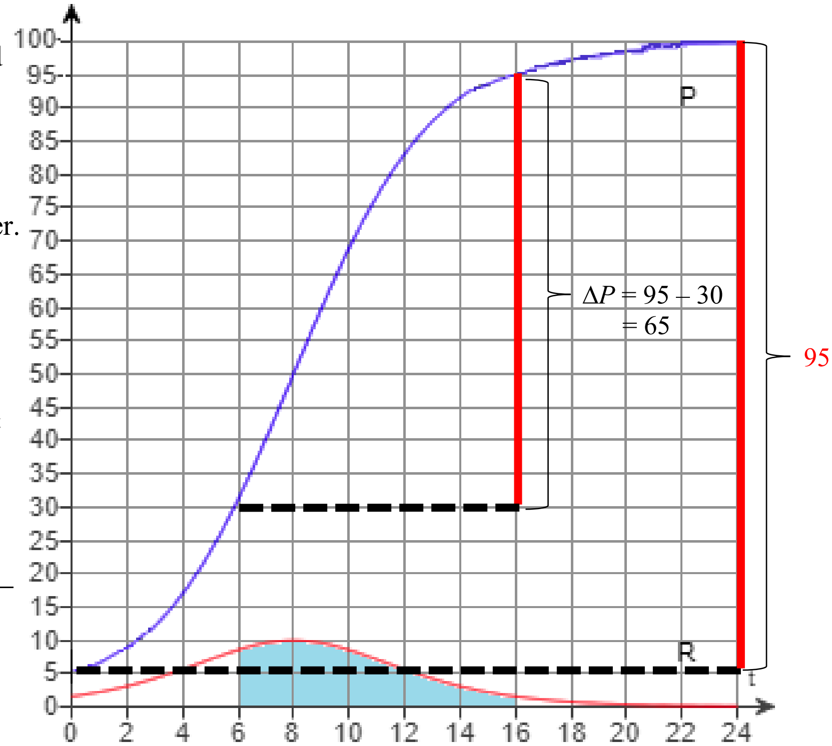
Definite Integrals (Section 5.2) and the *Fundamental* Theorem of Calculus (Section 5.3)



1. The rate $R = P'$ at which people are becoming infected with a contagious virus t weeks after 5 people were infected is graphed to the right, along with the total cumulative number of people, P, who have been infected over the life of the epidemic. It takes 24 weeks for the epidemic to run its course and be over.

a. Report the shaded area from week 6 to week 16.

$$\int_6^{16} R(t)\,dt = \int_6^{16} P'(t)\,dt = \boxed{65}$$ **people**

unit of measurement

$\Delta P = P(16) - P(6) = 95 - 30 = 65$

i. Interpret what this area represents in terms of the context of the epidemic.

65 additional people were infected from week 6 to 16.

ii. Sketch a segment on the graph of P to represent ΔP for your answer in part **a**.
Segment is vertical at $t = 16$ with length 65.
Starts at $P(6) = 30$ and ends at $P(16) = 95$.

b. What is the **total** area under R? 95

$$\int_0^{24} P'(t)dt = \Delta P = P(24) - P(0) = 100 - 5 = 95$$

i. Sketch a segment on the graph of P to represent ΔP for your answer in part **b.**
Segment is vertical at $t = 24$ with length 95. Starts at $P(0) = 5$ and ends at $P(24) = 100$.

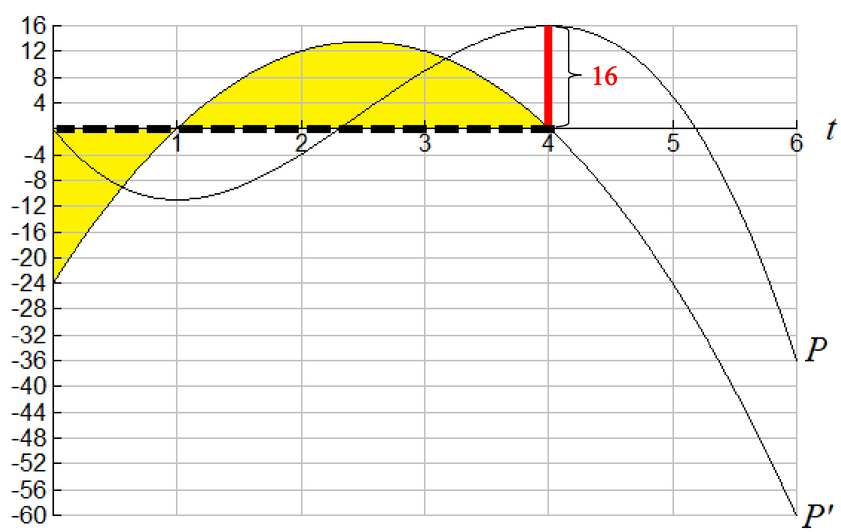
ii. Interpret what the total area represents in terms of the context of the epidemic.
95 additional people were infected after the initial 5 were infected.

c. When is the number of infected increasing the fastest? $t =$ 8 weeks **R is a maximum at $t = 8$.**

d. Complete with whole numbers.
From $0 < t < \boxed{8}$ the number infected by the virus {speeds up, slows down}

From $\boxed{8} < t < \boxed{24}$ the number infected by the virus {speeds up, slows down}

2. The graph shows a company's profit, P, in thousands, and marginal profit P' in thousands per year, for a 6 year interval.



a. $$\int_0^4 P'(t)\,dt = \boxed{16}$$

$\Delta P = P(4) - P(0) = 16 - 0 = 16$

b. Sketch the segment which represents ΔP for this interval.
Segment is vertical at $t = 4$ with length 10.
Starts at $P(0) = 5$ and ends at $P(24) = 100$.

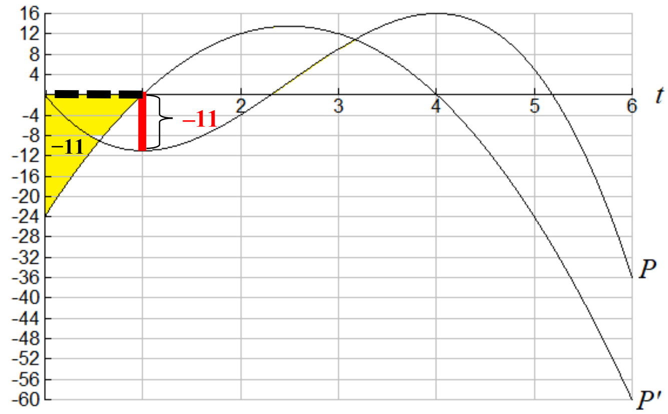
c. Interpret what this shaded area represents in the context of the company's profits.

The company's profits increased by a total of 16 thousand dollars in the first four years.

d. It is known that the shaded area under the curve P' from $t = 0$ to $t = 1$ is −$11 and that the curve P' is quadratic.

i. For the area to the right and each of the areas below:

- Sketch the segment which represents ΔP for the interval specified.
- Write the area as a definite integral and give its value.
- Interpret what the area means in terms of the company's profits.

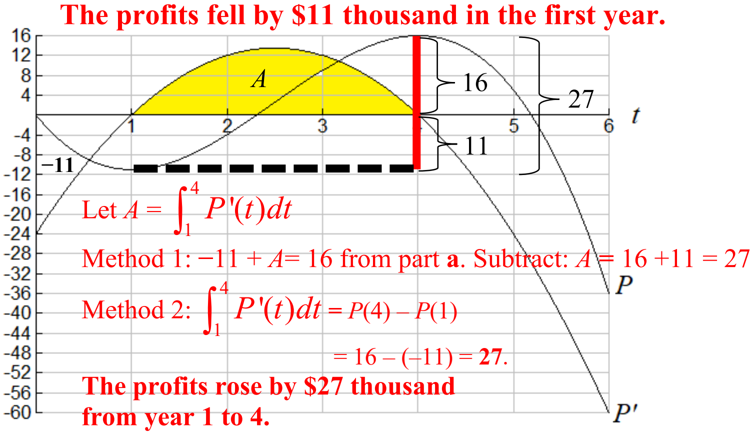


Since $\int_0^1 P'(t)dt = -11 = P(1) - P(0)$ and $P(0) = 0$, then $P(1) = -11$.

Report $P(1) =$ −11

The profits fell by $11 thousand in the first year.

ii.



Let $A = \int_1^4 P'(t)dt$

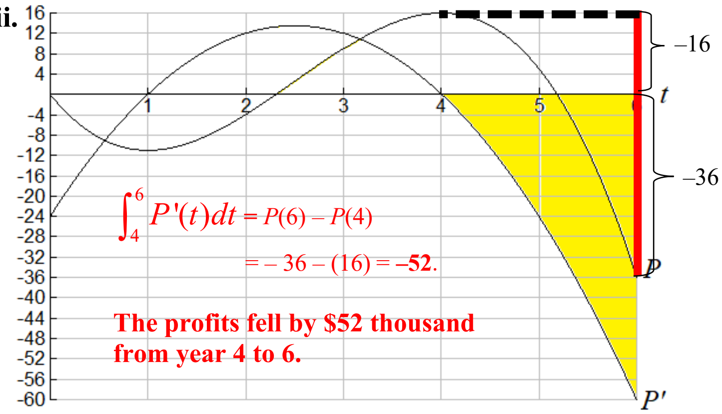
Method 1: $-11 + A = 16$ from part **a**. Subtract: $A = 16 + 11 = 27$

Method 2: $\int_1^4 P'(t)dt = P(4) - P(1)$

$= 16 - (-11) = \mathbf{27}$.

The profits rose by $27 thousand from year 1 to 4.

iii.

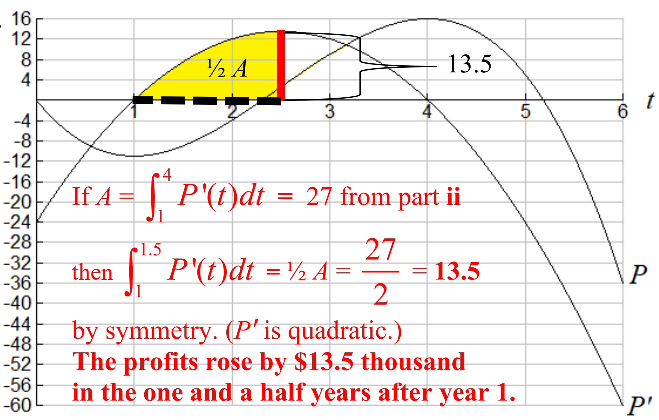


$\int_4^6 P'(t)dt = P(6) - P(4)$

$= -36 - (16) = \mathbf{-52}$.

The profits fell by $52 thousand from year 4 to 6.

iv.



If $A = \int_1^4 P'(t)dt = 27$ from part **ii**

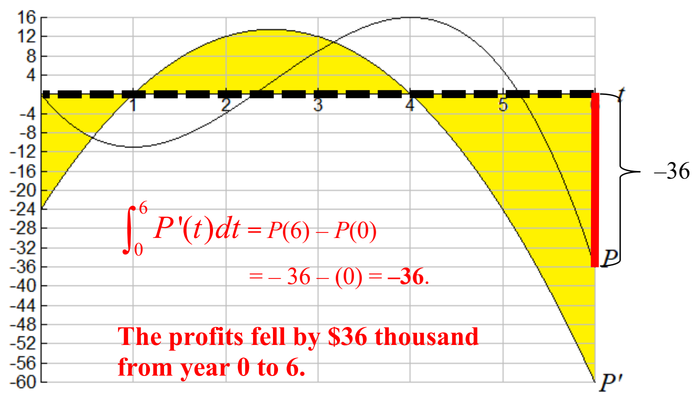
then $\int_1^{1.5} P'(t)dt = ½ A = \frac{27}{2} = \mathbf{13.5}$

by symmetry. (P' is quadratic.)

The profits rose by $13.5 thousand in the one and a half years after year 1.

Report $P(2.5) =$ **13.5**

v.

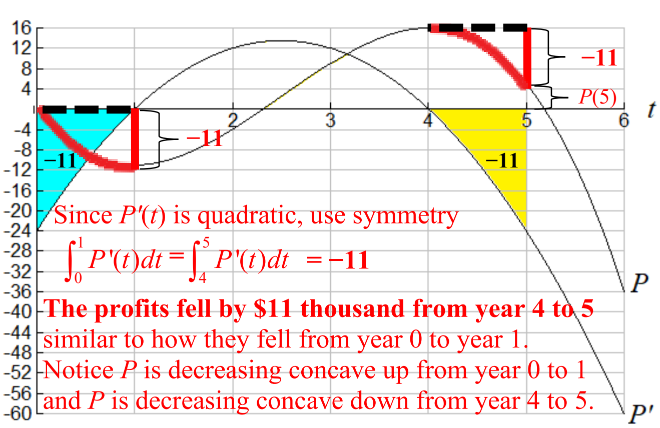


$\int_0^6 P'(t)dt = P(6) - P(0)$

$= -36 - (0) = \mathbf{-36}$.

The profits fell by $36 thousand from year 0 to 6.

vi.



Since $P'(t)$ is quadratic, use symmetry

$\int_0^1 P'(t)dt = \int_4^5 P'(t)dt = -11$

The profits fell by $11 thousand from year 4 to 5

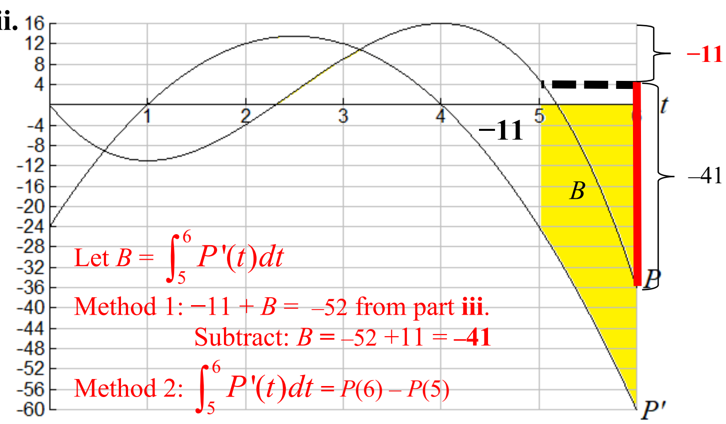
similar to how they fell from year 0 to year 1.

Notice P is decreasing concave up from year 0 to 1

and P is decreasing concave down from year 4 to 5.

Report $P(5) =$ 5 $P(5) + 11 = 16$ so $P(5) = 5$.

vii.



Let $B = \int_5^6 P'(t)dt$

Method 1: $-11 + B = -52$ from part **iii**.

Subtract: $B = -52 + 11 = \mathbf{-41}$

Method 2: $\int_5^6 P'(t)dt = P(6) - P(5)$

$= -36 - 5 = \mathbf{-41}$.

The profits fell by $41 thousand from year 5 to 6.

e. Give formulas for P and P'. See next page.

e. Give formulas for P and P'.

Method 1: The quadratic function P' has zeros at 1, 4 and passes through (0, –24)

$P' = a(t-1)(t-4)$ **so substitute (0, –24) and solve for a.**

$-24 = a(t-1)(t-4)$

$-24 = a(0-1)(0-4)$

$-24 = a(-1)(-4)$

$-24 = 4a$

$a = -6$

We have $\boxed{P' = -6(t-1)(t-4)}$

Method 2: The quadratic function P' has a vertex of (2.5, 13.5) and passes through (4, 0).

$P' = a(t-2.5)^2 + 13.5$ **so substitute (4, 0) and solve for a.**

$0 = a(4-2.5)^2 + 13.5$

$0 = a(1.5)^2 + 13.5$

$0 = 2.25a + 13.5$

$2.25a = -13.5$

$a = -6$

We have $\boxed{P' = -6(t-2.5)^2 + 13.5}$

We now integrate. Expanding $P' = -6(t-1)(t-4)$ or $P' = -6(t-2.5)^2 + 13.5$ we have $\boxed{P' = -6t^2 + 30t - 24}$

$$\int P'(t)dt = \int(-6t^2 + 30t - 24)dt$$

$$P(t) = -6\int t^2 dt + 30\int t\,dt - 24\int dt$$

$$= -6\frac{t^3}{3} + 30\frac{t^2}{2} - 24t + C$$

$$= -2t^3 + 15t^2 - 24t + C$$

Substitute $t = 0$, $P = 0$.

We have $C = 0$ so

$\boxed{P = -2t^3 + 15t^2 - 24t}$

TIP: Check the answers match the given graphs using a graphing calculator.