A natural invariant measure for polynomial semigroups, and its properties

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1 Preliminaries

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Example: Let f be a rational map on the Riemann sphere $\widehat{\mathbb{C}}$. Based on the behaviour of the points $x \in \widehat{\mathbb{C}}$, we have the dichotomy:

- The Fatou set of $f :=$ the set of normality (or equicontinuity) of $\{f^n : n \in \mathbb{Z}_+\}$.
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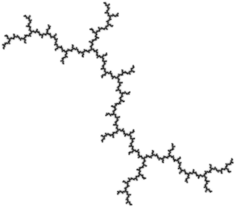
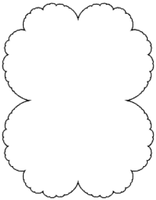
The dynamics on Fatou sets is tame and the structure of Fatou sets is well understood. On the other hand, the dynamics on Julia sets is chaotic and, in the generic case, Julia sets are fractals.

2 Some pictures



$z^2 + i$



$z - z^2$

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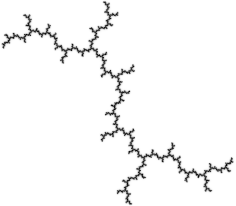

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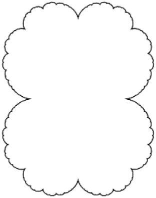

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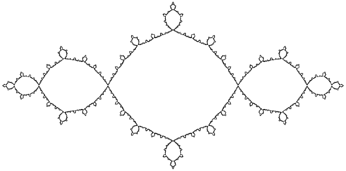
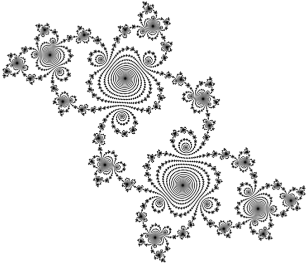


$z^2 - 1$



$z^2 - 0.1 + 0.651i$

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- There is considerable success in constructing analogues of the above-mentioned dynamically defined measures in higher dimensions even for multi-valued maps.

Mission statement of this talk: To study these measures from **potential-theoretic** points of view for the case of polynomial semigroups.

What conceptual framework do we have?

We study measures that describe the limiting distribution of the iterated pre-images of **any** point excluding, perhaps, a small set of exceptional points. In short, such a measure is the weak* limit of the sequence $\{\mu_n\}$ (if limit exists), where

$$\mu_n := \frac{1}{\sharp(f^{-n}\{a\})} \sum_{f^n(z)=a} \delta_z.$$

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What is an example of such a measure?

5 Brolin's theorem

Result (Brolin, 1965)

Let g be a polynomial of degree $d \geq 2$ and a be any point in the complex plane with (perhaps) one exception. Then

$$\mu_n := \frac{1}{d^n} \sum_{g^n(z)=a} \delta_z \xrightarrow{\text{weak*}} \mu_g \text{ as } n \to \infty,$$

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- Roughly speaking, an equilibrium measure gives the distribution of a unit charge, in the absence of any external field, on a conductor that minimizes energy.

6 Some terminology

A *rational semigroup* S is a semigroup consisting of non-constant rational maps on the Riemann sphere $\widehat{\mathbb{C}}$ with the function-composition as the semigroup operation.

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Example: $S = \langle z^2, z^2/2 \rangle$. Then $\mathbf{J}(S) = \{z : 1 \leq |z| \leq 2\}$.

Consider a generating set $\mathcal{G}$ of S. For $g \in S$, the expression $l(g) = n$ is the shorthand for the following implication:

$$l(g) = n \implies \exists\ g_{i_1}, \ldots, g_{i_n} \in \mathcal{G} \text{ such that } g = g_{i_n} \circ \cdots \circ g_{i_1}.$$

Measures associated with semigroups

Result (Boyd, 1999; special case of Dinh–Sibony, 2006)

Let S be a finitely generated rational semigroup. Assume that:

- (Boyd) *every element of S is of degree at least 2; or*
- (Dinh–Sibony) *S has an element of degree at least 2: i.e., as in* $(*)$.

Let $\mathcal{G} = \{g_1, \dots, g_N\}$ be a generating set and $D := \sum_{i=1}^{N} \deg(g_i)$. Then there exists a Borel probability measure $\mu_{\mathcal{G}}$ such that for every a outside some polar set

$$\mu_n := \frac{1}{D^n} \sum_{\substack{g(z)=a \\ l(g)=n}} \delta_z \xrightarrow{\text{weak*}} \mu_{\mathcal{G}} \quad \text{as } n \to \infty.$$

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- We assume that degree 1 elements in S have ∞ as an attracting fixed point — we refer to such semigroups as *polynomial semigroups*.

Logarithmic potentials

Let σ be a Borel probability measure on $\mathbb{C}$ with compact support. Its *logarithmic potential* is the function $U^\sigma : \mathbb{C} \to (-\infty, \infty]$ defined by

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- $\mathcal{E}(S)$:=$\{z \in \widehat{\mathbb{C}} : \cup_{g \in S} g^{-1}\{z\}$ is a finite set$\}$.
- (Arsove, 1960) U^σ is finite and continuous at z_0 if σ satisfies

$$\sigma(D(z,r)) \leq Cr^\alpha \quad \forall r \in (0, r_0),$$

where $|z - z_0| < \delta$ and C, α, r_0, δ are positive constants depending only on σ and z_0.

External fields and our first theorem

Let Σ be a compact subset of $\mathbb{C}$ and $Q : \Sigma \to (-\infty, \infty]$ be lower semi-continuous and $Q(z) < \infty$ on a set of positive capacity. The function Q is called an *external field*. We define the *energy* as

$$I_Q(\sigma) := \int_{\mathbb{C}} \int_{\mathbb{C}} \log \frac{1}{|z-t|} d\sigma(z) d\sigma(t) + 2 \int_{\mathbb{C}} Q d\sigma.$$

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Remark: The hypothesis can be made "canonical"; don't have time to discuss it.

SKETCHES OF A FEW PROOFS . . .

A counting lemma

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$$M := \max\{|g_i'(z)| : z \in \mathbf{J}(S),\ i \in \{1, \ldots, N\}\},$$

$$R := \frac{D}{N} \quad \text{and} \quad \lambda := \frac{\log R}{\log M},$$

where $D := \sum_{i=1}^{N} \deg(g_i)$. Thus $M = R^{\frac{1}{\lambda}}$. Note, $R > 1$ and $M > 1$.

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Assume $\sharp\big(\boldsymbol{C}(g_1, \ldots, g_N)\big) > 1$. Then there exist $r_0 > 0$ and $\kappa \in \mathbb{Z}_+$ such that for any $r \in (0, r_0]$ and $y \in \mathbf{J}(S)$, we have

$$\sharp((F^n)^\dagger(y) \cap D(z, r))^\bullet \leq \max\left(D^{n - \frac{\nu}{\kappa} + 1} N^{\frac{\nu}{\kappa} - 1}, (D - \tfrac{1}{2})^n\right) \qquad (1)$$

for all $n \in \mathbb{N}$ and $z \in \mathbb{C}$, where $\nu \in \mathbb{Z}_+$ is the unique integer such that

$$r \in I(\nu) := \left(r_0 R^{\frac{-2\nu}{\lambda}}, r_0 R^{\frac{-2(\nu-1)}{\lambda}}\right].$$

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- Let $\delta_4 > 0$ be the Lebesgue number of the following cover: $\{D(\xi, r(\xi)) : \xi \in \bar{\mathbf{J}}^{\delta_2}\}$, where $r(\xi) > 0$ is such that $g_i|_{D(\xi, r(\xi))}$ is injective for $i = 1, 2, \ldots, N$.

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- Write:

$$r_0 := \frac{\min\{\delta_2, \delta_3, \delta_4\}}{4} \text{ and } \kappa = 1.$$

With this choice of r_0 and κ, the inequality follows by induction on n.

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With this choice of r_0 and κ, the inequality follows by induction on n.

What if $C(g_1, \ldots, g_N) \cap \mathbf{J}(S) \neq \emptyset$?

Hint: Choose $\delta_1 > 0$ appropriately and $\kappa \in \mathbb{Z}_+$ such that base-case of induction holds given $\sum_{i: g_i'(x)=0} \mathrm{ord}_x(g_i) \not\equiv 0$ on $C(g_1, \ldots, g_N) \cap \mathbf{J}(S)$.

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- *Case 2:* $\sharp\big(\boldsymbol{C}(\mathcal{G})\big) = 1$ and $\boldsymbol{C}(\mathcal{G}) \cap \mathbf{J}(S) \neq \emptyset$.

  Consider the **list** of polynomials $\mathcal{G}^2 := \{g_i \circ g_j : 1 \leq i, j \leq N\}^\bullet$.
  $\boldsymbol{C}(\mathcal{G}) \cap \mathbf{J}(S) \cap \mathcal{E}(S) = \emptyset$ implies that $\sharp\big(\boldsymbol{C}(\mathcal{G}^2)\big) > 1$.

Sketch of the proof of our first theorem

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$$G_{\mathcal{G}}(z) := \limsup_{n\to\infty} \frac{1}{D^n} \log\left(\prod_{l(g)=n} |g(z) - a|\right),$$

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  where $A = |\text{lead}(g_1) \times \text{lead}(g_2) \times \cdots \times \text{lead}(g_N)|$.
- As $U^{\mu_{\mathcal{G}}}$ is continuous, it follows that

$$U^{\mu_{\mathcal{G}}}(z) + G^*_{\mathcal{G}}(z) = \frac{\log A}{D - N} \quad \forall z \in \mathbb{C}.$$

Lower bound for capacity of the Julia set

Theorem (L.)

Let $(S, \mathcal{G})$ *be as in our main theorem and let* $Q_{\mathcal{G}}$ *denote the external field associated with* $(S, \mathcal{G})$*.*

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THANK YOU!