A HOPF'S LEMMA FOR HIGHER ORDER DIFFERENTIAL INEQUALITIES AND ITS APPLICATIONS

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ABSTRACT. We establish a sequential Hopf's Lemma for higher order differential inequalities in one variable and give some applications of this result.

1. INTRODUCTION

The Hopf's Lemma is one of the fundamental tools in the study of elliptic partial differential equations [3]. There have been many variations and generalizations of this lemma, for example [4], [5], and [6]. But there appears to be no work in the literature on the Hopf's lemma for third or higher order equations, perhaps partially because the maximum principle fails for higher order equations.

In this paper we study this question in the one dimensional case and prove a sequential Hopf's lemma of higher order in one variable. One application of this result is the following comparison theorem for nth order nonlinear differential operators.

Theorem 1.1. *Assume that $K(z_1, ..., z_{n+2}) \in C^1(\mathbf{R}^{n+2})$ and $\frac{\partial K}{\partial z_{n+2}} > 0$, where $n \geq 2$. Suppose $u(x)$ and $v(x)$ are two functions in $C^n((a,b))$ that satisfy*

(1) $$K\left(x, u(x), u'(x), ..., u^{(n)}(x)\right) \leq K\left(x, v(x), v'(x), ..., v^{(n)}(x)\right) \qquad \text{for all } x \in (a,b)$$

and

$$u(x_0) = v(x_0), \quad u'(x_0) = v'(x_0), \quad \dots, \quad u^{(n-1)}(x_0) = v^{(n-1)}(x_0) \qquad \text{for some } x_0 \in (a,b).$$

If n is even, then there exists $\delta > 0$ such that $u(x) \leq v(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$.
If n is odd, then there exists $\delta > 0$ such that $u(x) \geq v(x)$ for $x \in (x_0 - \delta, x_0)$ and $u(x) \leq v(x)$ for $x \in (x_0, x_0 + \delta)$.

This theorem shows that if u and v have $(n-1)$-th order of contact at a point x_0, then they intersect only once in a small neighborhood of x_0. The crucial ingredient in the proof is a higher order sequential version of Hopf's lemma.

Theorem 1.2. *Let $u \in C^n((a,b)) \bigcap C^{n-1}([a,b))$ be a function which satisfies*

(2) $$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0 \quad \text{for } \; x \in (a,b),$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C([a,b))$. Suppose u satisfies

(3) $$u(a) \; = \; u'(a) \; = \; \cdots \; = u^{(n-2)}(a) \; = \; 0,$$

and

(4) *there exists a sequence $\{x_i\}$ such that $a < x_i < b, \; x_i \to a, \;$ and $\; u(x_i) > 0$.*

Then $u^{(n-1)}(a) > 0$. Furthermore, $u > 0$ in a neighborhood of a.

When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

The Taylor's expansion of u at a and Condition (4) easily imply that $u^{(n-1)}(a) \geq 0$, so the key is that it is strictly positive. At the right side endpoint of an interval, we have

Theorem 1.3. *Let $u \in C^n((a,b)) \bigcap C^{n-1}((a,b])$ be a function which satisfies*

$$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0 \quad \textit{for} \;\; x \in (a,b),$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C((a,b])$. Suppose u satisfies

$$u(b) \;=\; u'(b) \;=\; \cdots \;= u^{(n-2)}(b) \;=\; 0. \tag{5}$$

If n is even and

$$\textit{there exists a sequence } \{x_i\} \textit{ such that } a < x_i < b, \;\; x_i \to b, \;\; \textit{and} \;\; u(x_i) > 0, \tag{6}$$

then $u^{(n-1)}(b) < 0$ and $u > 0$ in a neighborhood of b.

If n is odd and

$$\textit{there exists a sequence } \{x_i\} \textit{ such that } a < x_i < b, \;\; x_i \to b, \;\; \textit{and} \;\; u(x_i) < 0, \tag{7}$$

then $u^{(n-1)}(b) < 0$ and $u < 0$ in a neighborhood of b.

When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

In the subsequent sections we will prove the above theorems and discuss some applications.

2. Proof of the Comparison Theorem

Since u must be negative or 0 near a if condition (4) is not met, an equivalent statement of Theorem 1.2 is

Theorem 2.1. *Let $u \in C^n((a,b)) \bigcap C^{n-1}([a,b))$ be a function which satisfies*

$$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0 \quad \textit{for} \;\; x \in (a,b),$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C([a,b))$.
If

$$u(a) \;=\; u'(a) \;=\; \cdots \;= u^{(n-2)}(a) \;= u^{(n-1)}(a) \;=\; 0,$$

then $u(x) \leq 0$ for all x sufficiently close to a.
When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

Similarly, an equivalence of Theorem 1.3 is

Theorem 2.2. *Let $u \in C^n((a,b)) \bigcap C^{n-1}((a,b])$ be a function which satisfies*

$$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0 \quad \textit{for} \;\; x \in (a,b),$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C((a,b])$.
Suppose

$$u(b) \;=\; u'(b) \;=\; \cdots \;= u^{(n-2)}(b) \;= u^{(n-1)}(b) \;=\; 0.$$

If n is even, then $u(x) \leq 0$ for all x sufficiently close to b.

If n is odd, then $u(x) \geq 0$ for all x sufficiently close to b.

When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

Note that

$$\begin{aligned} & K\left(x, u(x), u'(x), ..., u^{(n)}(x)\right) - K\left(x, v(x), v'(x), ..., v^{(n)}(x)\right) \\ = \;& c_0(x)(u - v) + \cdots + c_{n-1}(u^{(n-1)} - v^{(n-1)}) + c_n(u^{(n)} - v^{(n)}) \end{aligned}$$

where

$$\begin{aligned} c_0(x) &= \int_0^1 \frac{\partial K}{\partial z_2}\left(x, tu(x) + (1-t)v(x), ..., tu^{(n)}(x) + (1-t)v^{(n)}(x)\right) dt, \\ &\;\vdots \\ c_{n-1}(x) &= \int_0^1 \frac{\partial K}{\partial z_{n+1}}\left(x, tu(x) + (1-t)v(x), ..., tu^{(n)}(x) + (1-t)v^{(n)}(x)\right) dt, \\ c_n(x) &= \int_0^1 \frac{\partial K}{\partial z_{n+2}}\left(x, tu(x) + (1-t)v(x), ..., tu^{(n)}(x) + (1-t)v^{(n)}(x)\right) dt. \end{aligned}$$

Let $w(x) = u(x) - v(x)$. By (1) we have

$$c_0 w + c_1 w' + \cdots + c_{n-1} w^{(n-1)} + c_n w^{(n)} \leq 0.$$

If $\frac{\partial K}{\partial z_{n+2}} > 0$, then $c_n > 0$, so

$$w^{(n)}(x) + \frac{c_{n-1}}{c_n} w^{(n-1)}(x) + \cdots + \frac{c_1}{c_n} w'(x) + \frac{c_0}{c_n} w(x) \leq 0.$$

The initial condition implies that

$$w(x_0) = 0, \quad w'(x_0) = 0, \quad ... \quad w^{(n-1)}(x_0) = 0.$$

By Theorem 2.1, there exists $\delta > 0$ such that $w(x) \leq 0$ for $x \in (x_0, x_0 + \delta)$.

If n is even, applying Theorem 2.2 and choosing a smaller δ if necessary, we know that $w(x) \leq 0$ for $x \in (x_0 - \delta, x_0)$.

If n is odd, applying Theorem 2.2 and choosing a smaller δ if necessary, we know that $w(x) \geq 0$ for $x \in (x_0 - \delta, x_0)$.

Therefore,
if n is even, then $u(x) \leq v(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$;
if n is odd, then $u(x) \geq v(x)$ for $x \in (x_0 - \delta, x_0)$ and $u(x) \leq v(x)$ for $x \in (x_0, x_0 + \delta)$.
This completes the proof of Theorem 1.1.

3. The Sequential Form of the Second Order Hopf's Lemma

Next, we will establish the higher order sequential versions of Hopf's lemma which are crucial in the proof of Theorem 1.1. We first need to prove the following sequential Hopf's lemma in second order.

Theorem 3.1. *Let $u \in C^2((a,b)) \bigcap C^1([a,b))$ be a function which satisfies*

$$u''(x) + a_1(x)u'(x) + a_0(x)u(x) \le 0 \quad \text{for } x \in (a,b)$$

where $|a_1(x)|$ and $|a_0(x)|$ are bounded by some constant $C > 0$. Assume that u satisfies

$$u(a) = 0,$$

and Condition (4).
Then $u'(a) > 0$. Furthermore, $u > 0$ in a neighborhood of a.

The classical second order Hopf's lemma requires that $u(x) > 0$ for all x greater than and sufficiently close to a, that is, $u(a)$ is a local minimum. But here we only need the weaker assumption that u is positive at a sequence of points approaching a, and we can show that then u must be actually positive at all points near the boundary a. In other words, $u(x)$ cannot oscillate around the y-axis as x approaches a.

In this section we present a proof of Theorem 3.1 that relies on the following maximum principle on small intervals. An alternative proof is given in the Appendix.

Lemma 3.2. *Suppose $g \in C^2((a,b)) \bigcap C^1([a,b))$ satisfies*

$$L[g] = g''(x) + a_1(x)g'(x) + a_0(x)g(x) \le 0 \quad \text{for } x \in (a,b),$$

where $|a_1(x)|, |a_0(x)|$ are bounded by some constant $C > 0$. Then there exists a constant $\delta = \delta(C) > 0$ such that on any interval $[c,d] \subseteq [a,b)$ with $|d-c| < \delta$, we have $g \ge 0$ provided $g(c) \ge 0$ and $g(d) \ge 0$.

Proof: Without loss of generality we can assume $c = 0$. Define

$$h(x) = e^{\gamma\delta} - e^{\gamma x} \qquad \text{and} \qquad w(x) = \frac{g(x)}{h(x)},$$

where $\gamma, \delta > 0$ are to be chosen. Then

$$\begin{aligned} L[g] &= \frac{d^2}{dx^2}\left(w(x)h(x)\right) + a_1(x)\frac{d}{dx}\left(w(x)h(x)\right) + a_0(x)\left(w(x)h(x)\right) \\ &= hw'' + \left(2h' + a_1 h\right) w'(x) + L[h]w(x). \end{aligned} \tag{8}$$

Suppose the minimum of w is negative and achieved at some $x_0 \in (0,d)$. Then

$$w''(x_0) \ge 0, \qquad w'(x_0) = 0, \qquad \text{and} \qquad w(x_0) < 0.$$

By definition

$$h(x) > 0 \qquad \text{if} \qquad 0 \le x \le d < \delta.$$

Direct computation shows that

$$\begin{aligned} L[h] &= e^{\gamma x} \cdot \left(-\gamma^2 - a_1\gamma + a_0\left(e^{\gamma\delta - \gamma x} - 1\right)\right) \\ &\le e^{\gamma x}\left(-\gamma^2 + C\gamma + C(e^{\gamma\delta} - 1)\right) \quad \text{when } 0 \le x \le d < \delta. \end{aligned}$$

We first choose $\gamma > 0$ sufficient large so that $-\gamma^2 + C\gamma + 2C < 0$, then we choose $0 < \delta < \frac{\ln 3}{\gamma}$ so $0 < e^{\gamma\delta} - 1 < 2$. Thus

$$L[h] \leq e^{\gamma x}\left(-\gamma^2 + C\gamma + 2C\right) < 0$$

when $0 \leq x \leq d < \delta$.

Then by (8) it follows that $L[g](x_0) > 0$. This contradiction proves that the minimum of w on $[0, d]$ must be nonnegative, thus $g(x) \geq 0$ on $[0, d]$ since $h(x) > 0$.

□

Next, we use Lemma 3.2 to prove Theorem 3.1.

Proof: Without loss of generality we can assume $a = 0$.

Denote

$$L[u] := u''(x) + a_1(x)u'(x) + a_0(x)u(x).$$

Let

$$g(x) = u(x) - \epsilon\left(e^{\lambda x} - 1\right),$$

where $\epsilon > 0$ will be chosen later.

For $x \geq 0$ and $\lambda > 0$,

$$\begin{aligned} L[e^{\lambda x} - 1] &= \lambda^2 e^{\lambda x} + a_1(x)\lambda e^{\lambda x} + a_0(e^{\lambda x} - 1) \\ &= e^{\lambda x}\left(\lambda^2 + a_1\lambda + a_0\left(1 - e^{-\lambda x}\right)\right) \\ &\geq e^{\lambda x}\left(\lambda^2 - C\lambda - C\right) \\ &> 0 \end{aligned}$$

when λ is chosen to be sufficiently large. Thus we know

$$\begin{aligned} L[g] &= L[u] - \epsilon L[e^{\lambda x} - 1] \\ &< 0. \end{aligned}$$

By definition $g(0) = 0$. Since the sequence $x_i \to 0$, we may choose an index i_0 such that $0 < x_{i_0} < \delta$, where δ is chosen as in Lemma 3.2. Because $u(x_{i_0}) > 0$, we can choose

$$\epsilon = \frac{u(x_{i_0})}{e^{\lambda x_{i_0}} - 1} > 0$$

in the definition of $g(x)$. Then we have $g(x_{i_0}) = 0$.

Now Lemma 3.2 implies

$$g(x) \geq 0 \qquad \text{on} \qquad [0, x_{i_0}].$$

The Taylor expansion of g at 0 gives

$$g(x) = g'(0)x + O(x^2),$$

thus $g'(0) \geq 0$. Consequently

$$u'(0) = g'(0) + \epsilon\lambda > 0.$$

□

Lemma 3.2 shows that if g is nonnegative at the two endpoints of a sufficiently small interval, then $g \geq 0$ in that interval. For third and higher order differential inequalities, it no longer holds. To see this, consider the sequence of functions

$$g_i(x) = \left(x - \frac{1}{i}\right)^2 - \frac{1}{i^2}.$$

Each function satisfies the differential equation $u_i^{(k)} = 0$ for all $k = 3, 4, ...$ Although $g_i(0) = g_i(\frac{2}{i}) = 0$ and $\frac{2}{i} \to 0$, $g_i(x)$ is negative on $(0, \frac{2}{i})$.

The classical maximum principle also fails in the higher order case. For example, the function $u(x) = \sin x$ satisfies

$$\begin{aligned} u^{(3)} + u' + 0 \cdot u &= 0 \\ u^{(4)} + u'' + 0 \cdot u &= 0 \\ &\vdots \end{aligned}$$

and $u(0) = u(2\pi) = 0$, but $u \leq 0$ on $[\pi, 2\pi]$.

Therefore, there exists a very interesting distinction between the Hopf's lemma and maximum principle in higher orders. Although for the second order inequalities the Hopf's lemma can be used to prove the maximum principle, in the higher order case the maximum principle fails, but the Hopf's lemma still holds.

4. The Higher Order Hopf's Lemmas

Now we are ready to prove the higher order Hopf's Lemma, Theorems 1.2 and 1.3.

Proof of Theorem 1.2:
We will employ a reduction of order technique and use mathematical induction. The case $n = 2$ is provided by Theorem 3.1. Suppose the theorem is true for $n = k \geq 2$, we will show that it is also true for $n = k + 1$, i.e. assume u satisfies

$$u^{(k+1)}(x) + a_k(x)u^k(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0,$$

where $a_k(x), ..., a_1(x), a_0(x)$ are in $C([a, b))$,

$$u(a) = u'(a) = \cdots = u^{(k-1)}(a) = 0, \tag{9}$$

and Condition (4), we need to show that $u^{(k)}(a) > 0$.
Let

$$v := fu + u', \tag{10}$$

where f is to be chosen. We then have

$$
\begin{aligned}
v' &= f'u + fu' + u'' \\
v'' &= f''u + 2f'u' + fu'' + u^{(3)} \\
v^{(3)} &= f^{(3)}u + 3f''u' + 3f'u'' + fu^{(3)} + u^{(4)} \\
&\vdots \\
v^{(k-2)} &= f^{(k-2)}u + \binom{k-2}{1} f^{(k-3)}u' + \cdots + \binom{k-2}{k-3} f'u^{(k-3)} + fu^{(k-2)} + u^{(k-1)} \\
v^{(k-1)} &= f^{(k-1)}u + \binom{k-1}{1} f^{(k-2)}u' + \cdots + \binom{k-1}{k-2} f'u^{(k-2)} + fu^{(k-1)} + u^{(k)} \\
v^{(k)} &= f^{(k)}u + \binom{k}{1} f^{(k-1)}u' + \cdots + \binom{k}{k-1} f'u^{(k-1)} + fu^{(k)} + u^{(k+1)}.
\end{aligned}
\tag{11}
$$

We would like to choose appropriate functions $b_0(x), b_1(x), ..., b_{k-1}(x) \in C([a, b))$, such that

$$
\begin{aligned}
& u^{(k+1)}(x) + a_k(x)u^k(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \\
= \; & v^{(k)}(x) + b_{k-1}(x)v^{k-1}(x) + \cdots + b_1(x)v'(x) + b_0(x)v(x).
\end{aligned}
\tag{12}
$$

Because of (10) and (11), the right hand side of (12) becomes

$$
\begin{aligned}
& f^{(k)}u + \binom{k}{1} f^{(k-1)}u' + \cdots + \binom{k}{k-1} f'u^{(k-1)} + fu^{(k)} + u^{(k+1)} \\
& + b_{k-1}\left[f^{(k-1)}u + \binom{k-1}{1} f^{(k-2)}u' + \cdots + \binom{k-1}{k-2} f'u^{(k-2)} + fu^{(k-1)} + u^{(k)} \right] \\
& + b_{k-2}\left[f^{(k-2)}u + \binom{k-2}{1} f^{(k-3)}u' + \cdots + \binom{k-2}{k-3} f'u^{(k-3)} + fu^{(k-2)} + u^{(k-1)} \right] \\
& + \cdots + b_2\left(f''u + 2f'u' + fu'' + u^{(3)}\right) + b_1\left(f'u + fu' + u''\right) + b_0(fu + u'),
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
& u^{(k+1)} + (f + b_{k-1})\, u^{(k)} + \left[\binom{k}{k-1} f' + b_{k-1}f + b_{k-2} \right] u^{(k-1)} \\
& + \left[\binom{k}{k-2} f'' + b_{k-1}\binom{k-1}{k-2} f' + b_{k-2}f + b_{k-3} \right] u^{(k-2)} + \cdots \\
& + \left[\binom{k}{2} f^{(k-2)} + b_{k-1}\binom{k-1}{2} f^{(k-3)} + b_{k-2}\binom{k-2}{2} f^{(k-4)} + \cdots + b_2 f + b_1 \right] u'' \\
& + \left[\binom{k}{1} f^{(k-1)} + b_{k-1}\binom{k-1}{1} f^{(k-2)} + b_{k-2}\binom{k-2}{1} f^{(k-3)} + \cdots + b_1 f + b_0 \right] u' \\
& + \left(f^{(k)} + b_{k-1}f^{(k-1)} + b_{k-2}f^{(k-2)} + \cdots + b_1 f' + b_0 f \right) u.
\end{aligned}
$$

In light of (12), we want to choose $b_0(x), b_1(x), ..., b_{k-1}(x)$ such that

$$
\begin{aligned}
a_k &= f + b_{k-1} \\
a_{k-1} &= \binom{k}{k-1} f' + b_{k-1} f + b_{k-2} \\
a_{k-2} &= \binom{k}{k-2} f'' + b_{k-1}\binom{k-1}{k-2} f' + b_{k-2} f + b_{k-3} \\
&\vdots \\
a_2 &= \binom{k}{2} f^{(k-2)} + b_{k-1}\binom{k-1}{2} f^{(k-3)} + b_{k-2}\binom{k-2}{2} f^{(k-4)} + \cdots + b_2 f + b_1 \\
a_1 &= \binom{k}{1} f^{(k-1)} + b_{k-1}\binom{k-1}{1} f^{(k-2)} + b_{k-2}\binom{k-2}{1} f^{(k-3)} + \cdots + b_1 f + b_0 \\
a_0 &= f^{(k)} + b_{k-1} f^{(k-1)} + b_{k-2} f^{(k-2)} + \cdots + b_1 f' + b_0 f.
\end{aligned} \tag{13}
$$

Solving for $b_{k-1}, ..., b_1, b_0$ from the first k equations, we obtain

$$
\begin{aligned}
b_{k-1} &= a_k - f \\
b_{k-2} &= a_{k-1} - \binom{k}{k-1} f' - b_{k-1} f \\
b_{k-3} &= a_{k-2} - \binom{k}{k-2} f'' - b_{k-1}\binom{k-1}{k-2} f' - b_{k-2} f \\
&\vdots \\
b_1 &= a_2 - \binom{k}{2} f^{(k-2)} - b_{k-1}\binom{k-1}{2} f^{(k-3)} - b_{k-2}\binom{k-2}{2} f^{(k-4)} - \cdots - b_2 f \\
b_0 &= a_1 - \binom{k}{1} f^{(k-1)} - b_{k-1}\binom{k-1}{1} f^{(k-2)} - b_{k-2}\binom{k-2}{1} f^{(k-3)} - \cdots - b_1 f.
\end{aligned} \tag{14}
$$

If the first equation in (14) is substituted into the second equation, b_{k-2} can be expressed as $a_{k-1} - \binom{k}{k-1} f' - a_k f + f^2$, which is a polynomial in f and f' with coefficients comprised of a_k, a_{k-1} and universal constants. Similarly b_{k-3}, ... , b_1, b_0 all can be expressed as polynomials in f and its derivatives, with the coefficients given by $a_0(x)$, ... , $a_k(x)$ and universal constants.

Thus we can write

$$
\begin{aligned}
b_{k-1} &= P_{k-1}\left(a_k, f\right) \\
b_{k-2} &= P_{k-2}\left(a_k, a_{k-1}, f, f'\right) \\
b_{k-3} &= P_{k-3}\left(a_k, a_{k-1}, a_{k-2}, f, f', f''\right) \\
&\vdots \\
b_1 &= P_1\left(a_k, a_{k-1}, ..., a_2, f, f', ..., f^{(k-2)}\right) \\
b_0 &= P_0\left(a_k, a_{k-1}, ..., a_1, f, f', ..., f^{(k-1)}\right).
\end{aligned} \tag{15}
$$

Here P_{k-1}, P_{k-2}, ... , P_1, P_0 are polynomials in f and its derivatives, and their coefficients depend on the continuous functions $a_k(x)$, $a_{k-1}(x)$, ... , $a_1(x)$.

Then we substitute (15) into the last equation in (13), so the function f must satisfy the k-th order ODE

$$f^{(k)} + P_{k-1}f^{(k-1)} + P_{k-2}f^{(k-2)} + \cdots + P_1 f' + P_0 f = a_0. \tag{16}$$

Under the initial condition $f(a) = 1$, Equation (16) has a solution $f \in C^k([a, a+\epsilon))$ for some $\epsilon > 0$. With this choice of f, (12) holds, so we know that

$$v^{(k)}(x) + b_{k-1}(x)v^{k-1}(x) + \cdots + b_1(x)v'(x) + b_0(x)v(x) \leq 0.$$

Definition (15) implies that the coefficient functions $b_{k-1}(x)$, ... $b_1(x)$, $b_0(x)$ are all continuous.

Since $f(a) = 1$ and

$$u(a) = u'(a) = ... = u^{k-1}(a) = 0,$$

from (11) we know that

$$v(a) = v'(a) = ... = v^{(k-2)}(a) = 0.$$

Because there exists a sequence $x_i \to a$ with $u(x_i) > 0$ and $u(a) = 0$, we can choose a sequence $\tilde{x}_i \to a$ such that $u(\tilde{x}_i) > 0$ and $u'(\tilde{x}_i) > 0$. Since $f(a) = 1$, when i is sufficiently large we have $f(\tilde{x}_i) > 0$. Therefore

$$v(\tilde{x}_i) = f(\tilde{x}_i)u(\tilde{x}_i) + u'(\tilde{x}_i) > 0.$$

Thus by the inductive hypothesis we know

$$v^{(k-1)}(a) > 0.$$

Then the second last equation in (11) and the initial conditions (9) implies

$$u^{(k)}(a) > 0.$$

The proof of Theorem 1.2 is now completed by mathematical induction.

□

Proof of Theorem 1.3:

(i) If n is even, define

$$\hat{u}(x) := u(2b - x).$$

Then $\hat{u} \in C^n((b, 2b-a)) \bigcap C^{n-1}([b, 2b-a))$ and

$$\begin{aligned}
\hat{u}'(x) &= -u'(2b-x) \\
\hat{u}''(x) &= u''(2b-x) \\
&\vdots \\
\hat{u}^{(n-1)}(x) &= (-1)^{n-1}u^{(n-1)}(2b-x) \\
&= -u^{(n-1)}(2b-x) \\
\hat{u}^{(n)}(x) &= (-1)^n u^{(n)}(2b-x) \\
&= u^{(n)}(2b-x),
\end{aligned}$$

and $\hat{u}$ satisfies

$$\hat{u}^{(n)}(x) - a_{n-1}(2b-x)\hat{u}^{n-1}(x) + \cdots - a_1(2b-x)\hat{u}'(x) + a_0(2b-x)\hat{u}(x) \leq 0,$$

where the functions $a_0(2b-x), -a_1(2b-x), ..., -a_{n-1}(2b-x)$ are in $C([b, 2b-a))$.

The initial conditions (5) imply that

$$\hat{u}(b) \;=\; \hat{u}'(b) \;=\; \cdots \;=\; \hat{u}^{(n-2)}(b) \;=\; 0.$$

By (6), there exists a sequence $\{2b - x_i\}$, such that $b < 2b - x_i < 2b - a$, $2b - x_i \to b$, and $\hat{u}(2b - x_i) = u(x_i) > 0$.

Then by Theorem 1.2, $\hat{u}^{(n-1)}(b) > 0$ and $\hat{u} > 0$ in a neighborhood of b. Therefore, we have $u^{(n-1)}(b) < 0$ and $u > 0$ in a neighborhood of b.

(ii) If n is odd, define

$$\tilde{u}(x) := -u(2b - x).$$

Then $\tilde{u} \in C^n((b, 2b-a)) \bigcap C^{n-1}([b, 2b-a)$ and

$$\begin{aligned}
\tilde{u}'(x) &= u'(2b - x) \\
\tilde{u}''(x) &= -u''(2b - x) \\
&\vdots \\
\tilde{u}^{(n-1)}(x) &= (-1)^n u^{(n-1)}(2b - x) \\
&= -u^{(n-1)}(2b - x) \\
\tilde{u}^{(n)}(x) &= (-1)^{n+1} u^{(n)}(2b - x) \\
&= u^{(n)}(2b - x),
\end{aligned}$$

and $\tilde{u}$ satisfies

$$\tilde{u}^{(n)}(x) - a_{n-1}(2b - x)\tilde{u}^{n-1}(x) + \cdots + a_1(2b - x)\tilde{u}'(x) - a_0(2b - x)\tilde{u}(x) \leq 0,$$

where the functions $-a_0(2b - x), a_1(2b - x), ..., -a_{n-1}(2b - x)$ are in $C([b, 2b - a))$.

The initial conditions (5) imply that

$$\tilde{u}(b) \;=\; \tilde{u}'(b) \;=\; \cdots \;=\; \tilde{u}^{(n-2)}(b) \;=\; 0.$$

By (7), there exists a sequence $\{2b - x_i\}$, such that $b < 2b - x_i < 2b - a$, $2b - x_i \to b$, and $\tilde{u}(2b - x_i) = -u(x_i) > 0$.

Then by Theorem 1.2, $\tilde{u}^{(n-1)}(b) > 0$ and $\tilde{u} > 0$ in a neighborhood of b. Therefore, we have $u^{(n-1)}(b) < 0$ and $u < 0$ in a neighborhood of b.

□

5. Some Comments on the Proofs of Higher Order Hopf's Lemma

The proof of Theorem 1.2 shows that it is necessary to first obtain the sequential form of the second order Hopf's lemma (Theorem 3.1), as we only know the sign of the function v at a sequence of points after the reduction process, so the classical Hopf's lemma no longer applies.

It is worth pointing out that the conditions (4), (6), and (7) are sharp in the sense that if they are not satisfied, then the $(n-1)$-th derivative may vanish at the endpoints.

Example: For any $0 < \alpha < 1$ and $n \geq 3$, define

$$u = \begin{cases} (-1)^{n-1}\lambda_n(-x)^{\frac{n}{1-\alpha}}, & x < 0 \\ \quad -\lambda_n x^{\frac{n}{1-\alpha}}, & x \geq 0 \end{cases}$$

where

$$\lambda_n = [(\beta+n)\cdots(\beta+1)]^{\frac{1}{\alpha-1}}, \qquad \text{and} \qquad \beta = \frac{n}{1-\alpha} - n = \frac{n\alpha}{1-\alpha}.$$

Direct computation shows that

$$u^{(n)}(x) = -|u(x)|^{\alpha}, \qquad x \in (-\infty, \infty). \tag{17}$$

Therefore u satisfies the differential inequality

$$u^{(n)} \leq 0.$$

To simplify the expressions let us choose $\alpha = \frac{1}{2}$, then

$$u = \begin{cases} (-1)^{n-1}\left(\frac{n!}{(2n)!}\right)^2(-x)^{2n}, & x < 0 \\ \quad -\left(\frac{n!}{(2n)!}\right)^2 x^{2n}, & x \geq 0. \end{cases}$$

By definition

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0$$

and also

$$u^{(n-1)}(0) = 0.$$

Note that $u < 0$ on $(0,1)$, so Condition (4) is not satisfied on $(0,1)$.
If n is even, $u < 0$ on $(-1,0)$, so Condition (6) is not satisfied on $(-1,0)$.
If n is odd, $u > 0$ on $(-1,0)$, so Condition (7) is not satisfied on $(-1,0)$.

□

Theorems 1.2 and 1.3 need to assume that the coefficient functions $a_0(x)$,..., $a_{n-1}(x)$ are continuous, while in Theorem 3.1 they only need to be bounded. The continuity condition is assumed when $n \geq 3$ to ensure that Equation (16) possesses a solution f. It would be interesting to know whether this is merely a limitation of the technique used in the proof or this reflects an inherent difference between the second and higher order cases. When $n = 3$, the continuity requirement can be replaced by boundedness, if we assume an additional assumption that u be non-negative at all points near a.

Theorem 5.1. *Let $u \in C^3((a,b)) \bigcap C^2([a,b))$ be a function that satisfies*

$$u^{(3)}(x) + a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x) \leq 0 \quad \textit{for } \ x \in (a,b),$$

where $|a_0(x)|, |a_1(x), |a_2(x)| \leq C$ for some constant $C > 0$. Suppose $u(a) = u'(a) = 0$, $u(x) \geq 0$ for all x in a small neighborhood of a, and there exists a sequence $\{x_i\} \subset (a,b)$ such that $x_i \to a$ and $u(x_i) > 0$. Then $u''(a) > 0$.

Proof: If $a_0(x) \geq 0$ for x in a small neighborhood of a, then since $u(x) \geq 0$ near a, we have

$$v''(x) + a_2(x)v'(x) + a_1(x)v(x) \leq 0 \qquad \text{for} \quad x \in (a, a+\epsilon) \subset (a,b),$$

where

$$v(x) = u'(x) \qquad \text{and} \qquad v(a) = u'(a) = 0.$$

Suppose $v(x) \leq 0$ for all x near a, then $v(x) = u'(x)$ and $u(a) = 0$ imply $u(x) \leq 0$ on $(a, a+\epsilon)$, contradicting the assumption that $x_i \to a$ and $u(x_i) > 0$. Therefore there exists a sequence $\tilde{x}_i \to a$ such that $v(\tilde{x}_i) > 0$. By Theorem 3.1, we then have $v'(a) = u''(a) > 0$.

For general $a_0(x)$, let

$$m(x) = e^{\theta\eta} - e^{-\theta(x-a)} \quad \text{for} \quad a \leq x \leq a + \eta < b.$$

For each $\theta > 0$ we may choose η such that

$$e^{2\theta\eta} - 1 < \theta(b-a). \tag{18}$$

Then since $e^{\theta(\eta+x-a)} \leq e^{2\theta\eta}$, we have

$$e^{\theta(\eta+x-a)} = 1 + h(x), \qquad \text{where} \quad 0 < h(x) < \theta(b-a) \text{ for all } x \in (a, a+\eta).$$

Because $|a_2(x)|, |a_1(x)|, |a_0(x)| \leq C$, for $a < x < a + \eta$

$$\begin{aligned}
L[m] &:= m^{(3)}(x) + a_2(x)m''(x) + a_1(x)m'(x) + a_0(x)m(x) \\
&= \left(\theta^3 - a_2(x)\theta^2 + a_1(x)\theta + a_0(x)(e^{\theta(\eta+x-a)} - 1)\right) e^{-\theta(x-a)} \\
&= \left(\theta^3 - a_2(x)\theta^2 + a_1(x)\theta + a_0(x)h(x)\right) e^{-\theta(x-a)} \\
&\geq \left(\theta^3 - |a_2(x)|\theta^2 - |a_1(x)|\theta - |a_0(x)|\theta(b-a)\right) e^{-\theta(x-a)} \\
&\geq \left(\theta^3 - C\theta^2 - (1+b-a)C\theta\right) e^{-\theta(x-a)}.
\end{aligned}$$

We can choose θ to be sufficiently large such that

$$\theta^3 - C\theta^2 - (1+b-a)C\theta > 0.$$

With this θ, choose η as above to satisfy (18). Then we have

$$L[m] > 0.$$

For $x \in [a, a+\eta]$, $m(x) > 0$ by definition, so we may define

$$z(x) = \frac{u(x)}{m(x)}.$$

Applying the differential operator L to $u(x) = m(x)z(x)$,

$$\begin{aligned}
L[u] &= (m(x)z(x))^{(3)} + a_2(x)(m(x)z(x))'' + a_1(x)(m(x)z(x))' + a_0(x)(m(x)z(x)) \\
&= m(x)z^{(3)}(x) + [3m'(x) + a_2(x)m(x)]z''(x) \\
&\quad + [3m''(x) + 2a_2(x)m'(x) + a_1(x)m(x)]z'(x) + L[m]z(x).
\end{aligned}$$

Since $L[u] \leq 0$ and $m(x) > 0$, we have

$$z^{(3)}(x) + a_2^*(x)z''(x) + a_1^*(x)z'(x) + a_0^*(x)z(x) \leq 0, \tag{19}$$

where

$$\begin{aligned} a_2^*(x) &= \frac{3m'(x)}{m(x)} + a_2(x), \\ a_1^*(x) &= \frac{3m''(x) + 2a_2(x)m'(x)}{m(x)} + a_1(x), \\ a_0^*(x) &= \frac{L[m]}{m(x)} \end{aligned}$$

For fixed θ, $\frac{m'}{m}$, $\frac{m''}{m}$ and $\frac{L[m]}{m}$ are all bounded when $x \in (a, a+\eta] \subset (a, b)$, so there exists $C_1 > 0$ such that

$$|a_2^*(x)|, \ |a_1^*(x)|, \ |a_0^*(x)| \leq C_1.$$

Since $u'(a) = u(a) = 0$ and $m'(a) = \theta \neq 0$, we have

$$\lim_{x \to a+} z'(x) = \lim_{x \to a+} \frac{u'(x)m(x) - u(x)m'(x)}{m^2(x)} = \frac{u'(a)m(a) - u(a)m'(a)}{m^2(a)} = 0.$$

The function $z(x) \in \mathcal{C}^2([a, a+\eta])$ satisfies

$$z'(a) = z(a) = 0, \qquad z(x_i) = \frac{u(x_i)}{m(x_i)} > 0, \qquad \text{and} \quad z(x) \geq 0 \quad \text{for} \quad x \in (a, a+\eta].$$

Recall that $m > 0$ and $L[m] > 0$, so $a_0^*(x) > 0$ on $[a, a+\eta]$. Then by (19) and the discussion at the beginning of this proof we conclude that

$$z''(a) > 0.$$

Consequently,

$$\begin{aligned} u''(a) &= m''(a)z(a) + 2m'(a)z'(a) + m(a)z''(a) \\ &= m(a)z''(a) \\ &> 0. \end{aligned}$$

This completes the proof.

□

It is natural to ask if Theorems 1.2 and 1.3 can be generalized to include two or more variables. Generally speaking the answer is no. Even the second order sequential Hopf's lemma fails with two variables. For example, the function $u(x, y) = xy$ satisfies $\Delta u = 0$. Although $u(0, 0) = 0$ and we can find a sequence of points $(x_i, y_i) \to (0, 0)$ with $u(x_i, y_i) > 0$, all directional derivatives of u vanish at $(0, 0)$ because $\nabla u(0, 0) = (0, 0)$.

It also seems to be difficult to correctly formulate a multi-variable version of a higher order Hopf's lemma. When n is odd, Conditions (4) and (7) require $u(x_i)$ to assume different sign at the two endpoints, and $u^{(n-1)}(a)$ and $u^{(n-1)}(b)$ have opposite sign in Theorems 1.2 and 1.3.

This "boundary effect" is not an issue when $n = 2$ because it is an even number and $u'(b) = -D_\eta u(b)$, where η denotes the direction pointing toward the center of the interval. Therefore, Theorems 1.2 and 1.3 and be combined to state that $D_\eta u > 0$ on the boundary of the interval (a, b). When n is odd, however, we will not be able to unify the two derivatives at the two endpoints. In the multi-variable case, the boundary will be even more complicated, so it appears to be difficult to formulate a clear and unified expression for the derivatives like the one in the classical Hopf's lemma.

6. Applications of Higher Order Hopf's Lemmas

In this section we will give some additional applications of the higher order Hopf's lemmas.

Applying Theorem 2.1 to both functions u and $-u$ gives a new proof of the standard uniqueness theorem of linear ODEs:

Corollary 6.1. *Let $u \in C^n((a,b)) \bigcap C^{n-1}([a,b))$ be a function which satisfies*

$$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) = 0 \quad \textit{for} \;\; x \in (a,b)$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C([a,b))$. Assume that u satisfies

$$u(a) \;=\; u'(a) \;=\; \cdots \;= u^{(n-2)}(a) \;= u^{(n-1)}(a) \;=\; 0.$$

Then $u \equiv 0$.
When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

Another immediate consequence of Theorem 1.2 is a unique continuation theorem.

Corollary 6.2. *Suppose $u \in C^\infty([a,b))$ satisfies*

$$u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \leq 0,$$

where $n \geq 2$ is a positive integer and $a_{n-1}(x), ..., a_1(x), a_0(x)$ are in $C([a,b))$. If Condition (4) holds, then it cannot be true that $u^{(k)}(a) = 0$ for all $k = 0, 1, ...$
When $n = 2$, it suffices to assume that $a_1(x)$ and $a_0(x)$ are bounded functions.

When u is in $C^\infty([a,b))$, Theorem 1.2 also follows from Corollary 6.2, hence the two results are equivalent. Here is the proof.

Proof: Assume Corollary 6.2 holds and $u \in C^\infty([a,b))$ satisfies (2), (3), and (4), we need to show that $u^{(n-1)}(a) > 0$.
Condition (4) and the $(n-1)$-th degree Taylor's expansion of u near a implies that $u^{(n-1)}(a) \geq 0$.
Suppose $u^{(n-1)}(a) = 0$.
Then by the n-th degree Taylor's expansion of u near a we have $u^{(n)}(a) \geq 0$. On the other hand, (2) and (3) imply $u^{(n)}(a) \leq 0$. Therefore $u^{(n)}(a) = 0$. Again the $(n+1)$-th degree Taylor's expansion of u near a implies that $u^{(n+1)}(a) \geq 0$.

If $u^{(n+1)}(a) > 0$, then for x close to a,

$$\begin{aligned}
u(x) &= \frac{u^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + O\left((x-a)^{n+2}\right) \\
u'(x) &= \frac{u^{(n+1)}(a)}{n!}(x-a)^{n} + O\left((x-a)^{n+1}\right) \\
&\vdots \\
u^{(n-1)}(x) &= \frac{u^{(n+1)}(a)}{2!}(x-a)^{2} + O\left((x-a)^{3}\right) \\
u^{(n)}(x) &= \frac{u^{(n+1)}(a)}{1!}(x-a) + O\left((x-a)^{2}\right).
\end{aligned}$$

Therefore, since $a_0(x), ..., a_{n-1}(x)$ are bounded,

$$\begin{aligned}
& u^{(n)}(x) + a_{n-1}(x)u^{n-1}(x) + \cdots + a_1(x)u'(x) + a_0(x)u(x) \\
= \;& u^{(n+1)}(a)\Big[(x-a) + \frac{a_{n-1}(x)}{2!}(x-a)^2 + \cdots + \frac{a_2(x)}{(n-1)!}(x-a)^{n-1} \\
& + \frac{a_1(x)}{n!}(x-a)^n + \frac{a_0(x)}{(n+1)!}(x-a)^{n+1}\Big] + O\left((x-a)^2\right) \\
> \;& 0,
\end{aligned}$$

provided that $x-a>0$ is sufficiently small. This contradicts (2). Hence $u^{(n+1)}(a) = 0$. Next we can show by similar argument that $u^{(n+2)}(a) = 0$, then $u^{(n+3)}(a) = 0$, ... So $u^{(k)}(a) = 0$ for all $k = 0, 1, 2....$ But this contradicts Corollary 6.2. Therefore we must have $u^{(n-1)}(a) > 0$, and Theorem 1.2 holds.

□

The last application is about the boundary behavior of solutions to a type of non-linear ODEs. A similar "boundary estimate" concerning solutions of boundary-value problem for a semilinear Poisson PDE was given in [2].

Theorem 6.3. *Let $u \in C^n([a,b])$ satisfy*

(20) $$u^{(n)}(x) = f(u, u', ..., u^{(n-1)}) \quad \text{in } [a,b],$$

where $f(z_1, ..., z_n) : \mathbf{R}^n \to \mathbf{R}$ is Lipschitz continuous in all variables.

(i) Assume $u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0$ and $u > 0$ in a neighborhood of a. Then either

$$u^{(n-1)}(a) > 0$$

or

$$u^{(n-1)}(a) = 0, \quad u^{(n)}(a) > 0.$$

In either case, u is strictly increasing near a.

(ii) Assume $u(b) = u'(b) = \cdots = u^{(n-2)}(b) = 0$ and $u > 0$ in a neighborhood of b. Then either

$$(-1)^{n-1}u^{(n-1)}(b) > 0$$

or

$$u^{(n-1)}(b) = 0, \quad (-1)^n u^{(n)}(b) > 0.$$

In either case, u is strictly decreasing near b.

Proof:

(i) Assume $u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0$ and $u > 0$ in a neighborhood of a.

Case 1: $f(0, 0, ..., 0) \le 0$.

Since f is Lipschitz, it is differentiable almost everywhere. Then from (20) we have

$$\begin{aligned} f(0,0,...,0) &= \big(f(0,0,...,0) - f(u,u',...,u^{(n-1)})\big) + u^{(n)}(x) \\ &= -\left(\int_0^1 \frac{\partial f}{\partial z_1}(tu, tu', ..., tu^{(n-1)})\ dt\right) u \\ &\quad -\left(\int_0^1 \frac{\partial f}{\partial z_2}(tu, tu', ..., tu^{(n-1)})\ dt\right) u' \\ &\quad - \cdots \\ &\quad -\left(\int_0^1 \frac{\partial f}{\partial z_n}(tu, tu', ..., tu^{(n-1)})\ dt\right) u^{(n-1)} + u^{(n)}(x). \end{aligned}$$

Hence u satisfies

$$u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u = f(0,0,...,0) \le 0,$$

where

$$\begin{aligned} a_{n-1}(x) &= -\int_0^1 \frac{\partial f}{\partial z_n}(tu, tu', ..., tu^{(n-1)})\ dt, \\ &\vdots \\ a_0(x) &= -\int_0^1 \frac{\partial f}{\partial z_1}(tu, tu', ..., tu^{(n-1)})\ dt. \end{aligned}$$

By Theorem 1.2, we have

$$u^{(n-1)}(a) > 0.$$

The $(n-3)$-th degree Taylor expansion of $u'(x)$ near a gives

$$u'(x) = \frac{1}{(n-2)!} u^{(n-1)}(\theta)(x+1)^{n-2} \quad \text{for some } \ a < \theta < x.$$

When x is sufficiently close to a, $u^{(n-1)}(\theta) > 0$. Thus $u'(x) > 0$ and $u(x)$ is strictly increasing.

Case 2: $f(0, 0, ..., 0) > 0$.

Since $u(x) > 0$ near $x = a$ and $u(a) = u'(a) = \cdots = u^{(n-2)}(a) = 0$, from the Taylor expansion of u at a we know that $u^{(n-1)}(a)$ cannot be negative. If it is positive, we are done. Otherwise, suppose $u^{(n-1)}(a) = 0$, then by (20) we have

$$\begin{aligned} u^{(n)}(a) &= f(u(a), u'(a), ..., u^{(n-1)}(a)) \\ &= f(0, 0, ..., 0) \\ &> 0. \end{aligned}$$

It follows that $u(x)$ is strictly increasing near a by discussions similar to those in Case 1.

(ii) Assume $u(b) = u'(b) = \cdots = u^{(n-2)}(b) = 0$ and $u > 0$ in a neighborhood of b. Let $s = a + b - x$, define $\hat{u}(s) := u(x)$, then

$$\hat{u}'(s) := -u'(x), \quad \hat{u}''(s) := u''(x), \quad ..., \quad \hat{u}^{(n)}(s) := (-1)^n u^{(n)}(x).$$

Then $\hat{u}$ satisfies

$$\hat{u}^{(n)}(s) = (-1)^n f\left(\hat{u}(s), -\hat{u}'(s), ..., (-1)^{n-1}\hat{u}^{(n-1)}(s)\right),$$

with $\hat{u}(a) = \hat{u}'(a) = \cdots = \hat{u}^{(n-2)}(a) = 0$ and $\hat{u} > 0$ in a neighborhood of a. Then by the result in (i) we know that either

$$\hat{u}^{(n-1)}(a) > 0$$

or

$$\hat{u}^{(n-1)}(a) = 0, \quad \hat{u}^{(n)}(a) > 0.$$

In either case, $\hat{u}$ is strictly increasing near a. Therefore, either

$$(-1)^{n-1} u^{(n-1)}(b) > 0$$

or

$$u^{(n-1)}(b) = 0, \quad (-1)^n u^{(n)}(b) > 0.$$

In either case, u is strictly decreasing near b.

□

In this theorem, it is necessary to assume that f is Lipschitz. For example, in Equation (17) the function $f(z_1, ..., z_n) = z_1^\alpha$ is only Hölder continuous, but not Lipschitz continuous. The solution

$$u = \begin{cases} (-1)^{n-1}\lambda_n(-x)^{\frac{n}{1-\alpha}}, & x < 0 \\ \quad -\lambda_n x^{\frac{n}{1-\alpha}}, & x \geq 0 \end{cases}$$

satisfies $u^{(n-1)}(0) = u^{(n)}(0) = 0$, so the theorem does not hold in this case.

Appendix A. An Alternative Proof of Theorem 3.1

Here we give an alternative and elegant proof of Theorem 3.1 suggested by the referee of an earlier manuscript. This proof was inspired by [1].

Proof: Let

$$h_i = \sin\left(\frac{\pi}{2} + \frac{\pi}{9} \cdot \frac{x - y_i}{x_i - a}\right)$$

where $y_i = \frac{x_i + a}{2}$. Then

$$0 < \sin\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \leq h_i \leq 1 < \infty \quad \text{on } [a, x_i]$$

$$\begin{aligned}
h_i'(x) &= \frac{\pi}{9}\cdot\frac{1}{x_i-a}\cos\left(\frac{\pi}{2}+\frac{\pi}{9}\cdot\frac{x-y_i}{x_i-a}\right)\\
h_i''(x) &= -\left(\frac{\pi}{9}\cdot\frac{1}{x_i-a}\right)^2\sin\left(\frac{\pi}{2}+\frac{\pi}{9}\cdot\frac{x-y_i}{x_i-a}\right)\\
&= -\left(\frac{\pi}{9}\cdot\frac{1}{x_i-a}\right)^2 h_i.
\end{aligned}$$

It follows that on $[a,x_i]$,

$$\begin{aligned}
L[h_i] &:= h_i''(x)+a_1(x)h_i'(x)+a_0(x)h_i(x)\\
&= -\left(\frac{\pi}{9}\cdot\frac{1}{x_i-a}\right)^2 h_i + a_1(x)\cdot\frac{\pi}{9}\cdot\frac{1}{x_i-a}\cdot\cos\left(\frac{\pi}{2}+\frac{\pi}{9}\cdot\frac{x-y_i}{x_i-a}\right)+a_0(x)h_i(x)\\
&\le -\left(\frac{\pi}{9}\cdot\frac{1}{x_i-a}\right)^2\sin\left(\frac{\pi}{2}-\frac{\pi}{9}\right)+C\cdot\left(\frac{\pi}{9}\cdot\frac{1}{x_i-a}\right)+C\\
&\to -\infty \qquad \text{as} \quad i\to\infty,
\end{aligned}$$

where the last inequality is true because $x_i\to a$ as $i\to\infty$.
Therefore when i is large, $L[h_i]<0$ on $[a,x_i]$.
Define

$$w_i=\frac{u}{h_i}.$$

Then

$$\begin{aligned}
L[u] &= L[h_iw_i]\\
&= h_iw_i''+(2h_i+a_1h_i)\,w_i'+L[h_i]w_i.
\end{aligned}$$

On $[a,x_i]$, since $L[u]\le 0$ and $h_i>0$, we have

$$\tilde{L}_i[w_i]:=w_i''+\frac{2h_i+a_1h_i}{h_i}w_i'+\frac{L[h_i]}{h_i}w_i \ \le\ 0.$$

Since $L[h_i]<0$ and $h_i>0$, the linear term coefficient $\frac{L[h_i]}{h_i}<0$. Thus the classical maximum principle and Hopf's lemma both apply to $\tilde{L}_i[w_i]$.
Because $w_i(a)=0$ and $w_i(x_i)>0$, by the maximum principle we have

$$w_i(x)>0 \quad \text{in } (a,x_i).$$

Then by the Hopf's lemma

$$w_i'(a)>0.$$

Finally, since $w_i(a)=\frac{u(a)}{h_i(a)}=0$ and $h_i(a)>0$, we obtain

$$u'(a) \ = \ w_i'(a)h_i(a)+w_i(a)h_i'(a) \ >0.$$

This proves Theorem 3.1.

$\square$

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