Gauss-Lucas theorem and its dynamical consequences

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Complex dynamics

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Such a function extends naturally as a holomorphic map to $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.
We equip $\mathbb{C}_\infty$ with the chordal metric

$$\rho([X_1, Y_1], [X_2, Y_2]) = \frac{|X_1 Y_2 - X_2 Y_1|}{\sqrt{|X_1|^2 + |Y_1|^2}\sqrt{|X_2|^2 + |Y_2|^2}}.$$

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Complex dynamics studies behavior of the sequence of iterates $f^{\circ(n+1)} = f \circ f^{\circ n},\ (n \in \mathbb{N})$ in $\mathbb{C}_\infty$.

The Julia-Fatou dichotomy

We define the Fatou set F_f of f and the Julia set J_f of f as follows: F_f is the maximal open subset of $\mathbb{C}_\infty$ on which the sequence $\{f^{\circ n} : n \in \mathbb{N}\}$ is equicontinuous, and J_f is the complement of F_f in $\mathbb{C}_\infty$.

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The Julia set is closed and nonempty, in fact infinite. Both Julia and Fatou sets are totally f-invariant, i.e.,

$$f^{-1}(J(f)) = J(f) = f(J(f)),\ f^{-1}(F(f)) = F(f) = f(F(f)).$$

Dynamics of polynomials, I

For a polynomial p (of degree $d > 1$), the point $\infty = p(\infty) = p^{-1}(\infty)$ belongs to the Fatou set. Further, the Julia set $J_p \subset \mathbb{C}$ always has empty interior.

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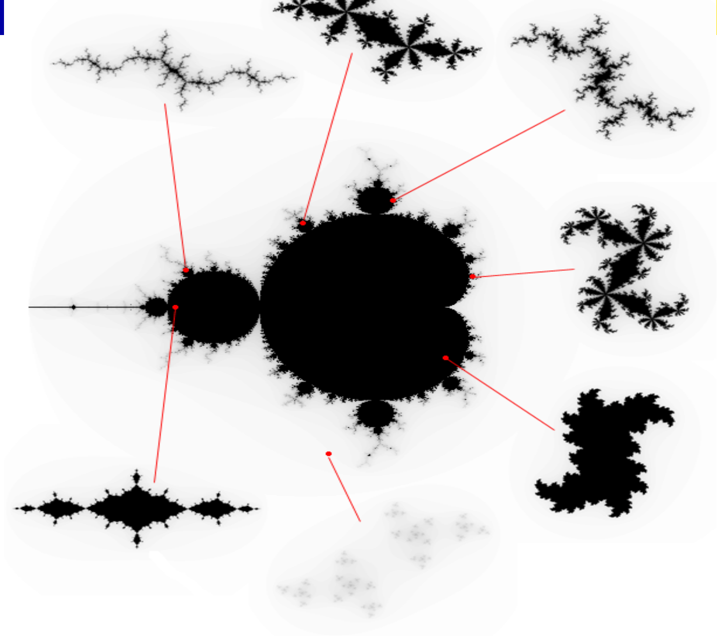
Some examples of Julia sets of quadratic polynomials $p_c(z) = z^2 + c$ can be seen in the next slide.



Dynamics of polynomials, II

Since

$$\lim_{|z|\to\infty} \frac{|p(z)|}{|z|^d} > 0,$$

there exists an $R > 0$ such that $p^{-1}(D_R) \subset D_R$, where $D_R := \{z : |z| \leqslant R\}$. Furthermore, for any such R and for each positive integer k_0 we have

$$\emptyset \neq K_p = \bigcap_{k \geqslant k_0} p^{-k}(D_R),$$

where $K_p := \{z \in \mathbb{C} : \{p^{\circ n}(z)\} \text{is bounded}\}$. We call K_p the **filled-in Julia set** of p. It is easy to show that $p^{-1}(K_p) = K_p = p(K_p)$ and that K_p is the union of $J_p = \partial K_p$ with bounded components of F_p.

A natural question

Do they exist nontrivial closed sets $Z \subset \mathbb{C}$ (other than J_p, K_p or D_R) containing J_p such that $p^{-1}(Z) \subset Z$?

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More specifically, for a complex polynomial p of degree $d \geqslant 2$, let $H_p = \mathrm{conv} J_p$ be the convex hull of the Julia set of p. Do we always have $p^{-1}(H_p) \subset H_p$?

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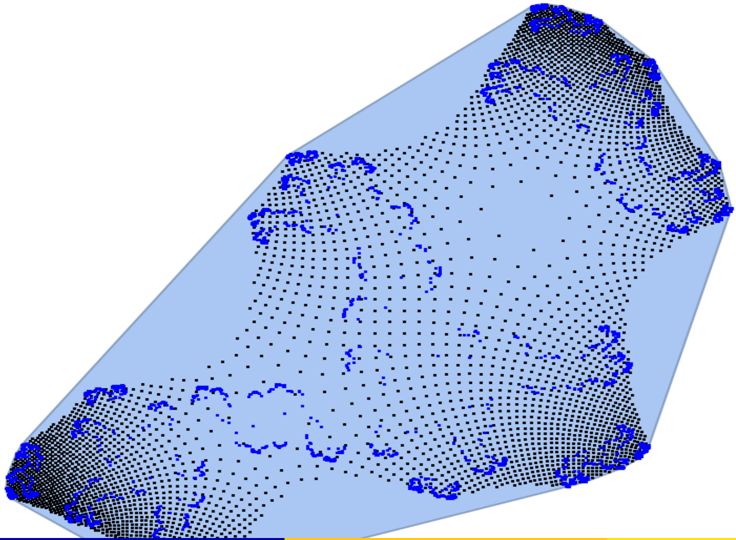
This was conjectured by Per Alexandersson and answered positively by the present author.

$f(z) = z^3 - iz + 0.2 + 0.4i$

Julia set and its convex hull (picture by P. Alexandersson)



A relation between convex sets and complex polynomials

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(Gauss-Lucas theorem) Every convex set in the complex plane containing all the zeros of a complex polynomial p also contains all critical points of p (solutions to $p'(z) = 0$).

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The following result due to W. P. Thurston is equivalent to the Gauss-Lucas theorem:

Theorem

Let p be any polynomial of degree at least two. Denote by $\mathcal{C}$ the convex hull of the critical points of p. Then $p : E \to \mathbb{C}$ is surjective for any closed half-plane E intersecting $\mathcal{C}$.

Other useful results, I

Lemma

(L. Hörmander) Let p be a complex polynomial and let B be a closed convex subset of $\mathbb{C}$ containing all zeros of p'. Then the set C_B of all $w \in \mathbb{C}$ such that all the zeros of $p(\cdot) - w$ are contained in B is a convex set.

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Proof.

Let $w_1, w_2 \in C_B$ and $n_1, n_2 \in \mathbb{N}$ and consider the polynomial (in one complex variable z) $P(z) := (p(z) - w_1)^{n_1}(p(z) - w_2)^{n_2}$. Then all zeros of P lie in B (by definition of C_B), so the convex hull of zeros of P is contained in B. By Gauss-Lucas, all zeros of P' are contained in B. The zeros of P' are respectively all the zeros of $p(z) - w_1$, all the zeros of $p(z) - w_2$ (if $n_1, n_2 > 1$), all the zeros of p' and all the zeros of $p(\cdot) - \left(\frac{n_2}{n_1+n_2} w_1 + \frac{n_1}{n_1+n_2} w_2\right)$. Hence $\frac{n_2}{n_1+n_2} w_1 + \frac{n_1}{n_1+n_2} w_2 \in C_B$ and finally $tw_1 + (1-t)w_2 \in B$ for all $0 \leqslant t \leqslant 1$.

Other useful results, II

Theorem

(hyperplane separation theorem) Let X be a convex and closed subset of a finite-dimensional vector space V. If $x_0 \notin X$, then there is an affine half-space containing x_0 which does not intersect X; that is, there is an affine function $f : V \to \mathbb{R}$ with $f(x_0) < 0 \leqslant f(x),\ x \in X$.

“Dynamical Gauss-Lucas”

We will prove Alexandersson’s conjecture using the following :

Lemma

Let p be any polynomial of degree at least two. Then all zeros of p' belong to $H_p = \mathrm{conv} J_p$.

Proof.

Suppose there is an $x_0 \notin H_p$ such that $p'(x_0) = 0$. By the hyperplane separation theorem (applied twice if necessary), there exists a closed half-plane E such that $x_0 \in E$ and $E \cap J_p = \emptyset$. By Thurston’s theorem, $p : E \to \mathbb{C}$ is surjective. Take a $z_0 \in J_p$. Then on one hand $p^{-1}(z_0) \subset J_p$, while on the other hand $p^{-1}(z_0) \cap E \neq \emptyset$, a contradiction. □

The main result

Theorem

Let p be a complex polynomial of degree $d \geqslant 2$. Then $p^{-1}(H_p) \subset H_p$.

Proof.

By “dynamical Gauss-Lucas”, $B = H_p$ satisfies the assumptions of Hörmander’s Lemma. Hence the set $C_p = \{w \in \mathbb{C} : p^{-1}(w) \in H_p\}$ is convex. Furthermore, for $w \in J_p$ we have $p^{-1}(w) \in J_p \subset H_p$, so $J_p \subset C_p$. Hence $H_p \subset C_p$, which implies $p^{-1}(H_p) \subset H_p$. □

A special case

Alexandersson's conjecture was motivated by the study of the quadratic family $p_c(z) = z^2 + c,\ c \in \mathbb{C}$ For this family is easy to check directly (without appealing to "dynamical Gauss-Lucas") that the critical point 0 is the center of symmetry of the Julia set J_c, so in particular it is a convex combination of two points in J_c.

Further results

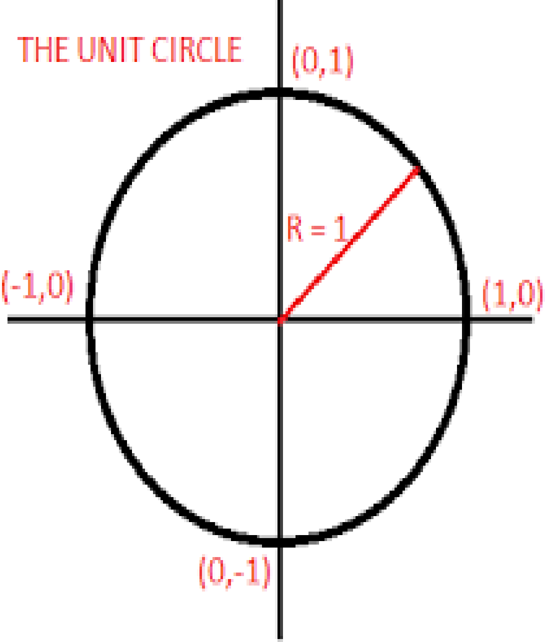
We can further prove that the equality $p^{-1}(H_p) = H_p$ is achieved if and only if J_p is either a line segment or a circle; that is, if and only if p is Möbius conjugated to the classical Chebyshev polynomial T_d of degree d, to $-T_d$ or the monomial cz^d with $|c| = 1$.

Further results

We can further prove that the equality $p^{-1}(H_p) = H_p$ is achieved if and only if J_p is either a line segment or a circle; that is, if and only if p is Möbius conjugated to the classical Chebyshev polynomial T_d of degree d, to $-T_d$ or the monomial cz^d with $|c| = 1$.
The distinction is according to whether $J_p = K_p$ or $J_p \subsetneq K_p$.

Special Julia sets

Let p be a polynomial of degree $d \geqslant 2$. Th (i) the real interval $[-1, 1]$ is both forward and backward invariant under p if and only if p is T_d or $-T_d$, where T_d is the Chebyshev polynomial of degree d;
(ii) the unit circle $\{|z| = 1\}$ is both forward and backward invariant under p if and only if $p(z) = \alpha z^d$, where $|\alpha| = 1$.


THE UNIT CIRCLE
(0,1)
R = 1
(-1,0)
(1,0)
(0,-1)

The case of a segment

Let p be a complex polynomial of degree $d \geqslant 2$ such that $H_p = p^{-1}(H_p) = J_p$. Then J_p is a line segment.

Proof.

Recall that for any polynomial p the Julia set J_p has empty interior. If $J_p = H_p$, then J_p is a closed convex set in $\mathbb{C}$ with empty interior, and hence it is a subset of a line. Being connected and compact, it must be a (closed) segment. $\square$

Auxiliary results

Lemma

For every complex polynomial p of degree $d \geqslant 2$, $K_p \subset H_p$.

Lemma

Let p be a complex polynomial of degree $d \geqslant 2$. *If* $H_p = p^{-1}(H_p)$, *then* $H_p \subset K_p$.

Convex hull vs. holomorphically convex hull

Let Ω be an open set in $\mathbb{C}$. Let $A(\Omega)$ denote the class of holomorphic functions in Ω. Let Z be an arbitrary compact subset of Ω. Then the holomorphically convex hull $\widehat{Z}$ of Z in Ω is defined as

$$\widehat{Z} = \widehat{Z}_\Omega = \{z \in \Omega : |f(z)| \leqslant \sup_Z |f| \ \forall f \in A(\Omega)\}.$$

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The hull $\widehat{Z}_\Omega$ is the union of Z and the connected components of $\Omega \setminus Z$ which are relatively compact in Ω. Furthermore we have $\widehat{Z} \subset \mathrm{conv} Z$.

The case of a circle

Let p be a complex polynomial of degree $d \geqslant 2$ such that $H_p = p^{-1}(H_p) \supsetneq J_p$. Then J_p is a circle.

Recall that the boundary of a convex set X with nonempty interior in $\mathbb{R}^2$ is homeomorphic to the unit circle. Furthermore, it has (positive and) finite length, hence is of Hausdorff dimension 1.

Theorem

(D. Hamilton, 1995) Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a rational function. Suppose that the Julia set J_f is a Jordan curve. Then $\dim(J_f) > 1$ or J_f is a circle/line.

What about non-polynomial rational maps?

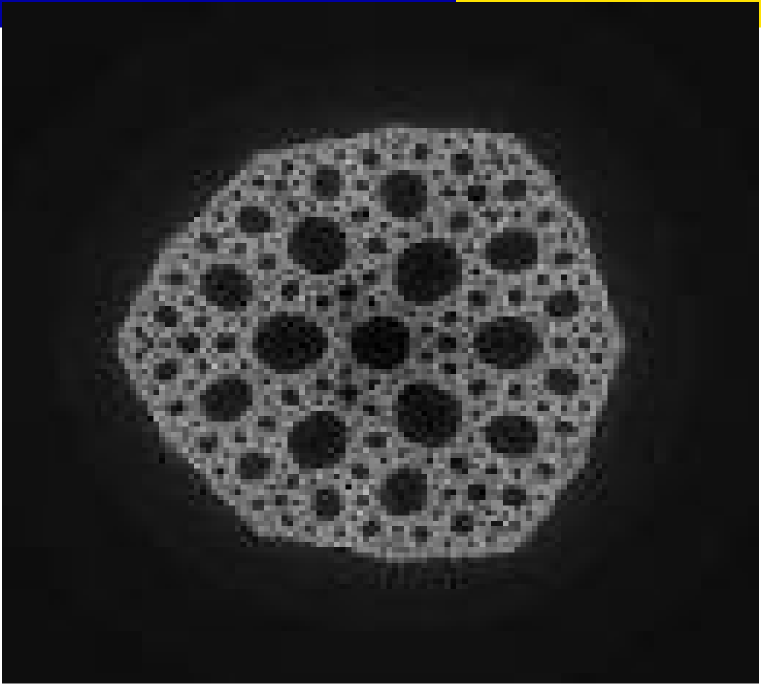
No complete answer is known yet, but let me present an example of a rational f with $f^{-1}(H_f) \subset H_f$ and two different examples with $f^{-1}(H_f) \not\subset H_f$.
A simple-minded example: $f(z) = 1/z^2$.
The Julia set J_f is $\{|z| = 1\}$, with $H_f = \{|z| \leqslant 1\}$. We have $f^{-1}(H_f) \not\subset H_f$. However, $f^{-2}(H_f) \subset H_f$.

Another example with $f^{-1}(H_f) \not\subset H_f$

Consider

$$f(z) = f_\rho(z) = \frac{-\rho^2 z^2}{z^3 - 1}$$

with $\rho > \rho_\epsilon = \sqrt{\frac{2-\epsilon(\epsilon^2-3\epsilon+3)}{1-\epsilon}}$, where $\epsilon \in (0,1)$ is fixed (after D. Look).
Here $f^{-n}(H_f) \not\subset H_f$ for any $n \in \mathbb{N}$.



In this example, 0 is a superattracting fixed point for f and $f^{-1}(0) = \{0, \infty\}$. The immediate attracting basin $\mathcal{O}$ of 0 is simply connected and $\partial\mathcal{O}$ is a simple closed curve. There is a neighborhood B of ∞ such that $f(B) = \mathcal{O}$ and $f^{-1}(\mathcal{O}) = \mathcal{O} \cup B$. Note that $\mathcal{O} \subset \widehat{J}_f \subset H_f$. Therefore $f^{-1}(\mathcal{O}) \subset f^{-1}(H_f)$, hence $B \subset f^{-1}(H_f)$ and $f^{-1}(H_f) \not\subset H_f$. Similarly, $f^{-n}(H_f) \not\subset H_f$ for any $n \in \mathbb{N}$.

If $f^{-1}(H_f) \subset H_f$, is ∞ necessarily in an attracting basin?

NO. Consider $f(z) := g(w^2)$, where $g(w) := \frac{3w+1}{w+3}$ (after A. F. Beardon).
The map f has no (super)attracting cycles, of any period. To see this, note that $f^{\circ n} \to 1$ on $\{|z| < 1\}$ and on $\{|z| > 1\}$. In particular, the orbits of both critical points of f (0 and ∞) tend to 1. But the point 1 is in the Julia set $J_f = \{|z| = 1\}$. Here $f^{-1}(H_f) = H_f$.

THANK YOU FOR YOUR ATTENTION!

Some references

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Proof of Thurston's theorem

If the convex hull of the roots of p does not contain the convex hull $\mathcal{C}$ of critical points of p , there would exist a closed half-plane E intersecting $\mathcal{C}$ but avoiding all roots of p. Hence $p|_E$ would not be surjective. For the converse, consider the convex hulls of roots of $p - w$ for all complex constants w.