ∎

It follows that if c is a critical point with $f(c) > 0$, then $f(c)$ is a global minimum. It further follows that either f is constant or there is at most one point c where $f'(c) = 0$ and $f(c) > 0$.

References

[C] A. Coffman, *Notes on first semester calculus*

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