Embark on the Voyage of Discovery with the TI-84 Plus C Silver Edition and the CCSS

26th Annual T^3 International Conference
Las Vegas, Nevada

Friday, March 7, 2014
10:00 a.m. – 11:30 a.m.
Rio Hotel, Amazon T

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Investigation 1: A Job Offer You Can't Refuse 

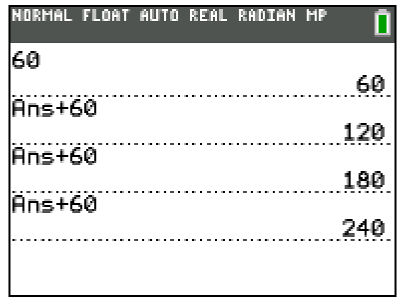
Congratulations! Your godfather Bugsy not only gives you a job offer, he lets you choose how you get paid.

- Option A: *Earn \$60 per day*
- Option B: *Earn 1 measly dollar for the first day, but \$2 for the second, \$4 for the third, and so on, so that each day's pay is double that of the previous day.*

What is the total amount earned after eight days of work? Should you take Option A or Option B?

1. Let's explore Option A. Complete the table. You can use the home screen to simulate your total.

Day, n	Total, T
1	60
2	120
3	180
4	240
5	
6	
7	
8	

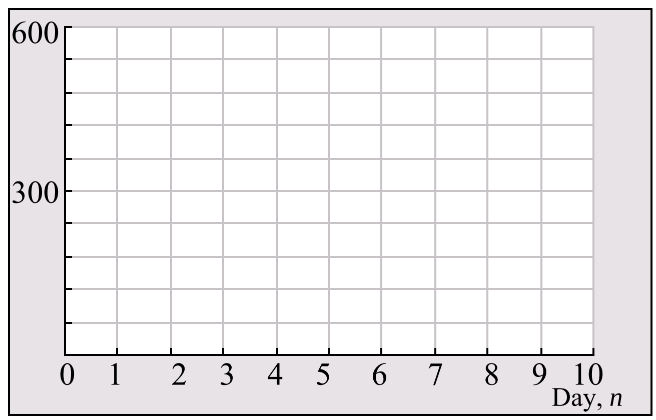


2. Look for connections and patterns between T and n.
   a. What kind of function would model (n, T)?
   i.e., linear, quadratic, exponential, etc.

   b. Write the formula for T in terms of n.

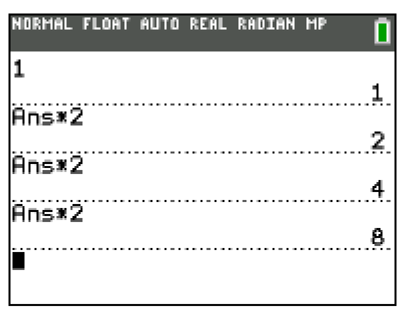
   $T(n)$ = ____________________

   c. Sketch a graph. Interpret the parameters in your formula in terms of the context.



3. Let's explore Option B. First, compute your earnings for **each day**. Complete the table.

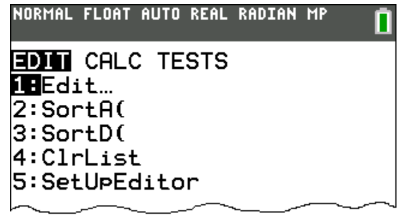
Day, n	Day's Payment D
1	1
2	2
3	4
4	8
5	
6	
7	
8	



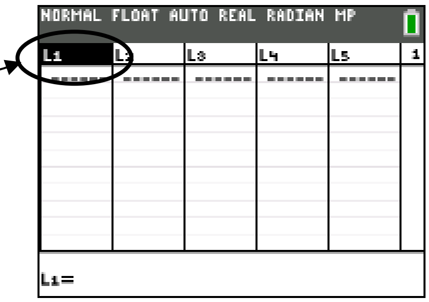
4. Look for connections and patterns between D and n.
   a. What kind of function would model (n, D)?
   i.e., linear, quadratic, exponential, etc.

   b. Write the formula for D as a function of n: $D(n)$ = ____________________

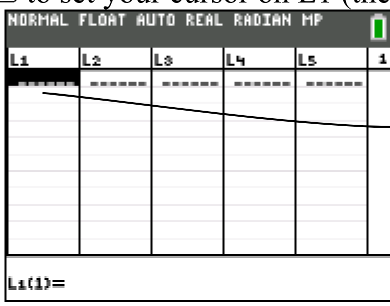
5. We can make a list which indexes the number of the day, another list for the amount you earned just that day, and finally a third list for the cumulative sum.



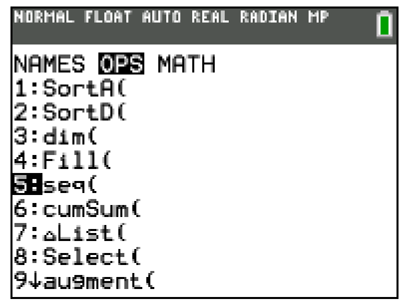
   a. Press STAT, followed by 1:Edit to get to the Stat Editor. **Note:** If there is data in the lists, clear your list by placing your cursor on the list name. Then press CLEAR and ENTER.



   Important step: To make a sequence {1, 2, 3, …., 8} in L1, first press ▲ to set your cursor on L1 (the top shelf!)

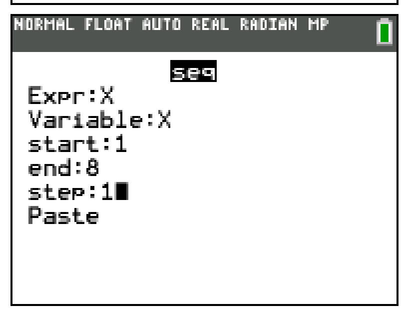


   Now press 2nd [LIST] ► and 5: seq(.
   The seq(wizard appears.

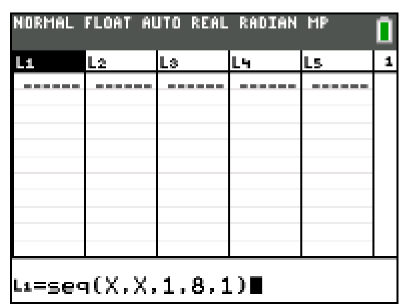


   Once the settings shown are entered, highlight Paste and press ENTER.

   This builds and inserts the command seq(X,X,1,8,1) into the L1 entry line.

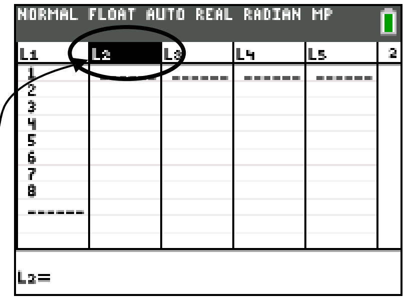


   Press ENTER once more to deliver the goods.

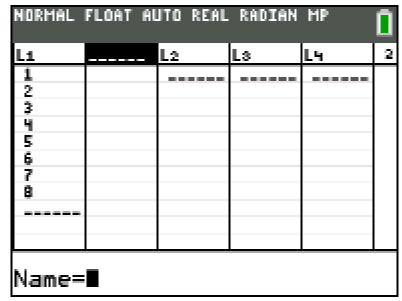


b. We can insert a column for a list named *D* to be used for the day's wage.

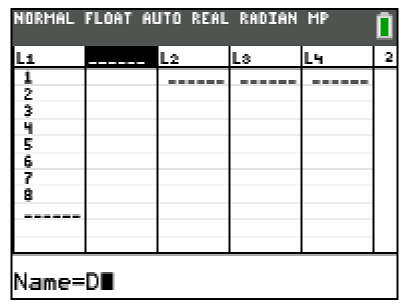
Sit your cursor on the top of L2.



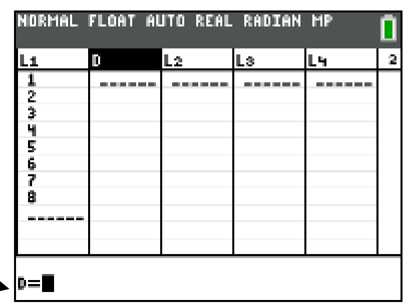
Press 2nd [INS].



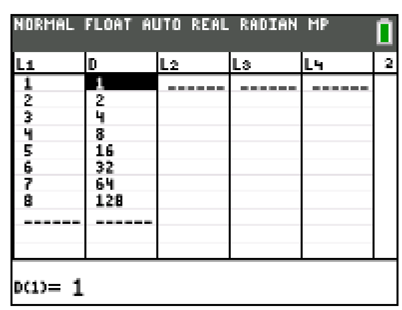
Press ALPHA [D] for the list name.
Press ENTER.



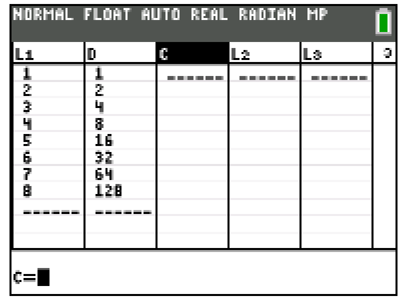
Press ENTER to get to the entry line.
Type your formula from Question 4 for *D*
(using L1 as the independent variable)
and press ENTER.



Validate your model matches your earlier work.
If not, make any necessary revisions.



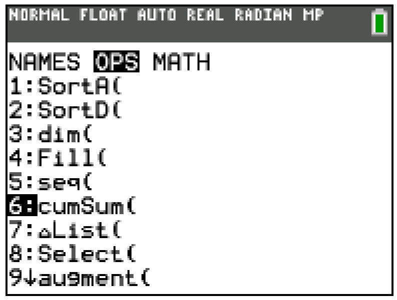
c. To insert a column for a list named *C* to be used for the cumulative sum, sit your cursor on the top of L2, press [2nd] [INS], press [ALPHA] [C] for the list name, then [ENTER].

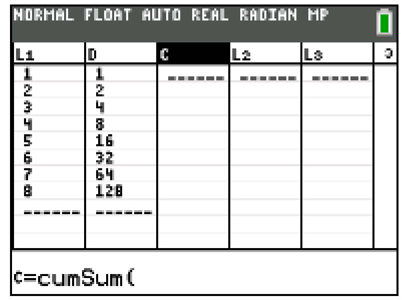


Press [ENTER] to take you to the entry line.

Press [2nd] [LIST] [▶] to select 6: cumSum(.

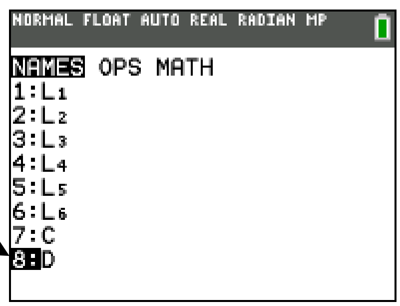
Press [ENTER].

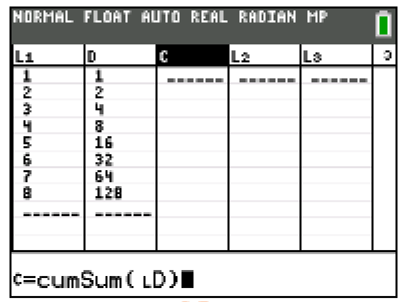




Press [2nd] [LIST] [▶] to select list *D*.

Press [)] [ENTER].





Note: The command `cumSum(D)` is **not** correct.  Be sure you use `cumSum(ʟD)`. 

6. Look for connections between *C* and *D*.

a. What patterns do you notice between the last two columns of data?

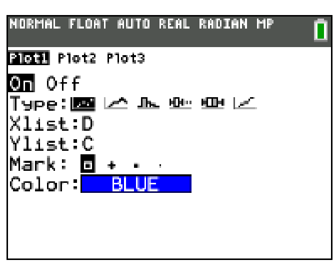
Day, *n*	Day's Payment *D*	Total (Cumulative) Amount You Have Been Paid, *C*
1	\$1	\$1
2	\$2	\$3
3	\$4	\$7
4	\$8	\$15
5	\$16	\$31
6	\$32	\$63
7	\$64	\$127
8	\$128	\$255

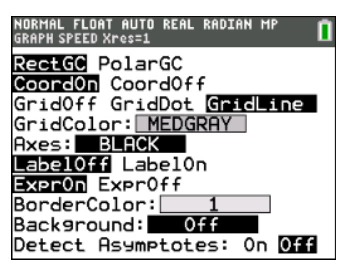
b. What kind of function would model (*D, C*)? ______________

i.e., linear, quadratic, exponential, etc. It may help to plot the data (*D, C*). You could use a standard window.

c. Write the formula for *C* as a function of *D*.

C(*D*) = ___________________________





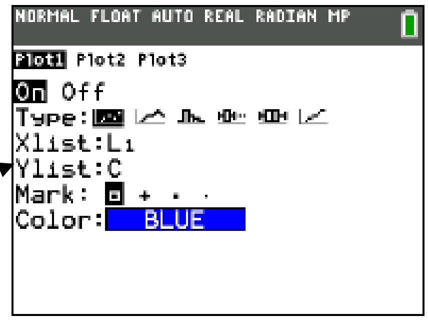
7. Find a formula for C in terms of n.

Hint: Substitute your formula for the function $D(n)$ into the formula for the function $C(D)$ and simplify.

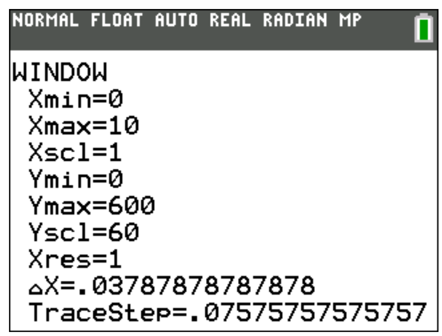
$C(n)$ = ____________________

Day, n	Day's Payment D	Total (Cumulative) Amount You Have Been Paid, C
1	\$1	\$1
2	\$2	\$3
3	\$4	\$7
4	\$8	\$15
5	\$16	\$31
6	\$32	\$63
7	\$64	\$127
8	\$128	\$255

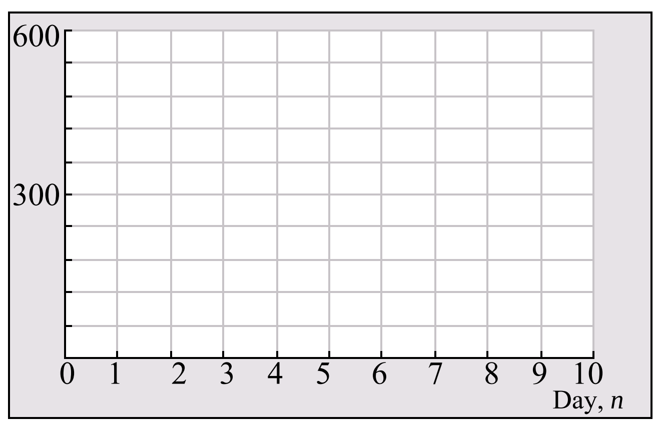
Plot the data ($L1$, C) in the specified window.
Enter your equation for $C(n)$ in the Y= editor to validate that your model matches the above.
If not, make any necessary revisions.



Note: You need only type ALPHA [C] here.
You do not need to use the "baby **L**" for a list.
It is not automatically in Alpha-LOCK on a TI84 Plus C so the F1-F5 soft keys do not interfere with the graphing keys such as Window and Zoom.



8. Use your calculator to create graphs of your total earnings under Plan A and Plan B.
Recopy your screen on the graph to the right.
Which plan should you choose?
Explain your reasoning.



9. Find the day that you will first exceed \$100,000 in total earnings (assuming Bugsy pays out) if you were compensated using Option B. About how many years would it take to earn \$100,000 if you were to choose Option A?

Teacher's Notes

Throughout this lesson, opportunities for these Common Core State Standards can appear:

A-SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★

A-SSE.4 Derive the formula for the sum of a finite geometric series (when *r* is not 1), and use the formula to solve problems.★

A-CED★.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

F-IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.★

F-IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. ★

F-BF.1 Write a function that describes a relationship between two quantities. ★
a. Determine an explicit expression, a recursive process, or steps for calculation from a context.
b. Combine standard function types using arithmetic operations.
c. Write a function that describes a relationship between two quantities.

F-BF.2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★

F-LE★.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.

F-LE★.5 Interpret the parameters in a linear or exponential function in terms of a context.

S★-ID.7 Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.

In addition, many of these Mathematical Practices can be fostered:

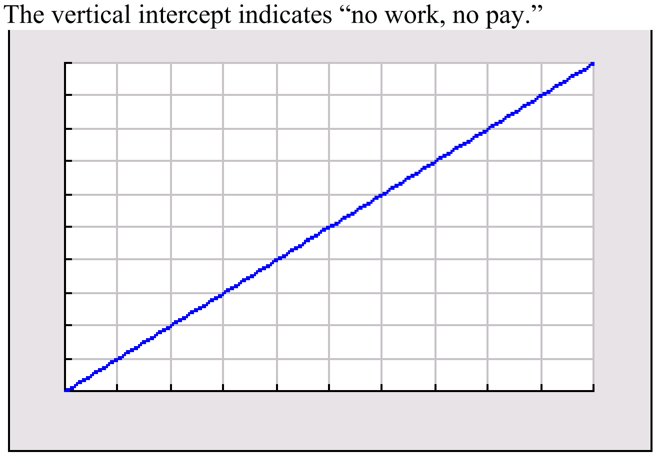
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics. ★
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Before they discuss or explore, ask students to conjecture which is the best plan. Have them write it down to force a commitment. If they choose Option B, ask them to conjecture which day Option B starts paying off more than Option A.

1.

Day, n	Total, T
1	60
2	120
3	180
4	240
5	300
6	360
7	420
8	560

2. a. linear. It grows by a constant amount. b. $T(n) = 60n$
c. The slope is the amount of dollars earned per day.
The vertical intercept indicates "no work, no pay."



3.

Day, n	Day's Payment D
1	$1=2^0$
2	$2=2^1$
3	$4=2^2$
4	$8=2^3$
5	$16=2^4$
6	$32=2^5$
7	$64=2^6$
8	$128=2^7$

4. a. exponential
b. Reasoning from the table (n, D), students may see that on day n the amount D earned that day would be $D = 2^{n-1}$ by writing each output as a power of 2. Along with a plot of data, students may see that it can be modeled by the formula $D = a \cdot b^n$ with growth factor $b = 2$ and vertical intercept $a = \frac{1}{2}$, so $D = \frac{1}{2} \cdot 2^n$ which is equivalent to $D = 2^{n-1}$ by laws of exponents:

$$D = \tfrac{1}{2} \cdot 2^n = 2^{-1} 2^n = 2^{-1+n} = 2^{n-1}$$

6. There are numerous connections between C and D.

   a. Recursively, in row n, $C(n) = D(n) + C(n-1)$.
   Since $D(n) = 1 + C(n-1)$,
   $C(n) = 1 + C(n-1) + C(n-1)$
   $\quad = 1 + 2C(n-1)$.
   In other words, NEXT = 1 + 2NOW.
   (Double the previous, then add 1)

D	***C***	***C***
1	1	1
2	3	3 = 1 + 2
4	7	7 = 3 + 4
8	15	15 = 7 + 8
16	31	31 = 15 + 16
32	63	63 = 31 + 32
64	127	127 = 63 + 64
128	255	255 = 127 + 128

   Alternatively, the last column shows that C is double the first column less 1.
   Students might explore the values of ΔC and ΔD, take their ratio, and see $\Delta C/\Delta D = 2$.

D	***C***
1	1
2	3
4	7
8	15
16	31
32	63
64	127
128	255

   b. linear

   c. The total (cumulative) sum C earned is a linear function of the amount earned that day, D, so $C = mD + b$. The average rate of change or slope $m = 2$ and the vertical intercept $b = -1$.
   Therefore $C = 2D - 1$.
   Students could also argue this from the patterns in the table as mentioned in part a.

7. By substitution: $C = 2D - 1$

$$= 2(\tfrac{1}{2} \cdot 2^n) - 1$$

$$= 2^n - 1$$

Alternatively: $C = 2(2^{n-1}) - 1 = 2^1(2^{n-1}) - 1 = 2^{1+n-1} - 1 = 2^n - 1$

Yet another way, there may be some students who will recognize the powers of 2 (especially after finding the formula for D):

n	C	$C + 1$
1	1	$2=2^1$
2	3	$4=2^2$
3	7	$8=2^3$
4	15	$16=2^4$
5	31	$32=2^5$
6	63	$64=2^6$
7	127	$128=2^7$
8	255	$256=2^8$

Thus $C + 1 = 2^n$ and so $C = 2^n - 1$.

If students are tempted to invoke the command `ExpReg L1,LC` they will find it does not give a perfect fit since the equation for C is not of the form `y=a*b^x`.

Students who have studied computers or base two arithmetic might recognize the base two representation of C as 11111111_2, which is a byte of 1's.
If we add 1 to C, we have $C + 1 = 100000000_2 = 2^8 = 256$. So $C = 2^8 - 1 = 255$.

n	C	C (as a series)	C (base 2)
1	1	2^0	1_2
2	3	$2^1 + 2^0$	11_2
3	7	$2^2 + 2^1 + 2^0$	111_2
4	15	$2^3 + 2^2 + 2^1 + 2^0$	1111_2
5	31	$2^4 + 2^3 + 2^2 + 2^1 + 2^0$	11111_2
6	63	$2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$	111111_2
7	127	$2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$	1111111_2
8	255	$2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$	11111111_2

Compare the above investigation with the question: *Find the sum of all of the positive divisors of 128.*
$128 = 2^7$ and has factors 1, 2, 2^2, 2^3, 2^4, 2^5, 2^6, and 2^7. The sum is $S = 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1$.

Precalculus students who have studied geometric series and sigma notation could use another approach.
On the nth day, the cumulative sum is $C = 1 + 2^1 + 2^2 + \ldots + 2^{n-1}$, a series of n terms.

This could be written in sigma notation $C = 1 + 2^1 + 2^2 + \ldots + 2^{n-1} = \sum_{k=1}^{n} 2^{k-1}$

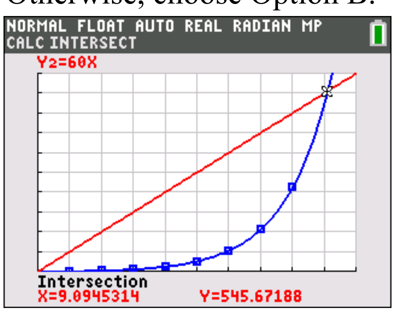
Bye!

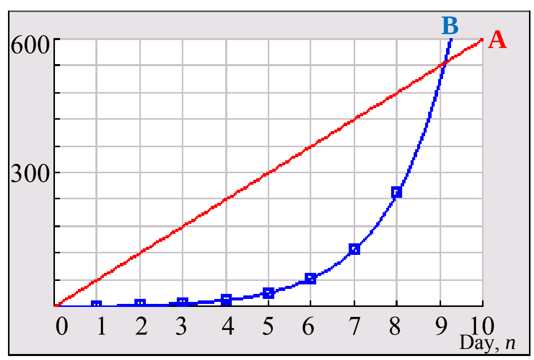
They could derive this sum using the classical approach shown:

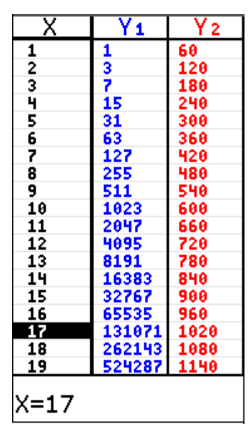
$$\begin{aligned} 2C &= \boxed{2^1 + 2^2 + \ldots + 2^{n-1}} + 2^n \\ -\quad C &= 1 + \boxed{2^1 + 2^2 + \ldots + 2^{n-1}} \\ \hline 2C - C &= -1 \qquad\qquad\qquad\qquad + 2^n \\ &= 2^n - 1 \end{aligned}$$

Or use the formula $a + ar^1 + ar^2 + \ldots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$ for $a = 1$ and $r = 2$, so $\frac{1(1-2^n)}{1-2} = \frac{1-2^n}{-1} = 2^n - 1$.

8. Option B overtakes Option A after 9 days.
If working for 9 days or less, choose Option A.
Otherwise, choose Option B.







X	Y1	Y2
1	1	60
2	3	120
3	7	180
4	15	240
5	31	300
6	63	360
7	127	420
8	255	480
9	511	540
10	1023	600
11	2047	660
12	4095	720
13	8191	780
14	16383	840
15	32767	900
16	65535	960
17	131071	1020
18	262143	1080
19	524287	1140

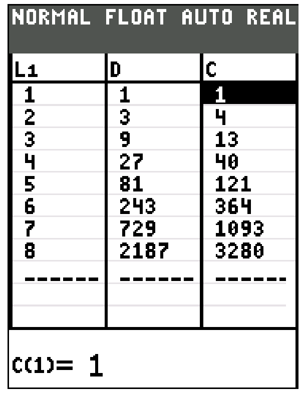
X=17

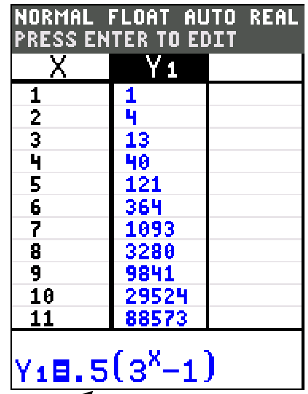
9. Solve $100{,}000 = 2^n - 1$ with a table, graph, or analytically with logs.
Option B takes only 17 days to earn \$100,000. Option A takes over 4.5 years.
It may surprise you to find when the sum will exceed 1 million, 1 billion, 1 trillion, etc.

Investigation 2: Tripling, Quadrupling, Quintupling, and beyond!

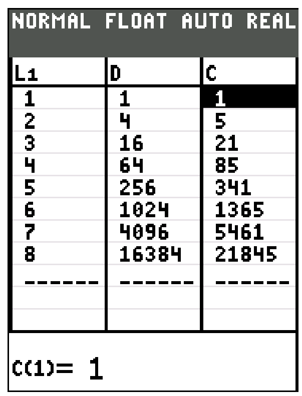
1. Suppose the job offer in the previous investigation was changed to the following:
   *Earn 1 measly dollar for the first day, but \$3 for the second, \$9 for the third, and so on, so that each day's pay is **triple** that of the previous day.*
   a. Create the list *D* as before,
      but for tripling.
      On the nth day, what is the daily wage *D*?
   b. Create the list *C* as before using `cumSum(ʟD)`
   c. Notice
      if $n = 1$, $C = 1$.
      if $n = 2$, $C = 1 + 3$.
      if $n = 3$, $C = 1+3+3^2$
      if $n = 4$, $C = 1+3+3^2+3^3$

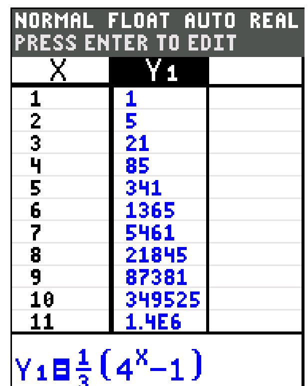
      Show using the formula for a geometric series $C = 1+3+3^2+\ldots+3^{n-1} = \frac{1}{2}(3^n - 1)$



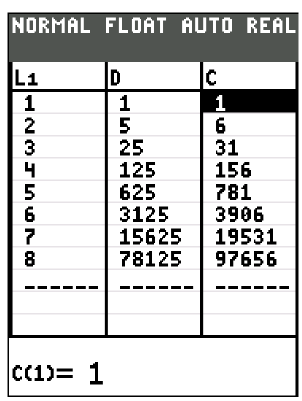


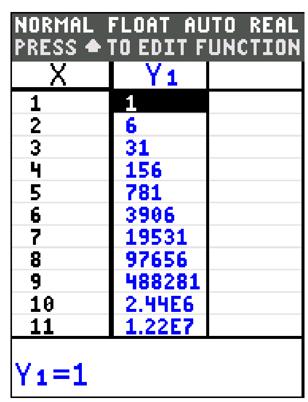
2. Replace **triple** with ***quadruple*** in Question 1.
   Show $C = 1+4+4^2+\ldots+4^{n-1} = \frac{1}{3}(4^n - 1)$





3. Conjecture what you think the formula is for *C* if $C = 1+5+5^2+\ldots+5^{n-1}$ and we ***quintupled*** the previous day's wage.
   Create a formula and test your conjecture.

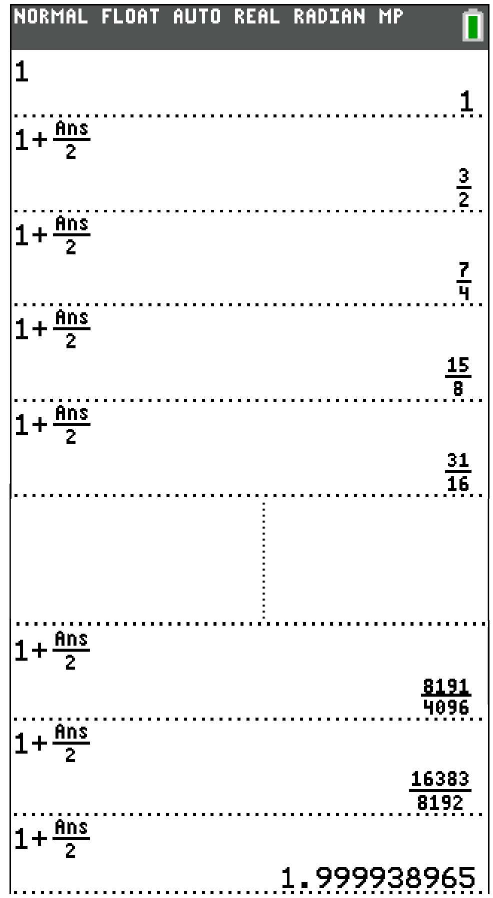




4. Does your pattern work if the next day was 10 times the amount of the previous day's wage? Generalize to any value.

Investigation 3: A gift that keeps on giving!

Reset the defaults on your calculator. (These are calculator settings it has when it first comes out of the box.)
Press [2nd] [MEM] 7:Reset… 2:Defaults… 2:Reset



1. Clear the home screen.
   Press the number 1 followed by [ENTER].
   This number will be the initial seed.

2. Next build the expression $1+\frac{\text{Ans}}{2}$
   For the shortcut FRAC menu, press [ALPHA] [F1] and use n/d for stacked fractions instead of the division key.

3. Once you build the expression, continue pressing [ENTER] to create the screen to the right. Describe any patterns.

4. Conjecture what three expressions will come next.

   $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}$, ______ , ______ , ______

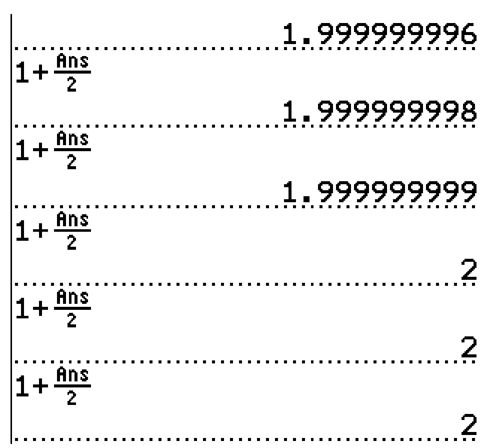
5. After pressing [ENTER] many, many times, the TI-84 Plus will eventually stop displaying a number as a stacked fraction. (It resigns from duty once the number's denominator exceeds 4 digits. Alas, using ▸Frac on 1.999938965 will not help.)

   a. If we were write the number which comes after $\frac{16383}{8192}$ as a stacked fraction, what would be its denominator?

   b. What would be its numerator?

   c. Verify your claim by entering your fraction on the home screen.

   d. A student had pressed the [ENTER] key 5 times to reach the number $\frac{31}{16}$. What is the least number of times they would have pressed [ENTER] to reach 1.999938965? Create a formula for the *n*th term.



6. Eventually the expression $1+\frac{\text{Ans}}{2}$ will converge to 2.
   This means when Ans = 2, then $1+\frac{\text{Ans}}{2} = \text{Ans}$.

   a. Solve the equation $1+\frac{x}{2}=x$ to show that $x = 2$ is the one and only value to which this expression converges.
   b. Conjecture what would happen if the initial seed were 2.
   c. Repeat the above with a seed of -1.
      After pressing [ENTER] 5 times, can you predict three more?
   d. Precalculus: Show your formula in 5d is equivalent to $2-\frac{1}{2^{n-1}}$.
      Then show why $2-\frac{1}{2^{n-1}} \to 2$ as $n \to \infty$.

Investigation 4: Exploring $1+\frac{Ans}{3}$, $1+\frac{Ans}{4}$, $1+\frac{Ans}{5}$, and beyond!

In the previous investigation, we have seen that the expression $1+\frac{Ans}{2}$ can build the sequence 1, $\frac{3}{2}$, $\frac{7}{4}$, $\frac{15}{8}$, $\frac{31}{16}$, … , 2, 2, ….

We now explore similar expressions.

1. Explore $1+\frac{Ans}{3}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks: 1, $\frac{4}{3}$, $\frac{13}{9}$, $\frac{40}{27}$, $\frac{121}{81}$, ______ , ______ , ______
   b. Predict the **denominator** of the 8th term ___________
   Predict the **denominator** of the *n*th term: _________

   c. Predict the **numerator** of the 8th term ___________
   Hint: Look back at Question 1 at the Stat Editor in Investigation 2 on page 10.

   Predict the **numerator** of the *n*th term: _________
   Hint: Look back at Question 1c in Investigation 2 on page 10.

   d. Solve the equation $1+\frac{x}{3}=x$ to find what this sequence converges to.
   Validate your claim by pressing ENTER on the home screen.

   e. Show that the *n*th term from parts b and c of this question is equivalent to $\frac{3}{2}\cdot\frac{3^n-1}{3^n}$.

   Complete the box: As $n\to\infty$, $\frac{3}{2}\cdot\frac{3^n-1}{3^n}\to\boxed{}$. Explain your reasoning.

2. Explore $1+\frac{Ans}{4}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks: 1, $\frac{5}{4}$, $\frac{21}{16}$, $\frac{85}{64}$, $\frac{341}{256}$, ______ , ______ , ______
   b. Predict the **denominator** of the 8th term ___________
   Predict the **denominator** of the *n*th term: _________

   c. Predict the **numerator** of the 8th term ___________
   Hint: Look back at Question 2 at the Stat Editor in Investigation 2 on page 10.

   Predict the **numerator** of the *n*th term: _________
   Hint: Look back at Question 2c in Investigation 2 on page 10.

   d. Solve the equation $1+\frac{x}{4}=x$ to find what this sequence converges to.
   Validate your claim by pressing ENTER continually on the home screen.

   e. Show that the *n*th term from parts b and c of this question is equivalent to $\frac{4}{3}\cdot\frac{4^n-1}{4^n}$.

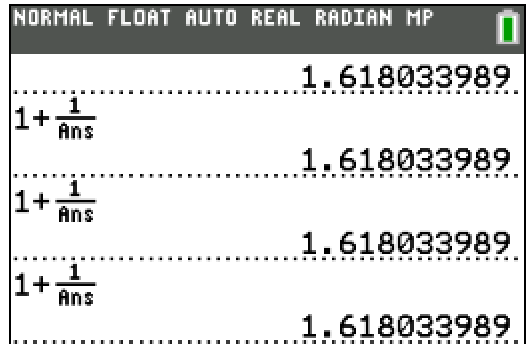
   Complete the box: As $n\to\infty$, $\frac{4}{3}\cdot\frac{4^n-1}{4^n}\to\boxed{}$. Explain your reasoning.

3. Similarly explore $1+\frac{Ans}{5}$. Then replace 5 with any real number and make a conjecture.
   For example, what would you expect for $1+\frac{Ans}{99}$? Show that your conjecture holds.

Investigation 5: Quadratic fun with $1+\frac{1}{\text{Ans}}$, $1+\frac{2}{\text{Ans}}$, $2+\frac{3}{\text{Ans}}$, $2+\frac{2}{\text{Ans}}$, and beyond!

In the previous investigation, we have seen that the expression $1+\frac{\text{Ans}}{2}$ can build the sequence 1, $\frac{3}{2}$, $\frac{7}{4}$, $\frac{15}{8}$, $\frac{31}{16}$, … , 2, 2, … We now explore similar expressions which lead to solving quadratic equations and building patterns..

1. Explore $1+\frac{1}{\text{Ans}}$ with 1 as an initial seed.
   a. Complete the next three blanks:
      1, 2, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{8}{5}$, $\frac{13}{8}$, ______ , ______ , ______
   b. Describe any patterns you see in this sequence.
   c. This will converges to a famous number. (Google "phi".)
      Solve the equation $1+\frac{1}{x}=x$ to find its exact value.
      (You will need the quadratic formula.)



2. Explore $1+\frac{2}{\text{Ans}}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks:
      1, 3, $\frac{5}{3}$, $\frac{11}{5}$, $\frac{21}{11}$, $\frac{43}{21}$, ______ , ______ , ______
   b. Consider this conjecture to predict the next numerator.
      For the 3rd term $\frac{5}{3}$, its numerator $5 = \mathbf{3} + 2\times\mathbf{1}$ where 3 and 1 are the two previous numerators.
      For the 4rd term $\frac{11}{5}$, its numerator $11 = \mathbf{5} + 2\times\mathbf{3}$ where 5 and 3 are the two previous numerators.
      For the 5rd term $\frac{21}{11}$, its numerator $21 = \mathbf{11} + 2\times\mathbf{5}$ where 11 and 5 are the two previous numerators.
      Does it hold for the next term?
   c. If 1 is the seed, this will converges to a positive integer. Solve the equation $1+\frac{2}{x}=x$ to find its value.
      Hint: Solve the equivalent equation $x+2=x^2$. It factors! ☺
      Validate your claim by pressing ENTER continually on the home screen.
   d. There are two solutions to $x+2=x^2$. What is the negative solution? What initial seed will have $1+\frac{2}{\text{Ans}}$ converge to this negative integer? (Hint: pick the solution as the seed.)

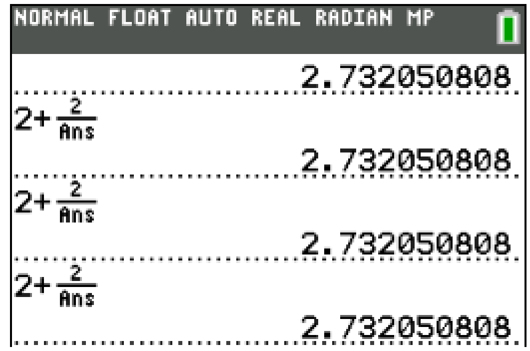
3. Explore $2+\frac{3}{\text{Ans}}$ with 1 as an initial seed.
   a. Use your calculator to complete the next three blanks:

      1, 5, $\frac{13}{5}$, $\frac{41}{13}$, $\frac{121}{41}$, $\frac{365}{121}$, ______ , ______ , ______
   b. Consider this conjecture to predict the next numerator.
      For the 3rd term $\frac{13}{5}$, its numerator $13 = 2\times\mathbf{5} + 3\times\mathbf{1}$ where 5 and 1 are the two previous numerators.
      For the 4rd term $\frac{41}{13}$, its numerator $41 = 2\times\mathbf{13} + 3\times\mathbf{5}$ where 13 and 5 are the two previous numerators.
      For the 5rd term $\frac{121}{41}$, its numerator $121 = 2\times\mathbf{41} + 3\times\mathbf{13}$ where 41 and 13 are the two previous numerators.
      Does it hold for the next term?
   c. If 1 is the seed, this will converges to a positive integer. Solve the equation $2+\frac{3}{x}=x$ to find its value.
      Hint: Solve the equivalent equation $2x+3=x^2$. It factors too! ☺ Then validate your claim by pressing ENTER continually on the home screen. What initial seed will have $2+\frac{3}{\text{Ans}}$ converge to a negative integer?

4. The expression $2+\frac{2}{\text{Ans}}$ with 1 as an initial seed will gives us the sequence of terms:

$$1,\ \frac{4}{1},\ \frac{(1+4)\times 2}{4}=\frac{5}{2},\ \frac{(5+2)\times 2}{5}=\frac{14}{5},\ \frac{(14+5)\times 2}{14}=\frac{19}{7},\ \frac{(19+7)\times 2}{19}=\frac{52}{19},\ \frac{(52+19)\times 2}{52}=\frac{71\times 2}{52}=\frac{71}{26},$$

a. What pattern do you see?



b. Use your pattern to conjecture the next three terms:
$1, \ldots, \frac{71}{26}$, ______ , ______ , ______
Validate with technology.

c. When we solve the equation $2+\frac{2}{x}=x$
to find the values at which it will converge,
we could solve the equivalent equation $2x+2=x^2$.
Solve this by completing the square.

d. Complete the boxes.
The solution in part c will be equivalent to $(x-1)^2=\boxed{}$
When 1 is the initial seed, $2+\frac{2}{\text{Ans}}$ will converge to the value $1+\sqrt{\boxed{}}$

e. It can be shown that the continued fraction $1+\cfrac{2}{2+\cfrac{2}{2+\cfrac{2}{2+\ldots}}} = \sqrt{3}$.

How does this investigation validate this amazing fact?
Hint: Add 1 to both sides of this equation and compare with 4d.

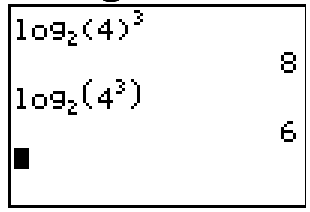
5. Find the value of the continued fraction $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}$. Start by exploring how $2+\frac{1}{\text{Ans}}$ converges.

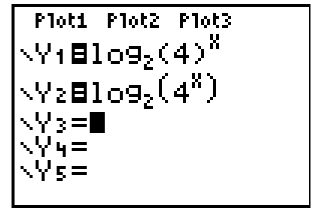
6. Find the value of the continued fraction $1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$. Hint: See Question 1.

7. Find the value of the continued fraction $1+\cfrac{3}{2+\cfrac{3}{2+\cfrac{3}{2+\ldots}}}$. Hint: See Question 3.

8. Make conjectures for what will happen for expressions of the form $b+\frac{c}{\text{Ans}}$ for any values of b and c (both positive and negative.)
For what values of b and c will you have the sequence converge to rational solutions?

Investigation 6: Explore Properties of Logarithms

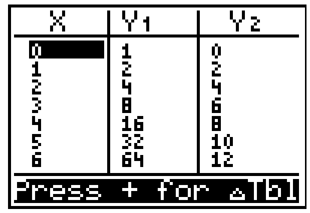


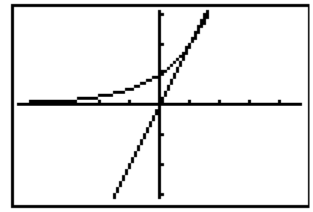


1. Compare the expressions on the screen to the right.
   Notice the usual order of operations are followed.
   Unveil $\log_2(4)^3 = (\log_2(4))^3 = (\log_2 2^2)^3 = (2)^3 = 8$
   and $\log_2(4^3) = \log_2(2^2)^3 = \log_2(2^6) = 6$

2. Explore with a table and a graph.
   What is the simplified form of each?

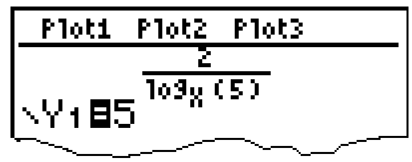
   Do they look more familiar now?
   Superimpose graphs of $y = 2^x$ and $y = 2x$ over each.





3. Facilitate a class discussion on logarithmic properties.

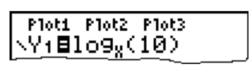
Investigation 7: Explore expressions of the form $a^{\frac{b}{\log_x a}}$



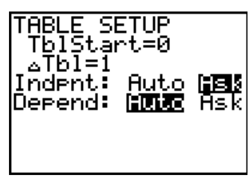
1. Insert the expression in Y1
   with $a = 5$ and $b = 2$ and
   explore the table.

2. Explore and discuss:
   - What happens when you change the parameter a to any positive number greater than 1?
   - What happens when you change the parameter b to 3? to 1? to 0? to 0.5? to −1?
   - Use properties of logarithms to explain.
     Hint: Take the logarithms to the base x of both sides of the equation $y = a^{\frac{b}{\log_x a}}$.

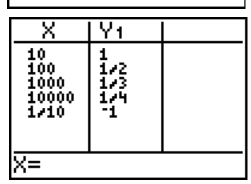
Investigation 8: What's My Rule?



1. Consider the function $y = \log_x 10$. Enter the expression in Y1.

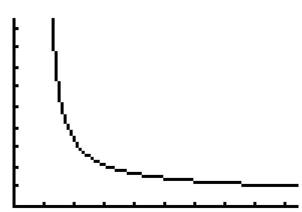


2. Press 2nd WINDOW to match the screen shown to the right,
   where Indpnt is set to Ask.



3. Explore with a table, where x is a power of 10.

   Discuss:
   - For the first four entries of the table,
     how is the denominator of the output
     related to the number of 0's of the input?
   - What relationship holds for negative integer powers of 10, such as $\frac{1}{10}$, $\frac{1}{100}$, etc.?



4. Explore with a graph after, say, ZQuadrant1.

5. Rewrite the function $y = \log_x 10$ so that x is not the logarithmic base.
   Hint: Let $y = \log_x 10$, write in exponential form, then take common logarithms
   of both sides of the equation. Compare tables and graph the result in the same window.

6. Follow up: In general, does $\log_a b = \frac{1}{\log_b a}$? Hint: Let $y = \log_a b$, solve for b, then take logs of both sides to the base b.