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AN INCREASING FUNCTION WITH INFINITELY CHANGING CONVEXITY

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Abstract. We construct a smooth, strictly increasing function with its second derivative changing signs infinitely many times near a point, then we extend the function to a surface increasing in radial direction with curvatures changing signs infinitely often near the origin. The interesting analytical properties of the functions may serve as examples to understand the reversed Hopf Lemma.

It is well known that there are many smooth functions vanishing at a point with infinite order. A good example is the function $f(t) = e^{-1/t^2}$ with $f^{(k)}(0) = \lim_{t\to 0} f^{(k)}(t) = 0$ for all k. In this study note, we are interested in finding a strictly increasing function with second derivative changing signs infinitely often near a point. Such function could be useful in understanding the reversed Hopf Lemma, as studied in [1] and [2]. To our best knowledge, this example does not seem to be presented in the current circulating publications. Hence in this note, we construct such a function along with its extension to a two dimensional surface strictly increasing in radial direction with curvature changing signs infinitely often near the origin.

1. An example

Example 1.1. Consider the function

$$u(t) \quad = \quad \int_0^t (t-s)e^{-\alpha/s}\left[\sin(1/s) + C\sin^2(1/s)\right] ds, \qquad t \in [0,1], \quad \alpha,\ C > 0.$$

The function u has the following properties:

(1) Vanishing at $t = 0$ of infinite order:
$u^{(k)}(0) = \lim_{t\to 0+} u^{(k)}(t) = 0$ for $k = 0, 1, 2, \cdots$.

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(2) Smoothness of infinite order: $u \in \mathcal{C}^\infty([0,1])$.
(3) Changing convexity infinitely often at 0:
u'' changes signs in $(0,\epsilon)$, for any $0<\epsilon<1$.
(4) Strictly increasing:
$u'(t)>0$ for $t\in(0,1]$ when $C \ge C_\alpha = 4e^{\alpha\pi}(9e^{\alpha\pi}-1)/\pi$.
(Note: C_α is an increasing function of α. In particular, $C_\alpha > 32/\pi$.)

Remarks: It is more straightforward to construct a function satisfying the first three properties. The main effort is to maintain monotonicity in property (4).

Proof of the properties in Example 1.1. By the general Leibniz rule in calculus,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,s)ds = b'(t)f(t,b(t)) - a'(t)f(t,a(t)) + \int_{a(t)}^{b(t)} \frac{d}{dt} f(t,s)ds,$$

we have

$$\begin{aligned} u'(t) &= \int_0^t e^{-\alpha/s}\left[\sin(1/s) + C\sin^2(1/s)\right] ds, \qquad t\in[0,1],\\ u''(t) &= e^{-\alpha/t}\left[\sin(1/t)+C\sin^2(1/t)\right], \qquad t\in(0,1]. \end{aligned}$$

The kth derivative $(k\ge 2)$ can be written as

$$u^{(k)}(t) = t^{-(k-1)}e^{-\alpha/t}H(t), \qquad t\in(0,1], \quad \textit{where} \quad H_k(t) = O(t) \quad \textit{as} \quad t\to 0.$$

That is, $H_k(t)$ are uniformly bounded for all $t\in(0,1],\ k\ge 2$, according to Taylor's Theorem. To show Property (1), use the fact that for any $s\in\mathbb{R}$, $t^s e^{-\alpha/t}\to 0$ as $t\searrow 0$, so $\lim_{t\to 0+} u^{(k)}(t) = 0$ for all k by the boundedness of sine functions.

By the infinite differentiability of $e^{-\alpha/t}$ and $\sin(1/t)$ on $t>0$, $u\in\mathcal{C}^\infty((0,1])$. Property (1) warrants the extension to $\mathcal{C}^\infty([0,1])$. Thus Property (2) holds.

To prove Property (3), we show that for any $\epsilon\in(0,1)$, there exist $t_1,t_2\in(0,\epsilon)$ with $u''(t_1)>0, u''(t_2)<0$. Choose n such that $2n\pi > 1/\epsilon$. Then $t_1 = (2n\pi + \pi/2)^{-1}\in(0,\epsilon)$, and

$$u''(t_1) = e^{-\alpha(2n\pi+\pi/2)}(1+C) > 0.$$

Select $\delta\in(0,\epsilon)$ such that $C\sin(\delta)<1$. Then $t_2 = ((2n+1)\pi+\delta)^{-1}\in(0,\epsilon)$, and

$$\begin{aligned} u''(t_2) &= e^{-\alpha((2n+1)\pi+\delta)}\left[\sin(\pi+\delta) + C\sin^2(\pi+\delta)\right]\\ &= e^{-\alpha((2n+1)\pi+\delta)}\sin(\delta)[-1+C\sin(\delta)] < 0. \end{aligned}$$

Therefore Property (3) holds.

To prove Property (4), for fixed $\alpha > 0$ and for any $t \in (0, 1]$, we write

$$u'(t) \quad = \quad \int_0^t u''(x)dx = \int_{1/t}^{\infty} \frac{u''(1/y)}{y^2} dy = \int_{1/t}^{\infty} \frac{e^{-\alpha y}}{y^2} \left(\sin y + C \sin^2 y\right) dy.$$

The above integral converges absolutely for any $t > 0$. Express the integral form of $u'(t)$ as

$$u'(t) = I_t + \sum_{n=n_o}^{\infty} I(n),$$

with

$$I_t = \int_{1/t}^{(2n_o+1)\pi} \frac{e^{-\alpha y}}{y^2} \left(C \sin^2 y + \sin y\right) dy,$$

$$n_o = n_o(t) = \lceil 1/t \rceil = \min\{n \in \mathbb{Z}, (2n+1)\pi \geq 1/t\},$$

and

$$I(n) = \int_{(2n+1)\pi}^{(2n+3)\pi} \frac{e^{-\alpha y}}{y^2} \left(\sin y + C \sin^2 y\right) dy.$$

For $t \in (0, 1]$, we prove $u'(t) > 0$ by showing $I(n) > 0$ for $n = 0, 1, \cdots$ and $I_t > 0$. Write

$$I(n) \quad = \quad C \int_{(2n+1)\pi}^{(2n+3)\pi} \frac{e^{-\alpha y}}{y^2} \sin^2 y \; dy - \left(- \int_{(2n+1)\pi}^{(2n+3)\pi} \frac{e^{-\alpha y}}{y^2} \sin y \; dy \right) = I_1(n) - I_2(n).$$

For the term $I_1(n)$, Shift the intervals to $(0, \pi)$ and use the monotone property of $e^{-\alpha y}$ and y^{-2},

$$\begin{aligned}
I_1(n) \quad &= \quad C \left(\int_{(2n+1)\pi}^{(2n+2)\pi} + \int_{(2n+2)\pi}^{(2n+3)\pi} \right) \frac{e^{-\alpha y}}{y^2} \sin^2 y \; dy \\
&= \quad C \int_0^{\pi} e^{-\alpha y - \alpha(2n+2)\pi} \left(\frac{e^{\alpha\pi}}{(y + (2n+1)\pi)^2} + \frac{1}{(y + (2n+2)\pi)^2} \right) \sin^2 y \; dy \\
&> \quad C \; e^{-\alpha(2n+2)\pi} \int_0^{\pi} e^{-\alpha y} \frac{e^{\alpha\pi} + 1}{(y + (2n+2)\pi)^2} \sin^2 y \; dy \\
&> \quad C \; e^{-\alpha(2n+2)\pi} \int_0^{\pi} e^{-\alpha\pi} \frac{e^{\alpha\pi} + 1}{(\pi + (2n+2)\pi)^2} \sin^2 y \; dy \\
&= \quad C \; \frac{e^{-\alpha(2n+2)\pi}}{((2n+3)\pi)^2} \left(1 + e^{-\alpha\pi}\right) \frac{\pi}{2}.
\end{aligned}$$

Similarly for the second term,

$$
\begin{aligned}
I_2(n) &= -\left(\int_{(2n+1)\pi}^{(2n+2)\pi} + \int_{(2n+2)\pi}^{(2n+3)\pi}\right) \frac{e^{-\alpha y}}{y^2} \sin y\ dy \\
&= \int_0^{\pi} e^{-\alpha y-\alpha(2n+2)\pi} \left(\frac{e^{\alpha\pi}}{(y+(2n+1)\pi)^2} - \frac{1}{(y+(2n+2)\pi)^2}\right) \sin y\ dy \\
&< e^{-\alpha(2n+2)\pi} \int_0^{\pi} e^{-\alpha y} \left(\frac{e^{\alpha\pi}}{((2n+1)\pi)^2} - \frac{1}{(\pi+(2n+2)\pi)^2}\right) \sin y\ dy \\
&= \frac{e^{-\alpha(2n+2)\pi}}{((2n+3)\pi)^2} \int_0^{\alpha\pi} e^{-\alpha y} \left(\frac{(2n+3)^2}{(2n+1)^2}\ e^{\alpha\pi} - 1\right) \sin y\ dy \\
&< \frac{e^{-\alpha(2n+2)\pi}}{((2n+3)\pi)^2} \int_0^{a\pi} (9e^{\alpha\pi}-1) \sin y\ dy \ = \ \frac{e^{-\alpha(2n+2)\pi}}{((2n+3)\pi)^2}\ (9e^{\alpha\pi}-1)\ 2.
\end{aligned}
$$

Therefore if we choose $C \geq C_\alpha = \dfrac{4(9e^{\alpha\pi}-1)}{\pi e^{-\alpha\pi}} > \dfrac{4(9e^{\alpha\pi}-1)}{\pi(e^{-\alpha\pi}+1)}$,

$$
I(n) = I_1(n) - I_2(n) > \frac{e^{-\alpha(2n+2)\pi}}{((2n+3)\pi)^2} \left[C\ \frac{\pi(1+e^{-\alpha\pi})}{2} - 2(9e^{\alpha\pi}-1)\right] > 0.
$$

Now consider the boundary interval $I_t = \displaystyle\int_{1/t}^{(2n_o+1)\pi} \frac{e^{-\alpha y}}{y^2}\left(C\sin^2 y + \sin y\right) dy$. If $1/t \in [2n_o\pi, (2n_o+1)\pi]$ for some $n_o \geq 0$, all integrands ≥ 0 on the interval of integration so $I_t > 0$. If $1/t \in [(2n_o-1)\pi, 2n_o\pi]$ for some $n_o > 0$, notice that $C\sin^2 y + \sin y = 0$ only when $\sin y = -1/C$, so $C\sin^2 y + \sin y < 0$ only on two sub-intervals

$$
\big(\ (2n_o-1)\pi,\ (2n_o-1)\pi + \sin^{-1}(1/C)\ \big) \quad \textit{and} \quad \big(\ 2n_o\pi - \sin^{-1}(1/C),\ 2n_o\pi\ \big)
$$

So it is only necessary to consider the "worst cases" when $1/t$ is the left endpoint of either subintervals. When $1/t = (2n_o-1)\pi$, the integral I_t is over $[(2n_o-1)\pi, (2n_o+1)\pi]$, an interval of length 2π. Then $I_t = I(n_o-1) > 0$ based on the above estimates for $I(n)$. Now we only need to consider the case when $1/t =$

$2n_o\pi - \sin^{-1}(1/C)$, then

$$\begin{aligned}
I_t &= \left(\int_{1/t}^{2n_o\pi} + \int_{2n_o\pi}^{(2n_o+1)\pi}\right) \frac{e^{-\alpha y}}{y^2}(C\sin^2 y + \sin y)\ dy \\
&> C\int_{2n_o\pi}^{(2n_o+1)\pi} \frac{e^{-\alpha y}}{y^2}\sin^2 y\ dy - \left[-\left(\int_{(2n_o-1)\pi}^{2n_o\pi} + \int_{2n_o\pi}^{(2n_o+1)\pi}\right)\frac{e^{-\alpha y}}{y^2}\sin y\ dy\right] \\
&> C\int_{2n_o\pi}^{(2n_o+1)\pi} \frac{e^{-\alpha y}}{y^2}\sin^2 y\ dy - I_2(n_o-1) \\
&> C\ \frac{e^{-\alpha 2n_o\pi}}{((2n_o+1)\pi)^2}\ \left(e^{-\alpha\pi}\ \frac{\pi}{2}\right) - \frac{e^{-\alpha 2n_o\pi}}{((2n_o+1)\pi)^2}\ (9e^{\alpha\pi}-1)\ 2.
\end{aligned}$$

In the last inequality we use similar derivations used earlier for $I_2(n)$ and $I_1(n)$. Therefore $I_t > 0$ when $C \geq C_\alpha = 4e^{\alpha\pi}(9e^{\alpha\pi}-1)/\pi$. Consequently

$$u'(t) = I_t + \sum_{n=n_o(t)}^{\infty} I(n) > 0 \qquad \forall t \in (0,1],$$

which proves Property (4). $\square$

2. Functions on the plane

Next we consider the two dimensional case, where curvature is determined by both the first and second derivatives of the function. Let $\mathbb{D}_R = \{(x,y) \in \mathbb{R}^2, \|(x,y)\| = \sqrt{x^2+y^2} < R\}$. As before, we use $\mathbb{D} = \mathbb{D}_1$ for simplicity. Define $f : \mathbb{D} \to \mathbb{R}$ by $f(x,y) = u(x^2+y^2)$ for any $u \in \mathcal{C}^2((0,1])$. Recall the Gaussian curvature of f

$$\kappa = \kappa(x,y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1+f_x^2+f_y^2)^2},$$

where $f_{xx} = f_{xx}(x,y)$ is the second order partial derivative of $f(x,y)$ with respect to x, and so on. Let $s = x^2+y^2$. The following proposition describes the behavior of the curvature near isolated critical points of the radial function f.

Proposition 2.1. *Assume $u'(s_0) = 0$, $s_0 \in (0,1)$, and $u'(s) \neq 0$ for $s \in (s_0-\delta, s_0+\delta) \setminus \{s_0\}$ for some $\delta \in (0, s_0)$. Then the Gaussian curvature of $f(x,y) = u(x^2+y^2)$ changes signs in a neighborhood of (x_0,y_0) for any $(x_0,y_0) = (s_0\cos\theta, s_0\sin\theta)$, $\theta \in [0,2\pi)$.*

Proof. The partial derivatives of f can be expressed as

$f_x = 2xu'(s),\ f_{xx} = 2u'(s) + 4x^2u''(s),\ f_{yy} = 2u'(s) + 4y^2u''(s),\ f_{xy} = 4xyu''(s).$

The denominator of κ is always > 0. The denominator of κ can be written in the forms

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4u'(s)^2 + 8su'(s)u''(s) = 4\frac{d}{ds}\left(su'(s)^2\right) = D(s). \tag{2.1}$$

For any $s_1 \in (s_0, s_0 + \delta)$,

$$\int_{s_0}^{s_1} D(s)ds = 4\int_{s_0}^{s_1} \frac{d}{ds}\left(su'(s)^2\right) ds = 4su'(s)^2\Big|_{s_0}^{s_1} = 4s_1u'(s_1)^2 > 0.$$

Therefore there must be $t_1 \in (s_0, s_1)$ such that $D(t_1) > 0$. Consequently for $(x_1, y_1) = (t_1 \cos\theta, t_1 \sin\theta)$, we have $\kappa(x_1, y_1) > 0$. Similarly, for any $s_2 \in (s_0 - \delta, s_0)$,

$$\int_{s_2}^{s_0} D(s)ds = 4su'(s)^2\Big|_{s_2}^{s_0} = -4s_2u'(s_2)^2 < 0.$$

So there must be $t_2 \in (s_2, s_0)$ such that $D(t_2) < 0$. Hence $\kappa(x_2, y_2) < 0$ for $(x_2, y_2) = (t_2 \cos\theta, t_2 \sin\theta)$.

Since $\delta > 0$ can be arbitrarily small, and both $(x_1, y_1), (x_2, y_2)$ are in the disk of raduis δ centered at (x_0, y_0), we have proved that $\kappa(x, y)$ changes signs in a neighborhood of (x_0, y_0) for any $(x_0, y_0) = (s_0 \cos\theta, s_0 \sin\theta),\ \theta \in [0, 2\pi)$. □

The following example is an extension of the function u in Example 1.1 to the two dimensional case.

Example 2.2. For $\alpha > 0$, let $v : \mathbb{D} \to \mathbb{R}$ be $v(x,y) = u(x^2 + y^2)$, where u is the function in Example 1.1 with $C > C_\alpha$. It is clear that $v(x,y)$ is strictly increasing in radial directions. Furthermore, the curvature of $v(x,y)$ changes signs infinite often near $(0,0)$: for any $\epsilon \in (0,1)$, there are $(x_1, y_1),\ (x_2, y_2) \in \mathbb{D}_\epsilon$ such that $\kappa(x_1, y_1) > 0,\ \kappa(x_2, y_2) < 0$.

Remarks: Similar to Example 1.1, it is not difficult to obtain a surface function with infinitely changing curvature. Thus the main attribute of the constructed function is having infinitely changing curvature while strictly increasing in radial directions.

Proof of the curvature property for Example 2.2. Fix $\alpha, C > C_\alpha$ and $\epsilon \in (0,1)$. Let $t = x^2 + y^2$ and

$$D(t) = 4u'(t)^2 + 8tu'(t)u''(t)$$

as defined in (2.1). It is sufficient to show that, for $\epsilon \in (0,1)$, there exist $t_+, t_- \in (0,\epsilon)$ with $D(t_+) > 0, D(t_-) < 0$. To find $D(t_+) > 0$ with $t_+ \in (0,\epsilon)$ is straightforward. Choose n such that $2n\pi > 1/\epsilon$. Then $t_+ = (2n\pi+\pi/2)^{-1} \in (0,\epsilon)$ and $\sin(1/t_+) = 1$, thus

$$u''(t_+) = e^{-\alpha(2n\pi+\pi/2)}(1+C) > 0 \qquad \Rightarrow \quad D(t_+) = 4u'(t_+)^2 + 8t_+u'(t_+)u''(t_+) > 0.$$

To attain $D(t_-) < 0$, we need to choose $t_- \in (0,\epsilon)$ such that

$$u''(t_-) < -\frac{u'(t_-)}{2t_-}$$

then consequently $D(t_-) = 8t_-u'(t_-)\left(u''(t_-) + \frac{u'(t_-)}{2t_-}\right) < 0$. Select $\delta \in (0,\epsilon)$ such that $0 < C\sin(\delta) < 1$. Define

$$\varepsilon_o = C\sin(\delta)[1 - C\sin(\delta)].$$

Next we examine a u' related term $J(t)$ to obtain an upper bound.

$$\begin{aligned} 0 < J(t) = e^{\alpha/t}\frac{u'(t)}{t} &= \frac{e^{\alpha/t}}{t}\int_0^t u''(x)dx = \frac{e^{\alpha/t}}{t}\int_{1/t}^{\infty}\frac{u''(1/y)}{y^2}dy \\ &= \frac{e^{\alpha/t}}{t}\int_{1/t}^{\infty} e^{-\alpha y}\ \frac{\sin y + C\sin^2 y}{y^2}dy \ = \ J_1(t) + CJ_2(t) \end{aligned}$$

Consider the first term $J_1(t)$. Notice that for any t_n such that $1/t_n \in ((2n+1)\pi, (2n+2)\pi)$, $n > 0$, we have

$$0 < \int_0^{1/t_n}\frac{\sin y}{y}dy < \int_0^{\infty}\frac{\sin y}{y}dy = \frac{\pi}{2}.$$

Therefore for such t_n,

$$\int_{1/t_n}^{\infty}\frac{\sin y}{y}dy = \frac{\pi}{2} - \int_0^{1/t_n}\frac{\sin y}{y}dy \quad \searrow 0 \qquad as \quad n\to\infty, \quad t_n \to 0.$$

We may choose $n > N_1$ such that

$$\begin{aligned} J_1(t_n) &= \frac{e^{\alpha/t_n}}{t_n}\int_{1/t_n}^{\infty} e^{-\alpha y}\frac{\sin y}{y^2}dy = \int_{1/t_n}^{\infty} e^{-\alpha(y-1/t_n)}\frac{1}{t_n}\ \frac{\sin y}{y^2}dy \\ &\le \int_{1/t_n}^{\infty} e^{-\alpha(y-1/t_n)}\ y\ \frac{\sin y}{y^2}dy = \int_{1/t_n}^{\infty} e^{-\alpha(y-1/t_n)}\ \frac{\sin y}{y}dy \qquad (since\ 1/t_n \le y) \\ &\le \int_{1/t_n}^{\infty}\frac{\sin y}{y}dy \le \frac{\varepsilon_o}{C} \qquad\qquad (since\ e^{-\alpha(y-1/t_n)} \le 1) \end{aligned}$$

Now look at the second term $CJ_2(t)$ in the u' related term $J(t)$. For t_n chosen above,

$$\begin{aligned}
0 < J_2(t_n) &= \frac{e^{\alpha/t_n}}{t_n}\int_{1/t_n}^{\infty} e^{-\alpha y}\ \frac{\sin^2 y}{y^2}dy = \int_{1/t_n}^{\infty} e^{-\alpha(y-1/t_n)}\ \frac{1}{t_n}\ \frac{\sin^2 y}{y^2}dy \\
&\le \int_{1/t_n}^{\infty} e^{-\alpha(y-1/t_n)}\ y\ \frac{\sin^2 y}{y^2}dy\ = \int_{0}^{\infty} e^{-\alpha z}\ \frac{\sin^2(z+1/t_n)}{z+1/t_n}dz \\
&\le \int_{0}^{\infty} e^{-\alpha z}dz\ \frac{1}{1/t_n}\ =\ \frac{t_n}{\alpha}
\end{aligned}$$

Choose N_2 such that $\frac{1}{(2N_o+1)\pi} < \varepsilon_o\frac{\alpha}{C^2}$. Then for $n \ge N_2$, we may choose $t_n = \frac{1}{(2n+1)\pi+\delta}$, where $\delta \in (0,\epsilon) \subset (0,1)$ was selected earlier and was used to define ϵ_o. Notice that such t_n satisfies the condition $1/t_n \in ((2n+1)\pi, (2n+2)\pi)$. For the selected sequence $\{t_n\}$, we have

$$J_2(t_n) \le \frac{t_n}{\alpha} < \frac{1}{\alpha(2n+1)\pi} \le \frac{1}{\alpha(2N_o+1)\pi} < \frac{\varepsilon_o}{C^2}.$$

Combining the results we have obtained, for $t_n = \frac{1}{(2n+1)\pi+\delta}$ and $n \ge \min\{N_1, N_2\}$, we have

$$e^{\alpha/t_n}\frac{u'(t_n)}{t_n} = J(t_n) = J_1(t_n) + CJ_2(t_n) < \frac{\varepsilon_o}{C} + C\frac{\varepsilon_o}{C^2} = \frac{2\varepsilon_o}{C}.$$

For the u'' term we can select $n \ge \max\{N_1, N_2\}$ such that $t_n = ((2n+1)\pi+\delta)^{-1} \in (0,\epsilon)$, then

$$\begin{aligned}
u''(t_n) &= e^{-\alpha/t_n}\left[\sin(1/t_n) + C\sin^2(1/t_n)\right] \\
&= e^{-\alpha/t_n}\left[\sin((2n+1)\pi+\delta) + C\sin^2((2n+1)\pi+\delta)\right] \\
&= e^{-\alpha/t_n}\left[\sin(\pi+\delta) + C\sin^2(\pi+\delta)\right]\ =\ e^{-\alpha/t_n}\sin(\delta)[-1+C\sin(\delta)] \\
&= -\ e^{-\alpha/t_n}C\sin(\delta)[1-C\sin(\delta)]\ \frac{1}{C}\ =\ -\ e^{-\alpha/t_n}\ \frac{\varepsilon_o}{C}.
\end{aligned}$$

Then

$$e^{\alpha/t_n}u''(t_n) = -\ \frac{\varepsilon_o}{C} < -\ \frac{1}{2}\ e^{\alpha/t_n}\frac{u'(t_n)}{t_n} \qquad \Longrightarrow \qquad u''(t_n) < -\ \frac{u'(t_n)}{2t_n}$$

Let $t_- = t_n$ for any of the $n \ge \max\{N_1, N_2\}$. Then $D(t_-) < 0$ with $t_- \in (0,\epsilon)$. This proves the claimed curvature property in Example 2.2. $\square$

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