# Minimum Riesz Energy Problem on the Hyperdisk


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## Abstract

We consider the minimum Riesz $s$-energy problem on the unit disk in the Euclidean space $\mathbb{R}^d$, $d \geq 3$, immersed into a smooth rotationally invariant external field $Q$. The charges are assumed to interact via the Riesz potential $1/r^s$, with $d-3 < s < d-1$, where $r$ denotes the Euclidean distance. The problem is solved by finding an explicit expression for the extremal measure. We then consider the applications of the obtained results to a monomial external field and an external field generated by a positive point charge, located at some distance above the disk on the polar axis.


## Minimum Riesz $s$-energy problem

Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$, and $\mathbb{D}_R := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = 0, x_2^2 + x_3^2 + \ldots + x_d^2 \leq R^2\}$ be the disk of radius $R$ in $\mathbb{R}^d$, with $d \geq 3$, and where $|\cdot|$ is the Euclidean distance. The ring $\mathscr{R}(a,b)$ in $\mathbb{R}^d$ is defined as $\mathscr{R}(a,b) := \{(0, r\bar{x}) \in \mathbb{R}^d : a \leq r \leq b, \bar{x} \in \mathbb{S}^{d-2}\}$, and the unit disk in $\mathbb{R}^d$ will be denoted by $\mathbb{D}$. Given a compact set $E \subset \mathbb{D}$, consider the class $\mathcal{M}(E)$ of unit positive Borel measures supported on $E$. For $0 < s < d$, the **Riesz $s$-potential** and **Riesz $s$-energy** of a measure $\mu \in \mathcal{M}(E)$ are defined respectively as

$$U_s^\mu(x) := \int \frac{1}{|x-y|^s}\, d\mu(y), \quad I_s(\mu) := \iint \frac{1}{|x-y|^s}\, d\mu(x) d\mu(y).$$

Let $W_s(E) := \inf\{I_s(\mu) : \mu \in \mathcal{M}(E)\}$. Define the **Riesz $s$-capacity** of $E$ as $\operatorname{cap}_s(E) := 1/W_s(E)$. When $\operatorname{cap}_s(E) > 0$, there is a unique $\mu_E$ such that $I_s(\mu_E) = W_s(E)$. Such $\mu_E$ is called the **Riesz $s$-equilibrium measure** for $E$.

An **external field** is defined as a continuous function $Q : E \to [0, \infty]$, such that $Q(x) < \infty$ on a set of positive surface area measure.

The **weighted energy associated with** $Q(x)$ is then defined by

$$I_Q(\mu) := I_s(\mu) + 2\int Q(x) d\mu(x).$$

The **minimum energy problem on $\mathbb{D}$ in the presence of the external field** $Q(x)$ refers to the minimal quantity

$$V_Q := \inf\{I_Q(\mu) : \mu \in \mathcal{M}(E)\}.$$

A measure $\mu_Q \in \mathcal{M}(E)$ such that $I_Q(\mu_Q) = V_Q$ is called the **$s$-extremal (or positive Riesz $s$-equilibrium) measure associated with** $Q(x)$. The potential $U_s^{\mu_Q}$ of the measure $\mu_Q$ satisfies the Gauss variational inequalities

$$U_s^{\mu_Q}(x) + Q(x) \geq F_Q \quad \text{on} \quad E, \tag{1}$$
$$U_s^{\mu_Q}(x) + Q(x) = F_Q \quad \text{for all} \quad x \in S_Q, \tag{2}$$

where $F_Q := V_Q - \int Q(x)\, d\mu_Q(x)$, and $S_Q := \operatorname{supp} \mu_Q$.

## Sufficient conditions on an external field $Q$ that guarantee that the support of the extremal measure $\mu_Q$ is a ring or a disk

### Theorem

*Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$. Assume that an external field $Q : \mathbb{D} \to [0, \infty]$ is invariant with respect to the rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \leq r \leq 1$. Further suppose that $Q$ is a convex function, that is $Q(r)$ is convex on $(0,1)$. Then the support of the extremal measure $\mu_Q$ is a ring $\mathscr{R}(a,b)$, contained in the disk $\mathbb{D}$. In other words, there exist real numbers $a$ and $b$ such that $0 \leq a < b \leq 1$, so that $\operatorname{supp} \mu_Q = \mathscr{R}(a,b)$.*

*Furthermore, if $Q(r)$ is, in addition, an increasing function, then then $a = 0$, which implies that the support of the extremal measure $\mu_Q$ is a disk of radius $b \leq 1$, centered at the origin. On the other hand, if $Q(r)$ is a decreasing function, then $b = 1$, that is the support of the extremal measure $\mu_Q$ will be a ring with outer radius 1.*

## Recovering of the extremal measure $\mu_Q$

### Theorem

*Suppose that the support of the extremal measure $\mu_Q$ is the disk $\mathbb{D}_R$, and the external field $Q$ is invariant with respect to rotations about the polar axis, that is $Q(x) = Q(r)$, where $x = (0, r\bar{x}) \in \mathbb{D}_R$, $\bar{x} \in \mathbb{S}^{d-2}$, $0 \leq r \leq R$. Also assume that $Q \in C^2(\mathbb{D}_R)$. Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$, and let*

$$F(t) = \frac{\sin(\lambda\pi)\, \Gamma((d-3)/2 + \lambda)}{\pi^{(d+1)/2}\, \Gamma(\lambda)} \frac{1}{t} \frac{d}{dt} \int_t^R \frac{g(r)\, r\, dr}{(r^2 - t^2)^{1-\lambda}}, \quad g(r) = \frac{1}{r^{d+2\lambda-4}} \frac{d}{dr} \int_0^r \frac{Q(u)\, u^{d-2}\, du}{(r^2 - u^2)^{1-\lambda}}, \quad 0 \leq r \leq R.$$

*Then for the extremal measure $\mu_Q$ we have*

$$d\mu_Q(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}_R, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \leq r \leq R,$$

where the density $f$ is explicitly given by

$$f(r) = C_Q\, (R^2 - r^2)^{\lambda-1} + F(r), \quad 0 \leq r \leq R,$$

with the constant $C_Q$ uniquely defined by

$$C_Q = \frac{2\, \Gamma((d-1)/2 + \lambda)}{\Gamma(\lambda)\, \Gamma((d-1)/2)} \frac{1}{R^{d+2\lambda-3}} \left\{ \frac{\Gamma((d-1)/2)}{2\, \pi^{(d-1)/2}} - \int_0^R F(t)\, t^{d-2}\, dt \right\}.$$

## Applications

### Definition

The **$\mathcal{F}$-functional** of a compact subset $E \subset \mathbb{D}$ of positive Riesz $s$-capacity is defined as

$$\mathcal{F}_s(E) := W_s(E) + \int Q(x)\, d\mu_E(x),$$

where $W_s(E)$ is the Riesz $s$-energy of the compact $E$ and $\mu_E$ is the equilibrium measure (with no external field) on $E$. (see [1])

### Proposition

*Let $Q$ be an external field on $\mathbb{D}$. Then $\mathcal{F}_s$-functional is minimized for $S_Q = \operatorname{supp} \mu_Q$.*

### Proposition

*If $E = \mathbb{D}_R$, then $\mathcal{F}_s$-functional is given by*

$$\mathcal{F}_s(\mathbb{D}_R) = \frac{\pi\, \Gamma((d+2\lambda-1)/2)}{\sin(\lambda\pi)\, \Gamma(\lambda)\, \Gamma((d-1)/2)} \frac{1}{R^{d+2\lambda-3}} \left\{ 1 + \frac{2\sin(\lambda\pi)}{\pi} \int_0^R Q(r)\, (R^2 - r^2)^{\lambda-1}\, r^{d-2}\, dr \right\}.$$

## External field generated by a monomial

Consider the situation when the disk $\mathbb{D}$ is immersed into an external field given by a monomial, namely

$$Q(x) = qr^m, \quad q > 0, \quad m > 1, \quad x = (0, r\bar{x}) \in \mathbb{D}, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \leq r \leq 1. \tag{3}$$

### Theorem

*Let $s = (d-3) + 2\lambda$, with $0 < \lambda < 1$. The extremal measure $\mu_Q$, corresponding to the monomial external field (3), is supported on the disk $\mathbb{D}_{R_*}$, where $R_*$ is defined as*

$$R_* = \left( \frac{(d+2\lambda-3)\pi\Gamma((d+m+2\lambda-1)/2)}{qm\sin(\lambda\pi)\Gamma(\lambda)\Gamma((d+m-1)/2)} \right)^{1/(d+m+2\lambda-3)}.$$

*For the extremal measure $\mu_Q$ we have*

$$d\mu_Q(x) = f(r)\, r^{d-2}\, dr\, d\sigma_{d-1}(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}_{R_*}, \quad \bar{x} \in \mathbb{S}^{d-2}, \quad 0 \leq r \leq R_*,$$

*with the density $f(r)$ is given by*

$$f(r) = \frac{\Gamma((d+2\lambda-1)/2)}{\pi^{(d-1)/2}\, \Gamma(\lambda)} \left\{ \frac{1}{R_*^{d+2\lambda-3}} + \frac{q\sin(\lambda\pi)\, \Gamma((d+m-1)/2)\, \Gamma(\lambda)}{\pi\, \Gamma((d+m+2\lambda-1)/2)} R_*^m \right\} (R^2 - r^2)^{\lambda-1} + F(r), \quad 0 \leq r \leq R_*,$$

*where*

$$F(r) = \frac{q\, \sin(\lambda\pi)\, \Gamma((m+d-1)/2)\, \Gamma((d+2\lambda-3)/2)}{\pi^{(d+1)/2}\, \Gamma((d+m+2\lambda-3)/2)} R_*^m\, (R_*^2 - r^2)^{\lambda-1} \times$$
$$\left\{ -\,{}_2F_1\left( -\frac{m}{2}, 1; \lambda+1; 1 - \left(\frac{r}{R_*}\right)^2 \right) + \frac{m}{2\lambda(\lambda+1)} \left( 1 - \left(\frac{r}{R_*}\right)^2 \right) {}_2F_1\left( 1 - \frac{m}{2}, 2; \lambda+2; 1 - \left(\frac{r}{R_*}\right)^2 \right) \right\}, \quad 0 \leq r \leq R_*.$$

## External field generated by a positive unit point charge corresponding to the Coulomb potential in $\mathbb{R}^3$

### Theorem

*Suppose the external field $Q$ is given by $Q(x) = 1/\sqrt{r^2 + h^2}$, $x = (0, r\bar{x}) \in \mathbb{D}$, $\bar{x} \in \mathbb{S}^1$, $0 \leq r \leq 1$ and where $h$ is chosen such that $h > h_+$, where $h_+$ is the unique positive root of the function*

$$p(h) = \frac{1}{2\pi} \left( 1 + \frac{2h \tan^{-1}(1/h)}{\pi\sqrt{1+h^2}} \right) - \frac{1}{\pi^2 h} - \frac{1}{\pi^2 h^2} \tan^{-1}(1/h).$$

*Then, under these assumptions $S_Q = \mathbb{D}$, and the extremal measure $\mu_Q$ is given by*

$$d\mu_Q(x) = f(r)\, r\, dr\, d\sigma_2(\bar{x}), \quad x = (0, r\bar{x}) \in \mathbb{D}, \quad \bar{x} \in \mathbb{S}^1, \quad 0 \leq r \leq 1,$$

*where the density $f(r)$ is*

$$f(r) = \frac{1}{2\pi} \left( 1 + \frac{2h \tan^{-1}(1/h)}{\pi\sqrt{1+h^2}} \right) \frac{1}{\sqrt{1-r^2}} - \frac{h}{\pi^2(h^2+r^2)} \frac{1}{\sqrt{1-r^2}} - \frac{h}{\pi^2} \frac{1}{(h^2+r^2)^{3/2}} \tan^{-1} \sqrt{\frac{1-r^2}{h^2+r^2}}, \quad 0 \leq r \leq 1.$$

This Theorem is a consequence of a **more general statement valid for the general Riesz $s$-potentials** generated by a positive point charge in higher dimensions (see also [2]).

## References


[1] Brauchart J., Dragnev P., Saff E.: Riesz extremal measures on the sphere for axis-supported external fields, J. Math. Anal. Appl., 356, pp. 769–792 (2009)

[2] Copson, E.T.: On the problem of the electrified disc, Proc. Edinburgh Math. Soc., 8, pp. 14–19 (1947)