Zeros of random linear combinations of orthogonal polynomials with complex Gaussian coefficients

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Abstract

We study zero distribution of random linear combinations of the form

$$P_n(z)=\sum_{j=0}^{n}\eta_j f_j(z),$$

in any Jordan region $\Omega\subset\mathbb{C}$. The basis functions f_j are entire functions that are real-valued on the real line, and $\eta_0,\dots,\eta_n$ are complex-valued iid Gaussian random variables. We derive an explicit intensity function for the number of zeros of P_n in Ω for each fixed n. Our main applications are to polynomials orthogonal on the real line and polynomials orthogonal on the unit circle. Using the Christoffel-Darboux formula, the intensity function in these cases takes a very simple shape. Moreover, we give the limiting values of the intensity function in these cases when the orthogonal polynomials are associated to Szegő weights.

The Explicit Formula for the Intensity Function

Let $\{f_j(z)\}_{j=0}^{n}$ be a sequence of entire functions that are real-valued on the real line. Let $\Omega\subset\mathbb{C}$ be a Jordan region. We will be studying the expectation of the number zeros in Ω, $\mathbb{E}[N_n(\Omega)]$, of random sums of the form

$$P_n(z)=\sum_{j=0}^{n}\eta_j f_j(z),\qquad z\in\mathbb{C},\tag{1}$$

where n is a fixed integer, and $\eta_j=\alpha_j+i\beta_j$, $j=0,1,\dots,n$, with $\{\alpha_j\}_{j=0}^{n}$ and $\{\beta_j\}_{j=0}^{n}$ being sequences of independent standard normal random variables. The formula we derive for the intensity is expressed in terms of the kernels

$$K_n(z,w)=\sum_{j=0}^{n}f_j(z)\overline{f_j(w)},\qquad K_n^{(0,1)}(z,w)=\sum_{j=0}^{n}f_j(z)\overline{f_j'(w)},\qquad\text{and}\qquad K_n^{(1,1)}(z,w)=\sum_{j=0}^{n}f_j'(z)\overline{f_j'(w)}.\tag{2}$$

Theorem

For each Jordan region $\Omega\subset\{z\in\mathbb{C}:K_n(z,z)\neq 0\}$, the intensity function h_n for the random sum (1) is given by

$$\mathbb{E}[N_n(\Omega)]=\int_{\Omega}h_n(x,y)\,dx\,dy,$$

with

$$h_n(x,y)=h_n(z)=\frac{K_n^{(1,1)}(z,z)K_n(z,z)-\left|K_n^{(0,1)}(z,z)\right|^2}{\pi\left(K_n(z,z)\right)^2},\tag{3}$$

where the kernels $K_n(z,z)$, $K_n^{(0,1)}(z,z)$, and $K_n^{(1,1)}(z,z)$, are defined in (2).

Applications to OPRL

We say that a collection of polynomials $\{p_j(z)\}_{j\geq 0}$ are *orthogonal on the real line* (OPRL) *with respect to* μ, with supp $\mu\subseteq\mathbb{R}$, if

$$\int p_n(x)p_m(x)d\mu(x)=\delta_{nm},\qquad\text{for all } n,m\in\mathbb{N}\cup\{0\}.$$

Setting $f_j(z)=p_j(z)$ in (1) for $j=0,1,\dots,n$, with the p_j's being OPRL, we use the Christoffel-Darboux formulas (Theorem 3.2.2 p. 43 of [2]),

$$K_n(z,\overline{w})=\sum_{j=0}^{n}p_j(z)p_j(w)=\frac{k_n}{k_{n+1}}\cdot\frac{p_{n+1}(z)p_n(w)-p_n(z)p_{n+1}(w)}{z-w},$$

$$K_n(z,\overline{z})=\sum_{j=0}^{n}(p_j(z))^2=\frac{k_n}{k_{n+1}}\cdot(p_{n+1}'(z)p_n(z)-p_n'(z)p_{n+1}(z)),$$

where k_n and k_{n+1} are the leading coefficients of p_n and p_{n+1} respectively, to obtain representations for kernels that make up the intensity function (3).

Theorem

Let $P_n(z)=\sum_{j=0}^{n}\eta_j p_j(z)$, where the η_j's are complex-valued iid Gaussian r.v., and the p_j's are OPRL. The intensity function $h_n(z)$ defined in (3) for $P_n(z)$ simplifies as

$$h_n(z)=\frac{1}{4\pi\left(\mathrm{Im}(z)\right)^2}-\frac{|p_{n+1}'(z)p_n(z)-p_n'(z)p_{n+1}(z)|^2}{4\pi\left(\mathrm{Im}(p_{n+1}(z)p_n(\overline{z})\right)^2},\qquad z\in\mathbb{C}.$$

Orthogonal Polynomials associated with Szegő Weights

We say that $f(\theta)\geq 0$ belongs to the *Szegő weight class*, denoted by G, if $f(\theta)$ is defined and measurable in $[-\pi,\pi]$, and the integrals

$$\int_{-\pi}^{\pi}f(\theta)\,d\theta,\qquad\int_{-\pi}^{\pi}|\log f(\theta)|\,d\theta$$

exist with the first integral assumed to be positive.

When $w(x)$ is a weight function supported on $[-1,1]$ with $w(\cos\theta)|\sin\theta|=f(\theta)\in G$, the *orthogonal polynomials associated to* $w(x)$ are polynomials $\{p_n(z)\}_{j\geq 0}$, where $z\in\mathbb{C}$, such that

$$\int_{-1}^{1}p_n(x)p_m(x)w(x)dx=\delta_{nm},\qquad\text{for all } n,m\in\mathbb{N}\cup\{0\}.$$

The Limiting Value of the Intensity Function for Random Orthogonal Polynomials spanned by OPRL associated with Szegő Weights

Theorem

Let $w(x)$ be a weight function on the interval $-1\leq x\leq 1$ such that $w(\cos\theta)|\sin\theta|=f(\theta)$ belongs to the weight class G. The intensity function for the random orthogonal polynomial $P_n(z)=\sum_{j=0}^{n}\eta_j p_j(z)$ with complex-valued iid Gaussian coefficients, where the p_j's are OPRL associated to $w(x)$, satisfies

$$\lim_{n\to\infty}h_n(z)=\frac{1}{4\pi\left(\mathrm{Im}(z)\right)^2}-\frac{\left|z+\sqrt{z^2-1}\right|^2}{4\pi|z^2-1|\left(\mathrm{Im}(z+\sqrt{z^2-1})\right)^2},\tag{4}$$

for all $z\in\mathbb{C}\setminus[-1,1]$. Furthermore, convergence in (4) holds uniformly on compact subsets of $\mathbb{C}\setminus[-1,1]$.

Applications to OPUC

The *orthogonal polynomials on the unit circle* (OPUC) *associated to a weight* $f(\theta)$, where $f(\theta)$ is a non-negative 2π periodic function that is Lebesgue integrable on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi}f(\theta)\,d\theta>0,$$

are polynomials $\{\phi_n(z)\}_{j\geq 0}$ that satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)\phi_n(e^{i\theta})\overline{\phi_m(e^{i\theta})}\,d\theta=\delta_{nm},\qquad\text{for all } n,m\in\mathbb{N}\cup\{0\}.$$

Remark: When we restrict the weight function $f(\theta)$ to be an even function, it follows that all the coefficients of each $\phi_j(z)$, $j=0,1,\dots,$ are real.

Taking $f_j(z)=\phi_j(z)$ in (1) for $j=0,1,\dots,n$, where the ϕ_j's are OPUC, using the Christoffel-Darboux formula for OPUC (Theorem 2.2.7, p. 124 of [1])

$$K_n(z,w)=\sum_{j=0}^{n}\phi_j(z)\overline{\phi_j(w)}=\frac{\overline{\phi_{n+1}^*(w)}\phi_{n+1}^*(z)-\overline{\phi_{n+1}(w)}\phi_{n+1}(z)}{1-\overline{w}z},$$

with $z,w\in\mathbb{C}$, $\overline{w}z\neq 1$, and $\phi_n^*(z)=z^n\overline{\phi_n\left(\frac{1}{\overline{z}}\right)}$, the kernels that make up intensity function (3) simplify to give the following:

Theorem

Let $f(\theta)$ be an even weight function associated to the OPUC $\{\phi_j(z)\}_{j=0}^{n}$. When $|z|\neq 1$, the intensity function $h_n(z)$ defined in (3) for the random orthogonal polynomial $P_n(z)=\sum_{j=0}^{n}\eta_j\phi_j(z)$ with complex-valued iid Gaussian coefficients, reduces to

$$h_n(z)=\frac{1}{\pi\left(1-|z|^2\right)^2}-\frac{|\phi_{n+1}^*(z)\phi_{n+1}'(z)-\phi_{n+1}^{*\,\prime}(z)\phi_{n+1}(z)|^2}{\pi\left(|\phi_{n+1}^*(z)|^2-|\phi_{n+1}(z)|^2\right)^2}.$$

The Limiting Value of the Intensity Function for Random Orthogonal Polynomials spanned by OPUC associated with Szegő Weights

Theorem

Let $\{\phi_j(z)\}_{j\geq 0}$ be OPUC associated to an even weight function $f(\theta)$ from the Szegő weight class. The intensity function $h_n(z)$ for the random orthogonal polynomial $P_n(z)=\sum_{j=0}^{n}\eta_j\phi_j(z)$ with complex-valued iid Gaussian coefficients has the property that

$$\lim_{n\to\infty}h_n(z)=\frac{1}{\pi\left(1-|z|^2\right)^2}\tag{5}$$

for $|z|\neq 1$. Moreover, the convergence in (5) holds uniformly on compact subsets of $\{z:|z|\neq 1\}$.

References

[1] B. Simon, *Orthogonal Polynomials on the Unit Circle*, American Mathematical Society Colloquium Publications, Vol. 54, Part I, Providence, RI, 2005.

[2] G. Szegő, *Orthogonal Polynomials* Fourth edition, American Mathematical Society, Providence, RI, 1975.

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