PROPER HOLOMORPHIC MAPS FROM DOMAINS IN $\mathbb{C}^2$ WITH TRANSVERSE CIRCLE ACTION

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Abstract. We consider proper holomorphic mappings between smoothly bounded pseudoconvex regions in complex 2-space, where the domain is of finite type and admits a transverse circle action. The main result is that the closure of each irreducible component of the branch locus of such a map intersects the boundary of the domain in the union of finitely many orbits of the group action.

1. Introduction

Let Ω be a bounded connected open subset of $\mathbb{C}^2$, with a smooth three-dimensional boundary $b\Omega$, and assume that Ω is pseudoconvex and of finite type. Suppose further there is a continuous, non-constant homomorphism from the circle S^1 to $Aut(\Omega)$, the Lie group of holomorphic automorphisms of Ω, with image denoted $\mathbf{T}$, so we say Ω is a domain with $\mathbf{T}$-action. If the $\mathbf{T}$-action, with identity element e, has the following two properties:

(1) The group action $\mathbf{T}\times\Omega\to\Omega$ extends smoothly to $\mathbf{T}\times\overline{\Omega}\to\overline{\Omega}$;
(2) For each point z_0 in $b\Omega$, the evaluation map $\psi_{z_0}:\mathbf{T}\to b\Omega:\theta\mapsto\theta(z_0)$ has differential map $d(\psi_{z_0}):T_e\mathbf{T}\to T_{z_0}b\Omega$ with image not contained in the complex tangent line $T^h_{z_0}b\Omega$;

then the $\mathbf{T}$-action is called transverse ([Barrett]).

For a holomorphic map $f:\Omega\to\mathbb{C}^2$, let $J_f(z)$ denote the Jacobian determinant of f at $z\in\Omega$, and let V_f denote the branch locus, $\{z\in\Omega: J_f(z)=0\}$. The main result is:

Theorem 1.1. *Given a smoothly bounded pseudoconvex finite type domain $\Omega\subseteq\mathbb{C}^2$ as above, with a transverse $\mathbf{T}$-action, and another smoothly bounded pseudoconvex region $D\subseteq\mathbb{C}^2$, if $f:\Omega\to D$ is a proper holomorphic map and W is an irreducible component of the analytic variety V_f, then the connected component of $\overline{W}\cap b\Omega$ containing a point z_0 is equal to the orbit $\psi_{z_0}(\mathbf{T})$.*

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We remark that this is similar to Theorem 1.1 of [CPS_1], which stated the additional hypothesis that D is a region of finite type, although in view of Proposition 2.2, this condition is superfluous. Our proof will be more elementary than that of [CPS_1] (modulo citation of well-known results of several complex variables from the 1980's), and we take the opportunity to give a more detailed presentation. The methods also allow the statement, in Section 5, of a new, local version of the main result, where we formulate a notion of "locally transverse" and replace the condition "finite type at every point" on the boundary with finite type at a single point.

We further remark that several widely studied classes of domains admit a $\mathbf{T}$-action: Hartogs domains, Reinhardt domains ([P]), and quasi-circular domains ([Barrett], [CPS_1], [CPS_2]), but we will not consider any specific applications or examples here.

Returning to the statement of Theorem 1.1, the smooth, pseudoconvex, finite type properties of Ω imply that Ω satisfies (the well-known) Condition R ([D]), which implies that for any $\mathbf{T}$-action, part (1) of the above definition of transverse is automatically satisfied ([$\mathrm{Bedford}_3$]). Together with the hypotheses on Ω and D in Theorem 1.1, Condition R also implies ([Bell_2], [BC_1], [DF_3]) that the proper map $f : \Omega \to D$ extends smoothly to the boundary, $\overline{\Omega} \to \overline{D}$. The Proof of Theorem 1.1 will proceed by analyzing the behavior of f on the boundary.

2. Geometry of the Domain

We begin by recalling some standard notions; for basic definitions and surveys of results regarding proper maps, see [Range], [Rudin] Chapter 15, [$\mathrm{Bedford}_2$], [BN], [F], [D] Chapter 7.

A bounded domain $\Omega \subseteq \mathbb{C}^2$ with smooth boundary $b\Omega$ has a smooth (C^∞) global defining function $r : \mathbb{C}^2 \to \mathbb{R}$ so that $\Omega = \{r < 0\}$, $b\Omega = \{r = 0\}$, and $dr \neq 0$ on $b\Omega$. Define the determinant of the Levi form, in terms of coordinates z_1, z_2 on $\mathbb{C}^2$, to be the function $\Lambda_r : \mathbb{C}^2 \to \mathbb{R}$,

$$\Lambda_r = -\det \begin{pmatrix} 0 & r_{\bar z_1} & r_{\bar z_2} \\ r_{z_1} & r_{z_1 \bar z_1} & r_{z_1 \bar z_2} \\ r_{z_2} & r_{z_2 \bar z_1} & r_{z_2 \bar z_2} \end{pmatrix}.$$

Define (as in [Bell_1], [$\mathrm{Bedford}_2$]) a function $\tau : b\Omega \to \mathbb{R} \cup \{+\infty\}$ by the rule that $\tau(p)$ is the order of vanishing of $\Lambda_r|_{b\Omega}$ at p (the smallest non-negative integer m such that there is a tangential differential operator T of order m on $b\Omega$ such that $T\Lambda_r(p) \neq 0$, or $+\infty$ if there is no such m). The function τ does not depend on the choice of r. For pseudoconvex Ω, we denote the set of points at which Ω is not strictly Levi pseudoconvex by:

$$wb\Omega = \{z \in b\Omega : \Lambda_r(z) = 0\} = \{z \in b\Omega : \tau(z) > 0\}.$$

Definition 2.1. The condition that Ω is of "finite type" refers to the property that $\tau(p)$ is finite at each point p in $b\Omega$. Equivalently ([D] §4.3.1), any complex curve in $\mathbb{C}^2$ meets $b\Omega$ with at most a finite order of contact at each point.

Under the finite type assumption, $\tau : b\Omega \to [0, +\infty)$ is upper semi-continuous: for each $q \geq 0$, the set $\{p \in b\Omega : \tau(p) < q\}$ is open, since its complement $\{z \in b\Omega : T\Lambda_r(z) = 0 \text{ for all } T \text{ with order } < q\}$ is closed, being the intersection over all T of closed zero sets of the smooth functions $T\Lambda_r$.

Proposition 2.2. *Given a proper holomorphic map $g : D_1 \to D_2$ between bounded pseudoconvex regions in $\mathbb{C}^2$ with smooth boundaries, if g extends smoothly to bD_1 in a neighborhood of $p \in bD_1$, then $\tau(p) \geq \tau(g(p))$, and when $\tau(p) \neq +\infty$, the following are equivalent:*

- *$\tau(p) = \tau(g(p))$,*
- *g extends to a local diffeomorphism at p,*
- *$p \notin \overline{V_g}$.*

Remark. A short proof is given in [Bell$_1$]. ∎

In particular, the Proposition applies to any automorphism $\theta \in \mathbf{T}$, which extends smoothly to $\overline{\Omega}$ by part (1) of the definition of transverse. So, for each $z_0 \in b\Omega$, τ is constant on the orbit $\psi_{z_0}(\mathbf{T})$ containing z_0. Each orbit is a smoothly embedded circle in $b\Omega$, by part (2) of the definition of transverse, and general properties of actions of Lie groups.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain of finite type. Then, for any point $p \in b\Omega$, there exists $\epsilon_0 > 0$ so that for any $\epsilon \in (0, \epsilon_0)$, there is some $\delta \in (0, \epsilon)$ with the following property: for any point $q \in \Omega \cap B(p, \delta)$, there exists a function h_q which is holomorphic on Ω, extends continuously to $\overline{\Omega}$, and such that $h_q(q) = 1$, and $|h_q(z)| \leq \frac{1}{2}$ for all $z \in \Omega \cap bB(p, \epsilon)$.*

Proof. Under the hypotheses of the Lemma, for any $p \in b\Omega$, there exists a global peak function ([BF] Remark 3.4, [FM] Theorem 4.1), i.e., a function h_p with the following properties: h_p is continuous on $\overline{\Omega}$ and holomorphic on Ω, $h_p(p) = 1$, and $|h_p(z)| < 1$ on $\overline{\Omega} \setminus \{p\}$. There is some $\epsilon_0 > 0$ so that $bB(p, \epsilon) \cap \overline{\Omega} \neq \emptyset$ for any $\epsilon \in (0, \epsilon_0)$. For any such ϵ, let $M = \max\{|h_p(z)| : z \in bB(p, \epsilon) \cap \overline{\Omega}\} < 1$. The set $\{z \in \overline{\Omega} : |h_p(z)| > M\}$ is an open neighborhood of p in $\overline{\Omega}$, so there is some $\delta > 0$ so that $0 \leq M < |h_p(z)|$ for all $z \in B(p, \delta) \cap \Omega$. For any $q \in B(p,\delta)\cap\Omega$, let $H_q(z) = h_p(z)/h_p(q)$, so $H_q(q) = 1$ and for $z \in \overline{\Omega}\cap bB(p,\epsilon)$, $|H_q(z)| = |h_p(z)|/|h_p(q)| \le M/|h_p(q)| < 1$. There is some integer N large enough so that $h_q(z) = (H_q(z))^N$ has the properties claimed by the Lemma. ∎

3. Geometry of the Branching Locus

First, for a proper map f as in Theorem 1.1, [Rudin] §15.1 shows J_f is not identically 0 on Ω, so $V_f \neq \Omega$. Further, it follows from Proposition 2.2 that for a proper map f as in Theorem 1.1, if $p \in \overline{V_f}\cap b\Omega$, then $0 \le \tau(f(p)) < \tau(p)$, so $\overline{V_f}\cap b\Omega \subseteq wb\Omega$. From the well-known ([D], [Rudin] §15.5) fact that for a bounded domain with smooth boundary Ω, $wb\Omega \neq b\Omega$, we can conclude $\overline{V_f}\cap b\Omega \neq b\Omega$ (giving another proof that $J_f \not\equiv 0$).

In the case where Ω is a strictly pseudoconvex domain, it is known ([Bedford$_2$], [DF$_2$]) that f must be locally biholomorphic, so V_f is empty and Theorem 1.1 is vacuously true.

Another case where one containment from Theorem 1.1 is easy is when $wb\Omega$ is a union of only finitely many orbits. Then, since each orbit is a connected component of $wb\Omega$, any connected component of any subset of $wb\Omega$ must be contained in exactly one orbit.

When $V_f \neq \emptyset$, V_f is an analytic set in Ω of pure complex dimension 1, and has a unique decomposition into irreducible components ([GR]). Each component W is also an analytic set in Ω of pure complex dimension 1 which contains at least one smooth point of V_f.

For such a component $W \subseteq \Omega$, its closure $\overline{W}$ in $\mathbb{C}^2$ meets the boundary $b\Omega$ in a closed set, $E = \overline{W}\cap b\Omega$. For a more precise description of how $\overline{V_f}$ and $\overline{W}$ meet $b\Omega$, we will need the following Lemmas.

Various versions of the following result appear in [Bedford$_1$], [DF$_2$], [P] Lemma 2.3, and [HJ] Lemma 3.2; we give a detailed proof here.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded, pseudoconvex domain of finite type. Given a holomorphic map $f : \Omega \to \mathbb{C}^2$ such that $J_f \not\equiv 0$ on Ω and f extends smoothly to $b\Omega$, for an irreducible component W of the branch locus V_f, let $E = \overline{W}\cap b\Omega$. Then there is a dense subset S of E such that for each $s \in S$, there is a neighborhood U_s of s in $\mathbb{C}^2$ and a smoothly embedded real 2-submanifold Y_s containing s in $\mathbb{C}^2$ so that $\overline{W}\cap U_s = Y_s \cap \overline{\Omega}\cap U_s$ and $Y_s \cap U_s$ meets $b\Omega$ transversely.*

Proof. Consider the non-vacuous case where $V_f \neq \emptyset$. Since J_f is holomorphic and $\not\equiv 0$ on Ω, there is some multi-index $\boldsymbol{\alpha}$ so that the holomorphic function $v = \dfrac{\partial^{\boldsymbol{\alpha}}}{\partial z^{\boldsymbol{\alpha}}} J_f$ is 0 on W, but the gradient $\vec{\nabla} v$ is not identically $\vec{0}$ on W. We can assume (by switching coordinate labels if necessary) that there is some point $w \in W$ where $\frac{\partial}{\partial z_1}v(w) \neq 0$. Let $W' = \{z \in W : \frac{\partial v}{\partial z_1}(z) = 0\}$, so (using the irreducibility assumption) W' is a zero-dimensional subvariety of W.

By hypothesis, the function f extends to some smooth function on an open set U containing $\overline{\Omega}$. Both v and $\frac{\partial v}{\partial z_1}$ also extend smoothly to U, and $v(p) = 0$ for all $p \in E$ by continuity.

Let $S_1 = \{z \in E : \frac{dv}{dz_1}(z) \neq 0\}$, so S_1 is open in E. To show S_1 is dense in E, it is enough to suppose, toward a contradiction, that there is some point $p \in E$ and some neighborhood U_0 of p in $\mathbb{C}^2$ so that $\frac{dv}{dz_1}$ is 0 at every point in $U_0 \cap E$. Corresponding to p, let $\epsilon_0 > 0$ be as in Lemma 2.3, so there is some $\epsilon < \epsilon_0$ with $\overline{B(p,\epsilon)} \subseteq U_0$, and a corresponding $\delta > 0$. The set $(W \setminus W') \cap B(p,\delta)$ is a neighborhood of some point w in W with $\frac{dv}{dz_1}(w) \neq 0$. Abbreviate $B = B(p,\epsilon)$ and $bB = \{|z-p| = \epsilon\}$, so by Lemma 2.3, there is some holomorphic function h on Ω, extending continuously to $\overline{\Omega}$, such that $h(w) = 1$ and $|h(z)| \leq \frac{1}{2}$ for all $z \in \Omega \cap bB$. Then, for any natural number N, the function $(h(z))^N \cdot \frac{dv}{dz_1}(z)$ is holomorphic when restricted to $W \cap B$, and extends continuously to the closure $\overline{W \cap B}$. It is elementary that $\overline{W \cap B}$ is contained in the union $(W \cap B) \cup (W \cap bB) \cup (E \cap \overline{B})$, so $h^N \cdot \frac{dv}{dz_1}$ is identically 0 on one part of the boundary, $E \cap \overline{B}$, and on the other part of the boundary, $W \cap bB$, has magnitude less than $|\frac{dv}{dz_1}(w)|$, the magnitude of the function on an interior point, for sufficiently large N. This means the continuous, non-constant function $\left|h^N \cdot \frac{dv}{dz_1}\right|$ attains its maximum value on the compact set $\overline{W \cap B}$ somewhere in $W \cap B$, but this contradicts the maximum principle for holomorphic functions on the subvariety $W \cap B$.

Continuing under the assumption that $S_1 \neq \emptyset$ (since otherwise, $E = \emptyset$ and the Lemma again holds vacuously), let q be some point in S_1, so $v(q) = 0$ and $\frac{dv}{dz_1}(q) \neq 0$. This implies that for some neighborhood U_q^1 of q in U, the extension of v to U restricts to $\tilde{v}_q$ on U_q^1, a smooth map $U_q^1 \to \mathbb{C} \cong \mathbb{R}^2$, with real Jacobian of constant rank 2. The set $Y_q = \{z \in U_q^1 : \tilde{v}_q(z) = 0\}$ is a smoothly embedded real surface in U_q^1 ([Boothby] §III.5), with $\overline{W} \cap U_q^1 \subseteq Y_q \cap \overline{\Omega}$.

The tangent space T_qY_q is a complex line (by continuity, since the tangent spaces are complex for points in Ω near q), so there is a complex affine coordinate transformation of $\mathbb{C}^2$ taking q to the origin and T_qY_q to the z_1-axis. Then there is some open polydisc $U_q^2 = D_q^2 \times H_q^2$ centered at q and contained in the (transformed) U_q^1 neighborhood so that the intersection of Y_q with U_q^2 is a graph over the disc $D_q^2 \times \{0\} = T_qY_q \cap U_q^2$. In the transformed coordinates, $Y_q \cap U_q^2 = \{(z_1, g_q(z_1)) : z_1 \in D_q^2\}$, for some smooth function $g_q : D_q^2 \to H_q^2$. Denote the graphing diffeomorphism $\Gamma_q : D_q^2 \to U_q^2 : z_1 \mapsto (z_1, g_q(z_1))$. The set $\Gamma_q^{-1}(\Omega \cap U_q^2) = \{z_1 \in D_q^2 : (z_1, g_q(z_1)) \in \Omega\}$ is open in D_q^2, and g_q and Γ_q are holomorphic on this set.

Let $\rho_q : \mathbb{C}^2 \to \mathbb{R}$ be a smooth defining function for the domain $\Omega = \{z : \rho_q(z) < 0\}$, chosen so that $-(-\rho_q)^{2/3}$ is strictly plurisubharmonic on $\Omega \cap U_q^3$, where U_q^3 is a neighborhood of q in $\mathbb{C}^2$ ([DF$_1$]). We can assume, by intersecting and shrinking if necessary, that U_q^3 is a polydisc $D_q^3 \times H_q^3$ centered at q and contained in U_q^2.

Define $\sigma_q : D_q^3 \to \mathbb{R} : \sigma_q(z_1) = \rho_q(z_1, g_q(z_1))$, so $\sigma_q = \rho_q \circ \Gamma_q|_{D_q^3}$ is smooth and $\sigma_q(z_1) = 0$ on the set $\{z_1 \in D_q^3 : (z_1, g_q(z_1)) \in b\Omega\}$. The real-valued function $\phi_q(z_1) = -(-\sigma_q(z_1))^{2/3}$ is continuous on D_q^3, and plurisubharmonic, and therefore subharmonic, on the open subset $\{z_1 \in D_q^3 : (z_1, g_q(z_1)) \in \Omega \cap U_q^3\}$, being a composite of the smooth, strictly plurisubharmonic function $-(-\rho_q)^{2/3}$ on $\Omega \cap U_q^3$ with a holomorphic map ([Range] Corollary II.4.10).

Consider the partition of D_q^3 into the open set $\{\sigma_q < 0\}$ and the closed set $\{\sigma_q \geq 0\}$ with common boundary Σ_q. It is elementary that the set of points s in D_q^3 such that there exists an open disc Δ_s with $\overline{\Delta_s} \subseteq D_q^3$ and $\overline{\Delta_s} \cap \{\sigma_q \geq 0\} = \{s\}$ is a dense subset of Σ_q. On such a disc $\overline{\Delta_s}$, the function ϕ_q attains its maximum value, 0, at a unique point, s. Let ν be the outward unit normal vector field on the boundary of Δ_s, so ν_s is the unit normal vector at s. By the Hopf Lemma ([Range] §II.4), there is some $C > 0$ so that $\phi_q(s - x\nu_s) \leq -Cx$ for x in some interval $(0, x_0)$. It follows that $\sigma_q(s - x\nu_s) \leq -(Cx)^{3/2}$, and since σ_q is smooth, this is enough to show the normal derivative at s, $\nu_s(\sigma_q)$, is non-zero. By construction, the pushforward vector field $(\Gamma_q)_*\nu$ on the boundary of $\Gamma_q(\Delta_s) \subseteq Y_q$, when applied to the function ρ_q at $(s, g_q(s)) \in E$, gives

$$((\Gamma_q)_*\nu)_{\Gamma_q(s)}(\rho_q) = \nu_s(\rho_q \circ \Gamma_q) = \nu_s(\sigma_q) \neq 0.$$

This shows Y_q meets $b\Omega$ transversely at $(s, g_q(s))$, and such points form a dense subset of $\Gamma_q(\Sigma_q) = E \cap U_q^3$.

The conclusion is that for any $q \in S_1$, $Y_q \cap U_q^3$ meets $b\Omega$ transversely at a point in E arbitrarily close to q, and since S_1 is open and dense in E, S is also dense in E. Since transverse intersection is an open condition, Y_q meets $b\Omega$ transversely in some neighborhood of each point in S. ∎

Lemma 3.2. *Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded domain. Given a holomorphic map $f : \Omega \to \mathbb{C}^2$ such that $J_f \not\equiv 0$ on Ω and f extends smoothly to $b\Omega$, with branch locus $V_f \subseteq \Omega$, the set $\overline{V_f} \cap b\Omega$ does not contain any totally real surface.*

Proof. A "totally real surface" is a smoothly embedded real surface in $\mathbb{C}^2$ such that none of its tangent planes is a complex line. We remark that this Lemma does not follow from Lemma 3.1 — if we write the branch locus as a union of irreducible components $V_f = \cup W_\alpha$, we could immediately conclude $\overline{V_f} \supseteq \cup \overline{W_\alpha}$, but this is not enough. Instead, we recall a result of Pinčuk ([Pinčuk], [Rudin], §10.6), which states that if the holomorphic function J_f extends continuously to $b\Omega$ and vanishes identically on some totally real surface in $b\Omega$, then $J_f \equiv 0$ on Ω. ∎

4. The Main Result

Recall the main result, Theorem 1.1, states that the connected component of $E = \overline{W} \cap b\Omega$ containing a point z_0 is equal to the orbit $\psi_{z_0}(\mathbf{T})$. By the definition of connected, the equality of sets will follow from showing the set $E \cap \psi_{z_0}(\mathbf{T})$ is both closed and open in both E and $\psi_{z_0}(\mathbf{T})$. Since E is closed and the orbit is an embedded circle, the intersection is closed. The remainder of this Section will show the intersection is open in both E and $\psi_{z_0}(\mathbf{T})$.

Theorem 4.1. *For Ω, D, f, W as in Theorem 1.1, let $E = \overline{W} \cap b\Omega$. For any point p in E, and any neighborhood U of p in $\mathbb{C}^2$, there is some $\epsilon > 0$, a point $v \in E \cap U$, and a smoothly embedded arc $\gamma_p : (-\epsilon, \epsilon) \to \mathbb{C}^2$ with $\gamma_p(0) = v$ and $\gamma_p((-\epsilon, \epsilon)) \subseteq E \cap \psi_v(\mathbf{T}) \cap U$.*

Proof. By Lemma 3.1, there is some $s \in S \cap U$, some neighborhood $U_s \subseteq U$ of s, and some smooth manifold Y_s meeting $b\Omega$ transversely in U_s so $Y_s \cap b\Omega \cap U_s = E \cap U_s$. In $U_s \cap b\Omega$, the function τ is positive on E and attains its minimum positive value at some point $v \in E \cap U_s$. Because τ is upper semicontinuous, the set $U_v = \{z \in U_s \cap b\Omega : \tau(z) < \tau(v) + 1\}$ is open, and so the set where $\tau(z)$ has constant value $\tau(v)$ is an open neighborhood of v in $E \cap U_s$. Because E is a transverse intersection in U_s, it is a smooth curve, admitting a local parametrization $\gamma_p : (-\epsilon_1, \epsilon_1) \to E \cap U_v$, with $\gamma_p(0) = v$ and nonvanishing velocity vector $\gamma_p'(t)$.

Consider a particular point $z_0 = \gamma_p(t_0)$ in $E \cap U_v$, which is also on its orbit curve under the group action on $\overline{\Omega}$, so $z_0 = \psi_{z_0}(e)$. Both tangent vectors $\gamma_p'(t_0)$ and $\psi_{z_0}'(e)$ are non-zero; suppose toward a contradiction that they are linearly independent. It is then elementary that the group action $(-\epsilon_1, \epsilon_1) \times \mathbf{T} \to b\Omega : (t, \theta) \mapsto \psi_{\gamma_p(t)}(\theta)$ parametrizes an embedded surface patch, for some neighborhood of (t_0, e) in $(-\epsilon_1, \epsilon_1) \times \mathbf{T}$ ([Boothby] Ch. III). This surface P in $b\Omega$ is totally real in $\mathbb{C}^2$; by part (2) of the transverse group action hypothesis, a tangent vector to an orbit is not contained in any complex line tangent to $b\Omega$. Further, P contains z_0, and meets E in some open arc in $\gamma_p((-\epsilon_1, \epsilon_1))$. Since τ is constant on the arc and on each orbit curve, τ has the constant value $\tau(z_0)$ on P.

By Proposition 2.2, $\tau(f(z_0)) < \tau(z_0)$, while $\tau(f(z)) = \tau(z)$ for all $z \notin \overline{V_f}$. Using the upper semicontinuity of τ on bD, the set $\{w \in bD : \tau(w) < \tau(f(z_0)) + 1\}$ is open in bD, so its inverse image under f, $\{z \in b\Omega : \tau(f(z)) \leq \tau(f(z_0))\}$, is open in $b\Omega$, and contains z_0, but no points q in $P \setminus \overline{V_f}$, which would have the property $\tau(f(q)) = \tau(q) = \tau(z_0) > \tau(f(z_0))$. The conclusion is that there is some totally real surface patch, a neighborhood of z_0 in P, contained in $\overline{V_f}$. However, this contradicts Lemma 3.2.

So, for each t in an interval around 0, the velocity vector $\gamma_p'(t)$ is tangent to some orbit $\psi_{\gamma_p(t)}(\theta)$, meaning the arc γ_p is an integral curve of the line field in $b\Omega$ tangent to the orbits. Therefore, the arc coincides with the single orbit $\psi_v(\theta)$ in some neighborhood of $v = \gamma_p(0)$ (a simple case of the Frobenius Theorem, [Boothby] §IV.8). ∎

We need one more fact about the geometry of Ω.

Lemma 4.2. *For Ω as in Theorem* 1.1*, and any point $p \in b\Omega$, let O denote the orbit $\psi_p(\mathbf{T})$. Then there exists a neighborhood U_O of O in $\mathbb{C}^2$, and a smooth analytic variety A_O in $U_O \cap \Omega$ such that $\overline{A_O} \cap b\Omega = O$.*

Proof. We begin by transforming a neighborhood of p in $b\Omega$ into a "rigid" hypersurface, using the hypotheses that Ω has smooth boundary and admits a transverse $\mathbf{T}$-action. In particular, $b\Omega$, considered as an abstract CR manifold, has a CR structure invariant under the transverse action of $\mathbf{T}$, and by a result of [BRT], there exists a neighborhood U_p of p in $\mathbb{C}^2$, and a smooth CR embedding $\tilde{F} : b\Omega \cap U_p \to \mathbb{C}^2$, so that in the target coordinate system $(\tilde{z}_1, \tilde{z}_2)$, $\tilde{z}_j = \tilde{x}_j + i\tilde{y}_j$, $\tilde{F}(p) = \vec{0}$ and the image of the embedding is defined by the smooth real graphing equation $\tilde{y}_2 = H(\tilde{x}_1, \tilde{y}_1) = H(\tilde{z}_1)$ in some neighborhood of the form $V = \{(\tilde{z}_1, \tilde{z}_2) : |\tilde{z}_1| < \eta, |\tilde{x}_2| < r_1, |\tilde{y}_2| < r_2\}$. The main property is that H does not depend on $\tilde{x}_2$. Further, for $q \in b\Omega \cap U_p$, the map $\tilde{F}$ takes orbits $O_q = \psi_q(\mathbf{T})$ to real lines parallel to the $\tilde{x}_2$-axis: $\tilde{F}(O_q \cap U_p) \cap V = \{(\tilde{z}_1, \tilde{z}_2) \in V : \tilde{z}_1 = c_q, |\tilde{x}_2| < r_1, \tilde{y}_2 = H(c_q)\}$, and in the special case $q = p$, denoting $O = O_p$, $\tilde{F}(O \cap U_p) \cap V$ is the $\tilde{x}_2$-axis, i.e., $\{\tilde{y}_2 = \tilde{z}_1 = 0\}$. (See also the survey [C], or [T] for the real analytic case.)

Since $b\Omega$ is a pseudoconvex hypersurface, and the Levi form is invariant under the CR isomorphism $\tilde{F}$, the image hypersurface is also pseudoconvex. By a result of [BC$_2$] (which also uses the hypothesis that Ω is of finite type), $\tilde{F}$ extends to a one to one map $F : U_p \to \mathbb{C}^2$ (possibly shrinking the neighborhood U_p of p) with the following properties. The map F is biholomorphic on $\Omega \cap U_p$, extends smoothly to $\overline{\Omega} \cap U_p$, and agrees with the diffeomorphism $\tilde{F}$ on $b\Omega \cap U_p$. In terms of the smooth graphing function H, (and, again possibly shrinking the polydisc V), the map F has the property $F(\Omega \cap U_p) \cap V = \{H(\tilde{z}_1) < \tilde{y}_2 < r_2\}$.

As a remark which will not be needed until later, we note that given a set V, the radius $\eta > 0$ can be shrunk arbitrarily while r_1 and r_2 are fixed, and the thinner neighborhood will still meet $F(b\Omega \cap U_p)$ in a graph with the claimed properties.

For each complex number c in the disc $\{|\tilde{z}_1| < \eta\}$, let A_c denote the set $\{\tilde{z}_1 = c, |\tilde{x}_2| < r_1, H(c) < \tilde{y}_2 < r_2\}$, which is a smooth, irreducible analytic variety in the open set $F(\Omega \cap U_p) \cap V$. It has the property that $\overline{A_c} \cap F(b\Omega \cap U_p) \cap V$ is a line segment, $\{\tilde{z}_1 = c, |\tilde{x}_2| < r_1, \tilde{y}_2 = H(c)\}$, equal to the image of some orbit, $F(O_q \cap U_p) \cap V$.

In fact, A_c is the unique irreducible analytic variety in $F(\Omega \cap U_p) \cap V$ meeting $F(b\Omega \cap U_p) \cap V$ along $F(O_q \cap U_p) \cap V$. Without loss of generality, consider the particular case $c = 0$, where A_0 is the intersection of the $\tilde{z}_2$-axis with $F(\Omega \cap U_p) \cap V$, and $\overline{A_0} \cap F(b\Omega \cap U_p) \cap V = F(O \cap U_p) \cap V$ is the $\tilde{x}_2$-axis in V. Suppose there is some point $\tilde{q} \in F(O \cap U_p) \cap V$, and some neighborhood V' of $\tilde{q}$ in V, and some irreducible one-dimensional analytic variety A' in $F(\Omega \cap U_p) \cap V'$ with closure $\overline{A'}$ in V' so that $\overline{A'} \cap F(b\Omega \cap U_p) \subseteq F(O \cap U_p)$. To show $A' \subseteq A_0$, we will use a maximum principle argument similar to that from the Proof of Lemma 3.1. The function $\tilde{z}_1$ is holomorphic on A', and extends continuously to $\overline{A'}$, where it is identically 0 on $\overline{A'} \cap F(b\Omega \cap U_p)$. If A' is not contained in A_0, then the set of points in A' with coordinates $(\tilde{z}_1, \tilde{z}_2)$, $\tilde{z}_1 = 0$, is zero-dimensional in A'. Corresponding to $p' = F^{-1}(\tilde{q})$ in $b\Omega$, let $\epsilon_0 > 0$ be as in Lemma 2.3, so there is some $\epsilon < \epsilon_0$ with $F(\overline{B(p', \epsilon)}) \subseteq V'$, and a corresponding $\delta > 0$. The set $A' \cap F(B(p', \delta))$ contains some point $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$, with $\tilde{w}_1 \neq 0$. Abbreviate $B = B(p', \epsilon)$ and $bB = \{|z - p'| = \epsilon\}$, so by Lemma 2.3, there is some holomorphic function h on Ω, extending continuously to $\overline{\Omega}$, such that $h(F^{-1}(\tilde{w})) = 1$ and $|h(z)| \leq \frac{1}{2}$ for all $z \in \Omega \cap bB$. Then, for any natural number N, the function $((h \circ F^{-1})(\tilde{z}_1, \tilde{z}_2))^N \cdot \tilde{z}_1$ is holomorphic when restricted to $A' \cap F(B)$, and extends continuously to the closure $\overline{A' \cap F(B)}$. Since $\overline{A' \cap F(B)}$ is contained in the union $(A' \cap F(B)) \cup (A' \cap F(bB)) \cup ((\overline{A'} \cap F(b\Omega \cap U_p)) \cap \overline{F(B)})$, $(h \circ F^{-1})^N \cdot \tilde{z}_1$ is identically 0 on one part of the boundary, $\overline{A'} \cap F(b\Omega \cap U_p) \cap \overline{F(B)}$, and on the other part of the boundary, $A' \cap F(bB)$, has magnitude less than $|\tilde{w}_1|$, the magnitude of the function on an interior point, for sufficiently large N. This means the continuous, non-constant function $\left|(h \circ F^{-1})^N \cdot \tilde{z}_1\right|$ attains its maximum value on the compact set $\overline{A' \cap F(B)}$ somewhere in $A' \cap F(B)$, but this contradicts the maximum principle for holomorphic functions on the subvariety $A' \cap F(B)$.

Since this construction can be carried out at any point p on O, O is covered by neighborhoods $F^{-1}(V)$, so that the open sets $F^{-1}(V) \cap \Omega$ contain analytic sets $F^{-1}(A_0)$ that patch together smoothly on overlaps near $b\Omega$ by the above uniqueness. By the compactness of O, it is covered by finitely many such neighborhoods $F^{-1}(V)$, whose union contains a neighborhood U_O of O, small enough so that its intersection with the union of varieties $F^{-1}(A_0)$ is a closed variety A in $U_O \cap \Omega$. ∎

At this point we have enough to establish the main result:

Proof of Theorem 1.1. Given $z_0 \in \overline{W} \cap b\Omega$, let $E = \overline{W} \cap b\Omega$, and let E_0 denote the connected component of E containing z_0.

Apply Lemma 4.2 to z_0, to get a one-sided holomorphic map $F : U_{z_0} \to \mathbb{C}^2$, and a neighborhood $V \subseteq F(U_{z_0})$ centered at $\vec{0} = F(z_0)$ and meeting $F(U_{z_0} \cap b\Omega)$ in a graph as in the Proof of Lemma 4.2. Fixing the lengths r_1, r_2 of V, consider any small $\eta > 0$ and a subset $V_\eta = \{(\tilde{z}_1, \tilde{z}_2) \in V : |\tilde{z}_1| < \eta\}$ as in the remark from the Proof of Lemma 4.2. From Theorem 4.1 applied to the neighborhood $F^{-1}(V_\eta)$ of z_0, there is some $v \in F^{-1}(V_\eta)$ and an arc $\gamma((-\epsilon, \epsilon))$ inside $E \cap \psi_v(\mathbf{T}) \cap F^{-1}(V_\eta)$. The composite $F \circ \gamma$ parametrizes a segment parallel to the $\tilde{x}_2$-axis in V_η, and part of the boundary of some irreducible variety A_c with $|c| < \eta$. However, this segment is also part of the boundary of the variety $F(W \cap U_{z_0})$, and by the uniqueness from the Proof of Lemma 4.2, the varieties A_c and $F(W \cap U_{z_0})$ meet in some open subset of A_c, so $A_c \subseteq F(W \cap U_{z_0})$. Since η can be arbitrarily small, it follows that every point on A_0 is a limit point of $F(W \cap U_{z_0})$, being within η of some point in A_c, and since $F(W \cap U_{z_0})$ is closed in $F(U_{z_0})$, $A_0 \subseteq F(W \cap U_{z_0})$.

This shows that any point $z_0 \in E$ is contained in some arc in $E \cap \psi_{z_0}(\mathbf{T})$, which is open in $\psi_{z_0}(\mathbf{T})$. Since E is closed, $E \cap \psi_{z_0}(\mathbf{T})$ is both open and closed in $\psi_{z_0}(\mathbf{T})$, and since the orbit is connected, $\psi_{z_0}(\mathbf{T}) \subseteq E_0$.

To show $\psi_{z_0}(\mathbf{T})$ is open in E, which as previously mentioned will complete the proof, it is enough to show that any point $z_0 \in E$ has a neighborhood U in $\mathbb{C}^2$ so that $U \cap E \subseteq \psi_{z_0}(\mathbf{T})$. Since this is a local statement, we again consider a transformation F and coordinate neighborhood V as in Lemma 4.2, and we need to show that there is some neighborhood of $\vec{0} \in V$ that meets $F(E \cap U_{z_0}) \cap V$ only in the $\tilde{x}_2$-axis.

The point $(\tilde{z}_1, \tilde{z}_2) = (0, \frac{i}{2}r_2)$ is in the open set $F(U_{z_0} \cap \Omega)$, so it is the center of some polydisc

$$P_I = \{|\tilde{z}_1| < \delta_1, |\tilde{z}_2 - \frac{i}{2}r_2| < \rho\} \subseteq F(U_{z_0} \cap \Omega) \cap V.$$

For $0 < \delta \leq \delta_1$, denote another polydisc

$$P_\delta = \{|\tilde{z}_1| < \delta, |\tilde{z}_2| < \rho\} \subseteq V,$$

centered at $\vec{0}$. Suppose, toward a contradiction, that for any δ, there is some point $\tilde{w} = (\tilde{w}_1, \tilde{w}_2) \in F(E \cap U_{z_0}) \cap P_\delta$ with $\tilde{w}_1 \neq 0$. Then, by the above argument, $A_{\tilde{w}_1} = \{\tilde{z}_1 = \tilde{w}_1, |\tilde{x}_2| < r_1, H(\tilde{w}_1) < \tilde{y}_2 < r_2\}$ is a subset of $F(W \cap U_{z_0}) \cap V$. It follows that for any complex number ξ in the disc $\{|\xi - \frac{i}{2}r_2| < \rho\}$, the point $(\tilde{w}_1, \xi)$ is an element of $P_I \cap F(W \cap U_{z_0})$, and since δ can be arbitrarily small, $(0, \xi)$ is an accumulation point of the analytic set $\{\tilde{z}_2 = \xi\} \cap P_I \cap F(W \cap U_{z_0})$ in the radius δ_1 disc $D_\xi = \{\tilde{z}_2 = \xi\} \cap P_I$. This implies each disc D_ξ is a subset of $F(W \cap U_{z_0})$, so the complex analytic variety $F(W \cap U_{z_0})$ contains the open set P_I, contradicting the fact that W is one-dimensional. ∎

Corollary 4.3. *For W as in Theorem 1.1, $\overline{W} \cap b\Omega$ is a finite union of orbits, and in a neighborhood of $b\Omega$, the set $\overline{W}$ is a submanifold with boundary.*

Proof. It follows from the construction in the above Proof that at each point $z_0 \in \overline{W} \cap b\Omega$, there is some neighborhood of z_0 where $\overline{W} \cap b\Omega$ meets exactly one orbit, and by compactness, $\overline{W} \cap b\Omega$ can be covered by finitely many such neighborhoods. The claim about $\overline{W}$ being a submanifold with boundary near $b\Omega$, in fact coinciding with some variety A_O near each boundary component, also follows from the observations in the previous Proof. ∎

5. A Local Version

The above method of proof of the main result allows for a local version, which improves on a claim of [CPS$_2$], its Theorem 2, which was stated without a complete proof.

On a smoothly bounded domain, if a point on the boundary is of finite type, then by upper semicontinuity, it has a neighborhood of points of finite type. Most of the above steps — including the transverse intersection Lemma 3.1, the arc intersections in Theorem 4.1, the transformation to rigid form in Lemma 4.2, and the topological and geometric arguments in the proof of Theorem 1.1 — are stated in local terms. The only use of a global object was in Lemma 2.3, where, in the interest of simplicity, we cited a result on the existence of a global peak function. However, we only need to apply the conclusion of Lemma 2.3 locally, and (as explained to the authors by G. Bharali) the method of [BF] in fact gives a local peak function at a point of finite type in the boundary of a pseudoconvex domain, so the Proof of Lemma 2.3 can be adapted to give the following:

Lemma 5.1. *Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain, and let p be a point of finite type in the boundary. Then there exists $\epsilon_0 > 0$ so that for any $\epsilon \in (0, \epsilon_0)$, there is some $\delta \in (0, \epsilon)$ with the following property: for any point $q \in \Omega \cap B(p, \delta)$, there exists a function h_q which is holomorphic on $\Omega \cap B(p, \epsilon)$, extends continuously to $\overline{\Omega} \cap \overline{B(p, \epsilon)}$, and such that $h_q(q) = 1$, and $|h_q(z)| \leq \frac{1}{2}$ for all $z \in \Omega \cap bB(p, \epsilon)$.* ∎

Using this version of the Lemma, but otherwise not significantly changing the argument, allows a modification of the main result. We continue to consider a domain Ω in $\mathbb{C}^2$ with a (global) $\mathbf{T}$-action, but which is not necessarily transverse everywhere. We also make the assumption that Ω satisfies Condition R. Recall Condition R implies part (1) of the definition of transverse $\mathbf{T}$-action, and conversely, it was shown by [Barrett] that if the $\mathbf{T}$-action is (globally) transverse, then the domain satisfies Condition R. We consider a particular point z_0 in the boundary where $d(\psi_{z_0}) : T_e\mathbf{T} \to T_{z_0}b\Omega$ has image not contained in the complex tangent line $T^h_{z_0}b\Omega$; this property is shared by points in a neighborhood of z_0, and since $\mathbf{T}$ acts by automorphisms, the property holds in a neighborhood of the orbit $\psi_{z_0}(\mathbf{T})$, and we say that the $\mathbf{T}$-action is locally transverse at z_0.

If there exists a proper map $f : \Omega \to D$, and D is pseudoconvex, then Ω is also pseudoconvex ([Range] §II.5). So, as before, it follows that any proper map from Ω to a smooth, pseudoconvex region D extends smoothly to the boundary.

Corollary 5.2. *Given a smoothly bounded domain Ω in $\mathbb{C}^2$ satisfying Condition R and admitting a $\mathbf{T}$-action, suppose $z_0 \in b\Omega$ is a point of finite type where the $\mathbf{T}$-action is locally transverse. Let $D \subseteq \mathbb{C}^2$ be a smoothly bounded pseudoconvex region. If $f : \Omega \to D$ is a proper holomorphic map and W is an irreducible component of the analytic variety V_f such that $z_0 \in \overline{W} \cap b\Omega$, then the connected component of $\overline{W} \cap b\Omega$ containing z_0 is equal to the orbit $\psi_{z_0}(\mathbf{T})$, and in a neighborhood of the orbit, $\overline{W}$ is a submanifold with boundary.* ∎

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