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A TAYLOR SERIES CONDITION FOR HARMONIC EXTENSION

Abstract

For a harmonic function on an open subset of real n-space, we propose a condition on the Taylor expansion that implies harmonic extension to a larger set, by a result on the size of the domain of convergence of its Taylor series. The result in the $n = 2$ case is due to M. Bôcher (1909), and the generalization to $n > 2$ is given a mostly elementary proof, using basic facts about multivariable power series.

1 Introduction

In this article, we use elementary methods to prove the following Theorem.

Theorem 1.1. *Suppose the function $u(\vec{x})$ is harmonic on some ball centered at the origin of $\mathbb{R}^n$, with coordinates $\vec{x} = (x', x_n)$. If the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ converge for $|x'| < r$, then the Taylor series of $u(\vec{x})$ converges for all $\vec{x} = (x', x_n)$ such that $|x'| + |x_n| < r$.* ∎

The above domain of convergence may be larger than the given ball, in which case the sum of the Taylor series is a harmonic extension of u. The next Section gives some examples showing why the hypotheses are necessary. In Section 5 we state some easy generalizations to other second-order PDEs.

The topic of power series expansions for harmonic functions in $\mathbb{R}^n$ has been approached using complex variable methods for a long time. The above Theorem in the $n = 2$ case follows from an elementary argument of [B], and our proof for the general case will use a similar line of reasoning, but with additional estimates for some combinatorial coefficients that arise when $n > 2$.

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The domain of convergence of the power series for a harmonic function has also been discussed by [Fugard], [Hayman], [Siciak], and [W_2]. The authors' investigation of this Theorem was motivated by the work of two of them in approximation theory ([LP]).

We also remark that the result can be given a less elementary proof, using the methods of complex analytic differential equations, which will be outlined in Section 6. Our approach uses only basic facts about multi-indexed real-variable series, with the exception of a Cauchy estimate for some derivatives of complex analytic functions. In addition to being in the classical spirit of Bôcher's approach, our proof shows how the set $\{|x'| + |x_n| < r\}$ naturally occurs as a subset of the domain of convergence for the series, and we give examples showing in which ways the general result is sharp.

2 Remarks on the main result

Since our topic is harmonic functions on domains in $\mathbb{R}^n$, we will fix some $n \geq 2$. The following coordinate system for $\mathbb{R}^n$ will be convenient.

Notation 2.1. For a vector $\vec{x} = (x_1, x_2, \ldots, x_{n-1}, x_n)$, let $x' = (x_1, \ldots, x_{n-1})$ and denote $\vec{x} = (x', x_n)$.

Notation 2.2. The standard Euclidean norm of a vector (real or complex) will be denoted

$$|\vec{x}| = (|x_1|^2 + \ldots + |x_n|^2)^{(1/2)},$$

and the same symbol is used for lengths of vectors in the hyperplane $x_n = 0$,

$$|x'| = |(x', 0)| = (|x_1|^2 + \ldots + |x_{n-1}|^2)^{(1/2)}.$$

The real ball with center $\vec{a} \in \mathbb{R}^n$ and radius $r > 0$ will be denoted

$$B(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n : |\vec{x} - \vec{a}| < r\},$$

and the ball in the hyperplane is denoted with a prime:

$$B'(a', r) = \{(x', 0) : |x' - a'| < r\}.$$

With this notation, we can state a Corollary of Theorem 1.1. Since the set $\{|x'| + |x_n| < r\}$ contains the ball with radius $r/\sqrt{2}$, we have:

Corollary 2.3. *Given $\epsilon > 0$, and a function $u(\vec{x})$ which is harmonic on $B(\vec{0}, 1)$, if the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ converge on $B'(0', \sqrt{2} + \epsilon)$, then $u(\vec{x})$ extends harmonically to $B(\vec{0}, 1 + \epsilon/\sqrt{2})$.* ∎

Note that u can have complex or real values, and a real-valued function on the ball will extend to another real-valued function on the larger ball. As r (or ϵ) is allowed to be arbitrarily large, there is another Corollary.

Corollary 2.4. *Given $\rho > 0$, and a function $u(\vec{x})$ which is harmonic on $B(\vec{0}, \rho)$, if the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ converge on the entire hyperplane $\{x_n = 0\}$, then $u(\vec{x})$ extends to a function which is harmonic on $\mathbb{R}^n$.* ∎

Example 2.5. The hypothesis on the derivative in the normal direction cannot just be dropped, as shown by this simple example in $\mathbb{R}^2$ with coordinates (x, y):

$$u(x, y) = \frac{y}{(x-1)^2 + y^2}.$$

This u is harmonic on the unit ball, and the Taylor series (with center 0) of $u(x, 0)$ has all zero coefficients, but the Taylor series (with center 0) of

$$u_y(x, 0) = \frac{1}{(x-1)^2}$$

diverges at $x = 1$, and u does not harmonically extend to any larger ball.

Example 2.6. It is well-known that if u is harmonic on a domain, then it is real analytic on the same domain ([ABR], [J]), so if $u(\vec{x})$ is harmonic on $B(\vec{0}, 1)$, then the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ converge in some neighborhood of the origin. However, even if $u(x', 0)$ and $u_{x_n}(x', 0)$ are real analytic on a large set, this is not enough to guarantee that the domain of convergence of the Taylor series is large enough to satisfy the hypotheses of the Theorem, as shown by this simple example in $\mathbb{R}^2$:

$$u(x, y) = \frac{x}{x^2 + (y-1)^2}.$$

This u is harmonic on the unit ball, and $u(x, 0) = \frac{x}{x^2+1}$ and $u_y(x, 0) = \frac{2x}{(x^2+1)^2}$ are real analytic on $\mathbb{R}^1$, but their Taylor series (with center 0) converge only for $-1 < x < 1$, and u does not harmonically extend to any larger ball.

For each of the functions from the above examples, Theorem 1.1 states that the corresponding Taylor series converges for $\{(x, y) : |x| + |y| < 1\}$, and it is shown by [B] and [Hayman] that the series does not converge on any larger open set. So in this sense, the claim of Theorem 1.1 on the size of the domain of convergence is sharp when $n = 2$.

The contrapositive formulation of Theorem 1.1 is also interesting.

Corollary 2.7. *Suppose the function $u(\vec{x})$ is harmonic on a ball centered at the origin of $\mathbb{R}^n$, with coordinates $\vec{x} = (x', x_n)$. If the Taylor series of $u(\vec{x})$ does not converge for all $\vec{x} = (x', x_n)$ such that $|x'| + |x_n| < r$, then the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ do not both converge for all x' such that $|x'| < r$.* ∎

Example 2.5 shows that one of the series may converge on a large set, while the other one does not. As the following Example shows, even if u extends continuously beyond $B(\vec{0}, 1)$, Corollary 2.7 still gives an upper bound for the radius of convergence of $u(x', 0)$ and $u_{x_n}(x', 0)$.

Example 2.8. Consider the continuous function $f(x, y) = |x|$, restricted from $\mathbb{R}^2$ to the unit circle $\{x^2 + y^2 = 1\}$. Using this as Dirichlet data, there exists a function $u(x, y)$ which is continuous on the closed ball $\{x^2+y^2 \leq 1\}$, harmonic on the open ball $B(\vec{0}, 1)$, and equal to f on the circle. Although the Taylor expansion converges to u on some neighborhood of $\vec{0}$, its domain of convergence cannot contain the points $(0, 1)$ or $(0, -1)$, where u is not differentiable, and so it does not contain any set of the form $\{|x| + |y| < r\}$, for $r > 1$. We can conclude that if R_1 and R_2 are the radii of convergence of $u(x, 0)$ and $u_y(x, 0)$, then $\min\{R_1, R_2\} \leq 1$. In fact, the Poisson formula gives:

$$u(x,0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{1-x^2}{1-2x\cos(\theta)+x^2}|\cos(\theta)|d\theta = \begin{cases} \frac{2(1+x^2)\tan^{-1}(x)}{\pi x}, & x \neq 0 \\ 2/\pi, & x = 0 \end{cases},$$

a real analytic function on $\mathbb{R}$ whose Taylor series converges exactly on $[-1, 1]$.

3 Some preliminary estimates and Propositions

The approach to the proof of the main result will be to work with the multivariable power series expansion of a harmonic function, and to describe its domain of convergence. Some references touching on the subject of elementary methods of multi-indexed series include [ABR], [D], [GF], [KP], [Range], and they mention various equivalent notions of "convergence" of power series in several variables, which could more specifically be described as "absolute convergence." In this Section, we briefly recall the results we will need. Our only appeals to the more advanced calculus will be to use Stirling's formula to get an estimate for a combinatorial quantity, and then to recall Cauchy's estimates for the derivatives of complex analytic functions in several variables.

Following the "prime" convention for real coordinates (Notation 2.1), similar notation will be used for the multi-indices that appear in power series expressions.

Notation 3.1. Let $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$ be the set of whole numbers. Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{W}^n$, let $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1})$, and then denote $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', \alpha_n)$.

We also recall some commonly used notation for multi-indices.

Notation 3.2. The degree of a multi-index is the sum $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$ (not to be confused with Notation 2.2), and the factorial of a multi-index is a product of factorials: $\boldsymbol{\alpha}! = \alpha_1! \cdot \alpha_2! \cdot \ldots \cdot \alpha_n!$. For vectors $\vec{y} = (y', y_n) \in \mathbb{R}^n$, the multi-indices can be used as exponents in a product: $(\vec{y})^{\boldsymbol{\alpha}} = y_1^{\alpha_1} \cdot \ldots \cdot y_n^{\alpha_n}$, and $(y')^{\boldsymbol{\alpha}'} = y_1^{\alpha_1} \cdot \ldots \cdot y_{n-1}^{\alpha_{n-1}}$. Given a multi-index $\boldsymbol{\beta}$, if $\alpha_i \geq \beta_i$ for $i = 1, \ldots, n$, denote this property of $\boldsymbol{\alpha}$ by "$\boldsymbol{\alpha} \geq \boldsymbol{\beta}$."

The following combinatorial enumeration is well-known and easy to establish ([KP] §1.5, [Stanley]).

Proposition 3.3. *Given $b \in \mathbb{W}$, the number of multi-indices $\boldsymbol{\beta} \in \mathbb{W}^n$ such that $|\boldsymbol{\beta}| = b$ is* $\binom{b+n-1}{n-1}$. ∎

The next Lemma will be used in the Proof of Theorem 4.3. The exact value of the upper bound will not be important, just the fact that it does not depend on b.

Lemma 3.4. *Given $n \in \mathbb{N}$ and $\boldsymbol{\beta} \in \mathbb{W}^n$, denote $b = |\boldsymbol{\beta}|$. Then,*

$$\frac{b! b^b (2\boldsymbol{\beta})!}{\boldsymbol{\beta}! \boldsymbol{\beta}^{\boldsymbol{\beta}} (2b)!} \leq (\sqrt{2})^{n-1}, \tag{3.1}$$

with equality if and only if $n = 1$.

Proof. In the $n = 1$ case, $\boldsymbol{\beta} = (\beta_1)$, so $b = \beta_1$ and the LHS of (3.1) cancels exactly to $1 = (\sqrt{2})^0$. The rest of the Proof will assume $n > 1$.

The following version of Stirling's formula with estimates is proved by [Robbins]. For $k \in \mathbb{N}$,

$$\begin{aligned} \ln(k!) &= \frac{1}{2}\ln(2\pi) + (k + \frac{1}{2})\ln(k) - k + r(k), \\ & \frac{1}{12k+1} < r(k) < \frac{1}{12k}. \end{aligned}$$

Since the formula does not apply to $k = 0$, we will begin with the case where all entries of the multi-index $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are positive. So, $n > 1$ implies $0 < \beta_j < b$, for $j = 1, \ldots, n$.

A straightforward application of the formula gives:

$$\begin{aligned}\ln\left(\frac{b!b^b(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!\boldsymbol{\beta}^{\boldsymbol{\beta}}(2b)!}\right) &= \frac{n-1}{2}\ln(2)+r(b)-r(2b)-\sum_{j=1}^{n}(r(\beta_j)-r(2\beta_j)) \\ &< \frac{n-1}{2}\ln(2)+\frac{1}{12b}-\frac{1}{24b+1}-\sum_{j=1}^{n}\left(\frac{1}{12\beta_j+1}-\frac{1}{24\beta_j}\right).\end{aligned}$$

The following inequalities can be checked by elementary methods:

$$\frac{1}{12\beta_j+1}-\frac{1}{24\beta_j}\geq\frac{1}{12(b-1)+1}-\frac{1}{24(b-1)}>\frac{1}{12b}-\frac{1}{24b+1}>0.$$

In particular, each of the terms in the above $j=1\dots n$ sum is large enough to more than cancel the $\frac{1}{12b}-\frac{1}{24b+1}$ quantity, and exponentiation results in:

$$\frac{b!b^b(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!\boldsymbol{\beta}^{\boldsymbol{\beta}}(2b)!}<(\sqrt{2})^{n-1}.$$

The remaining case is that some of the β_j components are 0. If all n components are 0, the claimed inequality (3.1) reduces to $1<(\sqrt{2})^{n-1}$. If $\boldsymbol{\beta}$ has exactly m non-zero components, $1\leq m<n$, and $\boldsymbol{\gamma}\in\mathbb{W}^m$ has the same positive components as $\boldsymbol{\beta}$, then $|\boldsymbol{\gamma}|=b$, and:

$$\frac{b!b^b(2\boldsymbol{\beta})!}{\boldsymbol{\beta}!\boldsymbol{\beta}^{\boldsymbol{\beta}}(2b)!}=\frac{b!b^b(2\boldsymbol{\gamma})!}{\boldsymbol{\gamma}!\boldsymbol{\gamma}^{\boldsymbol{\gamma}}(2b)!}\leq(\sqrt{2})^{m-1}<(\sqrt{2})^{n-1}.$$

∎

The proof of Theorem 4.3 will require some technical facts about multi-indexed series. The next three Propositions are unsurprising versions of well-known results, and we state them with just sketches of proofs. Notes giving complete and elementary proofs are available electronically from one of the authors, [C].

Proposition 3.5. *Given a multi-indexed sequence $c:\mathbb{W}^n\to\mathbb{C}$ and $\vec{a}\in\mathbb{R}^n$, if $\sum c_{\boldsymbol{\alpha}}(\vec{x}-\vec{a})^{\boldsymbol{\alpha}}$ converges for all $\vec{x}$ in a real ball, $B(\vec{a},r)$, then $\sum c_{\boldsymbol{\alpha}}(\vec{z}-\vec{a})^{\boldsymbol{\alpha}}$ and $\sum|c_{\boldsymbol{\alpha}}|(\vec{z}-\vec{a})^{\boldsymbol{\alpha}}$ converge on the complex ball with the same radius, $\{\vec{z}\in\mathbb{C}^n:|\vec{z}-\vec{a}|^2=\sum_{j=1}^{n}|z_j-a_j|^2<r^2\}$.*

Sketch of Proof. Given the convergence of a power series at $\vec{x}$, an element of $B(\vec{a},r)$ such that $\vec{x}-\vec{a}$ has all positive coefficients, the absolute convergence of the series $\sum c_{\boldsymbol{\alpha}}(\vec{z}-\vec{a})^{\boldsymbol{\alpha}}$ on an open complex polydisc with center $\vec{a}$ and polyradius $\vec{x}-\vec{a}$ is established by Abel's lemma. The complex ball is a union of such polydiscs. ∎

Proposition 3.6. *Given $c_{\boldsymbol{\alpha}}$, if $\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}}(\vec{x})^{\boldsymbol{\alpha}}$ converges on $B(\vec{0},r)$, and $\vec{a}\in B(\vec{0},r)$, then there is some multi-indexed sequence $c'_{\boldsymbol{\alpha}}$ so that for all $\vec{x}\in B(\vec{a},r-|\vec{a}|)$, $\sum c'_{\boldsymbol{\alpha}}(\vec{x}-\vec{a})^{\boldsymbol{\alpha}}$ is a convergent power series, with sum equal to $\sum c_{\boldsymbol{\alpha}}(\vec{x})^{\boldsymbol{\alpha}}$.*

Sketch of Proof. The interesting part of the Proposition is the radius, $r-|\vec{a}|$, which gives the largest ball centered at $\vec{a}$ contained in the ball centered at $\vec{0}$. The $n=1$ case, where the ball is an interval, is Proposition 1.2.2 of [KP]. For higher dimensions, a corresponding version of the statement for "polycylinders" (real polydiscs) is Theorem 9.3.1 of [D]. The statement for the ball follows; by some elementary geometry, for any element $\vec{x}\in B(\vec{a},r-|\vec{a}|)$, there is a polycylinder D_1 with center $\vec{a}$ and a polycylinder D_2 with center $\vec{0}$ such that $\vec{x}\in D_1\subseteq B(\vec{a},r-|\vec{a}|)$ and $D_1\subseteq D_2\subseteq B(\vec{0},r)$. The coefficients $c'_{\boldsymbol{\alpha}}$ could be found, for example, by Taylor's formula, and do not depend on D_1 or D_2. ∎

The following Proposition gives a criterion for the convergence of a multi-indexed series in terms of summations over subsets of the index set.

Proposition 3.7. *Given $n\geq 2$, a multi-indexed sequence $c:\mathbb{W}^n\to\mathbb{C}$, a sequence $b:\mathbb{W}\to\mathbb{C}$, and $\vec{y}\in\mathbb{C}^n$, if*

$$\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}}\left|c_{(\boldsymbol{\alpha}',\alpha_n)}(y')^{\boldsymbol{\alpha}'}\right| \tag{3.2}$$

forms a convergent multi-indexed series for each $\alpha_n\in\mathbb{W}$, and

$$\left\{\left(\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}}\left|c_{(\boldsymbol{\alpha}',\alpha_n)}(y')^{\boldsymbol{\alpha}'}\right|\right)\cdot b_{\alpha_n}\cdot y_n^{\alpha_n}:\alpha_n\in\mathbb{W}\right\}$$

is a bounded subset of $\mathbb{C}$, then, for all $\vec{x}$ such that $|x_i|<|y_i|$ for $i=1,\ldots,n$,

$$\sum_{\alpha_n\in\mathbb{W}}\left(\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}}c_{(\boldsymbol{\alpha}',\alpha_n)}(x')^{\boldsymbol{\alpha}'}\right)\cdot b_{\alpha_n}\cdot x_n^{\alpha_n}$$

and

$$\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n}c_{\boldsymbol{\alpha}}\cdot b_{\alpha_n}\cdot(\vec{x})^{\boldsymbol{\alpha}}$$

are both convergent, with the same sum.

Sketch of Proof. Again, the convergence of a power series at $\vec{x}$, as a consequence of the boundedness of terms involving $\vec{y}$, is essentially Abel's lemma. The interesting part of this Proposition is the need for the absolute value on the terms of (3.2). Although any convergent multi-indexed series can be re-written as an iterated sum of convergent series ([D] Theorem 9.1.4, [Range] §1.5), the converse is false in general. For example, it is easy to construct a double sum $\sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} c_{ij}\right)$ such that both single-indexed summations are absolutely convergent, but the re-arranged sum $\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} c_{ij}\right)$ and the multi-indexed sum $\sum_{(i,j)\in\mathbb{W}^2} c_{ij}$ both diverge. However, a converse does hold for series with all non-negative terms: if such an iterated series is convergent, then the terms form a convergent multi-indexed series ([Range] §1.5). The conclusion of the Proposition follows from comparison to the sum of the absolute values, and from the invariance of the sum under rearrangement of absolutely convergent series. ∎

The last result stated in this Section is a version of the well-known Cauchy estimates, for complex analytic functions on a ball in $\mathbb{C}^n$.

Lemma 3.8. *Given $\vec{a} \in \mathbb{C}^n$ and a ball $D = \{\vec{z} \in \mathbb{C}^n : |\vec{z} - \vec{a}| < r\}$, if $f : D \to \mathbb{C}$ is a complex analytic function and $0 < R < r$, then there exists a number M so that for each $\boldsymbol{\beta} \in \mathbb{W}^n$,*

$$\left| \left[\left(\frac{d}{dz}\right)^{\boldsymbol{\beta}} (f(\vec{z}))\right]_{\vec{z}=\vec{a}} \right| \leq \frac{\boldsymbol{\beta}! b^{b/2} M}{R^b \boldsymbol{\beta}^{\boldsymbol{\beta}/2}},$$

where $b = |\boldsymbol{\beta}|$.

Proof. The Cauchy estimates are usually stated for functions on a polydisc in $\mathbb{C}^n$ ([Range] Theorem 1.6), but can be adapted to some domains with other shapes ([Fuks] Theorem 3.8), including the ball D. The following argument is a special case of the proof given by [Fuks].

Let M be the maximum value of $|f(\vec{z})|$ on the closed ball

$$D_R = \{\vec{z} \in \mathbb{C}^n : |\vec{z} - \vec{a}| \leq R\} \subseteq D,$$

so the estimate holds for $b = 0$. Given a vector with positive components, $\vec{r} = (r_1, \ldots, r_n)$ such that $r_1^2 + \ldots + r_n^2 = R^2$, the polydisc $\{\vec{z} \in \mathbb{C}^n : |z_j - a_j| <$

$r_j, j = 1, \ldots, n\}$ is contained in D_R, and

$$\left| \left(\frac{d}{dz} \right)^{\boldsymbol{\beta}} (f(\vec{z})) \right]_{\vec{z}=\vec{a}} \right| \leq \frac{\boldsymbol{\beta}! M}{(\vec{r})^{\boldsymbol{\beta}}}.$$

To get the best estimate for each $\boldsymbol{\beta}$, we need to maximize the quantity $(\vec{r})^{\boldsymbol{\beta}} = r_1^{\beta_1} \cdots r_n^{\beta_n}$, subject to the constraint $r_1^2 + \ldots + r_n^2 = R^2$. For $b > 0$, the usual Lagrange multiplier technique works here, to show that the quantity is maximized when, for each $j = 1, \ldots, n$, $r_j = R \cdot \sqrt{\beta_j / b}$, and the claim of the Lemma follows. ∎

4 The proof of the main result

The following Theorem is an identity for power series in several variables.

Theorem 4.1. *If* $\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} (\vec{x})^{\boldsymbol{\alpha}}$ *converges for* $\vec{x}$ *in a neighborhood of* $\vec{0}$*, with sum equal to a harmonic function* $u(\vec{x})$*, then the following sum also converges to* u *in the same neighborhood:*

$$\begin{aligned} u(\vec{x}) \quad = & \sum_{(\boldsymbol{\alpha}', b) \in \mathbb{W}^n} \frac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b)!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (x')^{\boldsymbol{\alpha}'} x_n^{2b} \\ & + \sum_{(\boldsymbol{\alpha}', b) \in \mathbb{W}^n} \frac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b+1)!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} \right) (x')^{\boldsymbol{\alpha}'} x_n^{2b+1}. \end{aligned}$$

Proof. For $m \in \mathbb{W}$, $k = 1, \ldots, n$, denote $m_k = (0, 0, \ldots, 0, m, 0, \ldots, 0) \in \mathbb{W}^n$, with the m in the k^{th} place, and similarly, for $k = 1, \ldots, n-1$, denote $m'_k = (0, 0, \ldots, 0, m, 0, \ldots, 0) \in \mathbb{W}^{n-1}$.

$$u(\vec{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} (\vec{x})^{\boldsymbol{\alpha}}.$$

$$\begin{aligned} \Delta u(\vec{x}) &= \sum_{k=1}^{n} \sum_{\boldsymbol{\alpha} : \alpha_k \geq 2} c_{\boldsymbol{\alpha}} \alpha_k (\alpha_k - 1) (\vec{x})^{\boldsymbol{\alpha} - 2_k} \\ &= \sum_{\boldsymbol{\alpha} : \alpha_n \geq 2} \left(\left(\sum_{k=1}^{n-1} c_{(\boldsymbol{\alpha}' + 2'_k, \alpha_n - 2)} (\alpha_k + 2)(\alpha_k + 1) \right) + c_{\boldsymbol{\alpha}} \alpha_n (\alpha_n - 1) \right) (\vec{x})^{\boldsymbol{\alpha} - 2_n}. \end{aligned}$$

From $\Delta u = 0$, we get a recursive formula:

$$c_{\boldsymbol{\alpha}} = \frac{-1}{\alpha_n(\alpha_n - 1)} \sum_{k=1}^{n-1} (\alpha_k + 2)(\alpha_k + 1) c_{(\boldsymbol{\alpha}' + 2'_k, \alpha_n - 2)} \tag{4.1}$$

for $\alpha_n \geq 2$. A closed form expression for the coefficients holds for all $\boldsymbol{\alpha} \in \mathbb{W}^n$, as we will show by induction on $b \in \mathbb{W}$:

$$c_{\boldsymbol{\alpha}} = \begin{cases} \dfrac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b)!} \displaystyle\sum_{|\boldsymbol{\beta}'|=b} \dfrac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} & \text{if } \alpha_n = 2b \\ \dfrac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b+1)!} \displaystyle\sum_{|\boldsymbol{\beta}'|=b} \dfrac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} & \text{if } \alpha_n = 2b+1 \end{cases}.$$

The sums in these quantities are over all multi-indices $\boldsymbol{\beta}' \in \mathbb{W}^{n-1}$ with degree $\beta_1 + \ldots + \beta_{n-1} = b$. We'll write out the proof only for the case where α_n is even, the other case being similar. To start the induction, the $b = 0$ case is trivial:

$$\frac{1}{\boldsymbol{\alpha}'!} \sum_{|\boldsymbol{\beta}'|=0} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} = c_{(\boldsymbol{\alpha}',0)},$$

and the $b = 1$ case is a special case of Equation (4.1):

$$\begin{aligned} \frac{(-1)^1 1!}{\boldsymbol{\alpha}'! \cdot 2!} \sum_{|\boldsymbol{\beta}'|=1} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} &= \frac{-1}{2\boldsymbol{\alpha}'!} \sum_{k=1}^{n-1} \frac{(\boldsymbol{\alpha}' + 2'_k)!}{1} c_{(\boldsymbol{\alpha}'+2'_k,0)} \\ &= \frac{-1}{2} \sum_{k=1}^{n-1} (\alpha_k + 2)(\alpha_k + 1) c_{(\boldsymbol{\alpha}'+2'_k,0)} \\ &= c_{(\boldsymbol{\alpha}',2)}. \end{aligned}$$

The inductive step is to fix b, and assume that the formula (the even case, with index $2b$) holds for all multi-indices $\boldsymbol{\delta} = (\boldsymbol{\delta}', 2b)$. We want to show that for all multi-indices $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', 2(b+1))$:

$$c_{(\boldsymbol{\alpha}',2b+2)} = \frac{(-1)^{b+1}(b+1)!}{\boldsymbol{\alpha}'!(2b+2)!} \sum_{|\boldsymbol{\beta}'|=b+1} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)}.$$

The following steps begin with (4.1), and then use the inductive hypothesis.

$$
\begin{aligned}
& c_{(\boldsymbol{\alpha}',2b+2)} \\
= \quad & \frac{-1}{(2b+2)(2b+1)} \sum_{k=1}^{n-1} (\alpha_k+2)(\alpha_k+1) c_{(\boldsymbol{\alpha}'+2'_k,2b)} \\
= \quad & \frac{-\sum_{k=1}^{n-1} \left(\frac{(\alpha_k+2)(\alpha_k+1)(-1)^b b!}{(\boldsymbol{\alpha}'+2'_k)!(2b)!} \sum_{|\boldsymbol{\gamma}'|=b} \frac{(\boldsymbol{\alpha}'+2'_k+2\boldsymbol{\gamma}')!}{\boldsymbol{\gamma}'!} c_{(\boldsymbol{\alpha}'+2'_k+2\boldsymbol{\gamma}',0)} \right)}{(2b+2)(2b+1)} \\
= \quad & \frac{(-1)^{b+1} b!}{\boldsymbol{\alpha}'!(2b+2)!} \sum_{k=1}^{n-1} \sum_{|\boldsymbol{\gamma}'|=b} \frac{(\boldsymbol{\alpha}'+2(\boldsymbol{\gamma}'+1'_k))!}{\boldsymbol{\gamma}'!} c_{(\boldsymbol{\alpha}'+2(\boldsymbol{\gamma}'+1'_k),0)} \\
= \quad & \frac{(-1)^{b+1} b!}{\boldsymbol{\alpha}'!(2b+2)!} \sum_{|\boldsymbol{\beta}'|=b+1} \left(\sum_{\boldsymbol{\gamma}':\boldsymbol{\gamma}'+1'_k=\boldsymbol{\beta}'} \frac{1}{\boldsymbol{\gamma}'!} \right) (\boldsymbol{\alpha}'+2\boldsymbol{\beta}')! c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)}.
\end{aligned}
$$

For each $\boldsymbol{\beta}'$, the sum simplifies, using the common denominator:

$$
\sum_{\boldsymbol{\gamma}':\boldsymbol{\gamma}'+1'_k=\boldsymbol{\beta}'} \frac{1}{\boldsymbol{\gamma}'!} = \sum_{\boldsymbol{\gamma}':\boldsymbol{\gamma}'+1'_k=\boldsymbol{\beta}'} \frac{\gamma_k+1}{\boldsymbol{\beta}'!} = \sum_{k=1}^{n-1} \frac{\beta_k}{\boldsymbol{\beta}'!} = \frac{b+1}{\boldsymbol{\beta}'!},
$$

which proves the claim.

By the elementary theory of multi-indexed series (as mentioned in the sketch of Proposition 3.7), any convergent power series can be written as a sum of two disjoint subseries, which must also be convergent,

$$
\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}} (\vec{x})^{\boldsymbol{\alpha}} = \sum_{(\boldsymbol{\alpha}',b)\in\mathbb{W}^n} c_{(\boldsymbol{\alpha}',2b)} (x')^{\boldsymbol{\alpha}'} x_n^{2b} + \sum_{(\boldsymbol{\alpha}',b)\in\mathbb{W}^n} c_{(\boldsymbol{\alpha}',2b+1)} (x')^{\boldsymbol{\alpha}'} x_n^{2b+1},
$$

and the claim of the Theorem follows from the above formula for the coefficients. ∎

The following Corollary gives a series expression for $u(\vec{x})$ in terms of the restriction of u and u_{x_n} to $\mathbb{R}^{n-1} = \{\vec{x} : x_n = 0\}$ and the Laplacian on that hyperplane,

$$
\Delta' = \left(\frac{d}{dx_1}\right)^2 + \ldots + \left(\frac{d}{dx_{n-1}}\right)^2.
$$

Corollary 4.2. *If $u(\vec{x})$ is harmonic in a neighborhood U of the origin, then the following sum converges to u in some neighborhood of the origin, $U' \subseteq U$:*

$$\begin{aligned} u(\vec{x}) &= \sum_{b=0}^{\infty}\left(\frac{(-1)^b}{(2b)!}\left((\Delta')^b\,(u(x',0))\right)x_n^{2b}\right) \\ &\quad +\sum_{b=0}^{\infty}\left(\frac{(-1)^b}{(2b+1)!}\left((\Delta')^b\left(\left(\frac{du}{dx_n}\right)(x',0)\right)\right)x_n^{2b+1}\right). \end{aligned}$$

Proof. It was already mentioned that $u(\vec{x})$ is real analytic, so it is equal to the sum of its Taylor series, centered at $\vec{0}$, on some neighborhood, which we may assume is of the form $U' = B(\vec{0}, r) \subseteq U$. The formal derivatives of the Taylor series converge to the derivatives of u, on the same ball U'. The Taylor series of $u(x',0)$, in the variables x' with center $0'$, converges for $x' \in U' \cap \mathbb{R}^{n-1}$. On the same set, the following series for u_{x_n}, and the derivatives of u and u_{x_n} with respect to $x_1, \ldots, x_{n-1}$, are convergent:

$$\begin{aligned} u(x',0) &= \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}',0)}(x')^{\boldsymbol{\alpha}'}, \\ \left(\frac{d}{dx'}\right)^{2\boldsymbol{\beta}'}(u(x',0)) &= \sum_{\boldsymbol{\alpha}' \geq 2\boldsymbol{\beta}'} c_{(\boldsymbol{\alpha}',0)}\frac{\boldsymbol{\alpha}'!}{(\boldsymbol{\alpha}'-2\boldsymbol{\beta}')!}(x')^{\boldsymbol{\alpha}'-2\boldsymbol{\beta}'} \\ &= \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)}\frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(x')^{\boldsymbol{\alpha}'} \qquad (4.2) \\ u_{x_n}(x',0) &= \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}',1)}(x')^{\boldsymbol{\alpha}'}, \\ \left(\frac{d}{dx'}\right)^{2\boldsymbol{\beta}'}(u_{x_n}(x',0)) &= \sum_{\boldsymbol{\alpha}' \geq 2\boldsymbol{\beta}'} c_{(\boldsymbol{\alpha}',1)}\frac{\boldsymbol{\alpha}'!}{(\boldsymbol{\alpha}'-2\boldsymbol{\beta}')!}(x')^{\boldsymbol{\alpha}'-2\boldsymbol{\beta}'} \\ &= \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)}\frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(x')^{\boldsymbol{\alpha}'}. \qquad (4.3) \end{aligned}$$

The following formula for the multinomial coefficients is well-known (and can be established by an inductive argument similar to the previous proof's.)

$$(\Delta')^b = \left(\left(\frac{d}{dx_1}\right)^2 + \ldots + \left(\frac{d}{dx_{n-1}}\right)^2\right)^b = \sum_{|\boldsymbol{\beta}'|=b}\frac{b!}{\boldsymbol{\beta}'!}\left(\frac{d}{dx'}\right)^{2\boldsymbol{\beta}'}.$$

Applying this operator to the above series gives

$$
\begin{aligned}
(\Delta')^b(u(x',0)) &= \sum_{|\boldsymbol{\beta}'|=b} \frac{b!}{\boldsymbol{\beta}'!} \sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(x')^{\boldsymbol{\alpha}'} \\
&= \sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (x')^{\boldsymbol{\alpha}'}, \quad (4.4) \\
(\Delta')^b(u_{x_n}(x',0)) &= \sum_{|\boldsymbol{\beta}'|=b} \frac{b!}{\boldsymbol{\beta}'!} \sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(x')^{\boldsymbol{\alpha}'} \\
&= \sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} \right) (x')^{\boldsymbol{\alpha}'}.
\end{aligned}
$$

These convergent sums can be multiplied by $\frac{(-1)^b}{(2b)!}x_n^{2b}$ and $\frac{(-1)^b}{(2b+1)!}x_n^{2b+1}$, respectively, so that they look like the terms from the previous Theorem. It follows from the previously mentioned elementary theory of multi-indexed series that the power series appearing in Theorem 4.1 can be re-written as a convergent iterated series, so for $\vec{x} \in U'$, the following sum converges to $u(\vec{x})$:

$$
\begin{aligned}
&\sum_{b=0}^{\infty} \frac{(-1)^b}{(2b)!} \left(\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (x')^{\boldsymbol{\alpha}'} \right) x_n^{2b} \\
+ &\sum_{b=0}^{\infty} \frac{(-1)^b}{(2b+1)!} \left(\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} \right) (x')^{\boldsymbol{\alpha}'} \right) x_n^{2b+1}.
\end{aligned}
$$

The claim of the Corollary follows from the above series expression for the iterated Laplacian. ∎

Theorem 1.1 is a corollary of the following Main Theorem.

Theorem 4.3. *If there is an open neighborhood of* $\vec{0}$ *on which the series* $\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}}(\vec{x})^{\boldsymbol{\alpha}}$ *is convergent, with sum equal to a harmonic function, and for some* $r > 0$, *both of the series* $\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}',0)}(x')^{\boldsymbol{\alpha}'}$ *and* $\sum_{\boldsymbol{\alpha}'\in\mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}',1)}(x')^{\boldsymbol{\alpha}'}$ *are convergent for* $x' \in B'(0',r)$, *then the series* $\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}}(\vec{x})^{\boldsymbol{\alpha}}$ *is convergent for all* $\vec{x} = (x', x_n)$ *such that* $|x'| + |x_n| < r$.

Proof. Define

$$v(x') = \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} |c_{\boldsymbol{\alpha}',0}|(x')^{\boldsymbol{\alpha}'}.$$

By Proposition 3.5, the sum converges to a real analytic function on the real ball $B'(0', r)$. The derivatives are defined by convergent series on the same ball, with coefficients equal to the absolute value of those from Equation (4.2):

$$\left(\frac{d}{dx'}\right)^{2\boldsymbol{\beta}'} (v(x')) = \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} |c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)}| \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(x')^{\boldsymbol{\alpha}'}.$$

Given a vector $y' = (y_1, \ldots, y_{n-1}) \in B'(0', r)$ with all non-negative components, $v(x')$ has a series expansion centered at y', which by Proposition 3.6, converges on the ball $B'(y', r - |y'|)$. By Proposition 3.5, this extends to a complex analytic function $v(z')$ on the ball $\{z' \in \mathbb{C}^{n-1} : |z' - y'| < r - |y'|\}$ (this is where the convergent series hypothesis is used, and where "real analytic" would not be enough). Given $R \in (0, r - |y'|)$, Lemma 3.8 applies, so that there exists $M \geq 0$ (depending on v, y',and R) such that for a multi-index $\boldsymbol{\beta}' \in \mathbb{W}^{n-1}$ with $|\boldsymbol{\beta}'| = b$,

$$\left| \left[\left(\frac{d}{dz'}\right)^{2\boldsymbol{\beta}'} (v(z'))\right]_{z'=y'} \right| \leq \frac{(2\boldsymbol{\beta}')!(2b)^b M}{(2\boldsymbol{\beta}')^{\boldsymbol{\beta}'} R^{2b}} = \frac{(2\boldsymbol{\beta}')!b^b M}{(\boldsymbol{\beta}')^{\boldsymbol{\beta}'} R^{2b}}.$$

It follows that the iterated Laplacian at y' is bounded by the sum:

$$\left|(\Delta')^b v(y')\right| \leq \sum_{|\boldsymbol{\beta}'|=b} \frac{b!}{\boldsymbol{\beta}'!} \left| \left(\frac{d}{dz'}\right)^{2\boldsymbol{\beta}'} (v(y')) \right| \leq \sum_{|\boldsymbol{\beta}'|=b} \frac{b!(2\boldsymbol{\beta}')!b^b M}{\boldsymbol{\beta}'!(\boldsymbol{\beta}')^{\boldsymbol{\beta}'} R^{2b}}.$$

In terms of the series expansion at $0'$ of the derivatives of v, an expression resembling Equation (4.4) has the same bound:

$$\begin{aligned} &\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left| \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (y')^{\boldsymbol{\alpha}'} \right| \\ \leq \ &\sum_{|\boldsymbol{\beta}'|=b} \frac{b!}{\boldsymbol{\beta}'!} \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} |c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)}| \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\alpha}'!}(y')^{\boldsymbol{\alpha}'} \leq \sum_{|\boldsymbol{\beta}'|=b} \frac{b!(2\boldsymbol{\beta}')!b^b M}{\boldsymbol{\beta}'!(\boldsymbol{\beta}')^{\boldsymbol{\beta}'} R^{2b}}. \end{aligned}$$

Given $y_n \geq 0$,

$$\left| \frac{(-1)^b}{(2b)!} \left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left| \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (y')^{\boldsymbol{\alpha}'} \right| \right) y_n^{2b} \right|$$

$$\leq \frac{1}{(2b)!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{b!(2\boldsymbol{\beta}')!b^b M}{\boldsymbol{\beta}'!(\boldsymbol{\beta}')^{\boldsymbol{\beta}'} R^{2b}} \right) y_n^{2b} = M \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{b!(2\boldsymbol{\beta}')!b^b}{(2b)!\boldsymbol{\beta}'!(\boldsymbol{\beta}')^{\boldsymbol{\beta}'}} \right) \left(\frac{y_n}{R} \right)^{2b}.$$

Each of the terms in the last finite sum is bounded above by $(\sqrt{2})^{n-2}$, by Lemma 3.4, and there are $\binom{b+n-2}{n-2}$ terms, by Proposition 3.3. If $y_n < R$, the following quantity is bounded (as a function of b):

$$M(\sqrt{2})^{n-2} \binom{b+n-2}{n-2} \left(\frac{y_n}{R} \right)^{2b}, \tag{4.5}$$

which could be shown by using the ratio test on the series formed by these terms.

By Proposition 3.7, this boundedness is enough to show that for all (x', x_n) : $|x_j| < y_j$, $j = 1, \ldots, n$, the following series are convergent and equal:

$$\sum_{b=0}^{\infty} \frac{(-1)^b}{(2b)!} \left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \frac{b!}{\boldsymbol{\alpha}'!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (x')^{\boldsymbol{\alpha}'} \right) x_n^{2b}$$

$$= \sum_{(\boldsymbol{\alpha}',b) \in \mathbb{W}^n} \frac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b)!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',0)} \right) (x')^{\boldsymbol{\alpha}'} x_n^{2b}. \tag{4.6}$$

An entirely analogous argument shows that

$$\sum_{(\boldsymbol{\alpha}',b) \in \mathbb{W}^n} \frac{(-1)^b b!}{\boldsymbol{\alpha}'!(2b+1)!} \left(\sum_{|\boldsymbol{\beta}'|=b} \frac{(\boldsymbol{\alpha}' + 2\boldsymbol{\beta}')!}{\boldsymbol{\beta}'!} c_{(\boldsymbol{\alpha}'+2\boldsymbol{\beta}',1)} \right) (x')^{\boldsymbol{\alpha}'} x_n^{2b+1} \tag{4.7}$$

converges at the same points (x', x_n).

Since, for any (x', x_n) such that $|x'| + |x_n| < r$, there exist (y', y_n) and R such that $|x_j| < y_j$, $j = 1, \ldots, n$, and $0 \leq y_n < R < r - |y'|$, both quantities (4.6) and (4.7) converge for all (x', x_n) in the claimed set, and, by Theorem 4.1, their sum is the same formal sum as $\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} (\vec{x})^{\boldsymbol{\alpha}}$. ∎

The result on harmonic extension from a small ball to a larger one, Corollary 2.3, follows from the Main Theorem.

Corollary 4.4. *If $u(\vec{x})$ is harmonic in a connected open set U such that $\vec{0} \in U \subseteq B(\vec{0}, 1)$, and its series expansion satisfies the hypotheses of the previous Theorem with $r = \sqrt{2}+\epsilon$, $\epsilon > 0$, then there is a harmonic function on $B(\vec{0}, 1+\epsilon/\sqrt{2})$, which agrees with $u(\vec{x})$ on the set U.*

Proof. By the previous Theorem, the series $\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}}$ converges for all $\vec{x} = (x', x_n)$ such that $|x'| + |x_n| < r$, defining a real analytic function $\tilde{u}$ such that $\tilde{u} = u$ on a neighborhood $U' \subseteq U$ (from the Proof of Corollary 4.2), and therefore on all of U. The real analytic function $\Delta\tilde{u}$ vanishes on the set U, and therefore for all $\vec{x}$ such that $|x'| + |x_n| < r$.

To prove the claim about the ball, note that $B(\vec{0}, r/\sqrt{2})$ is contained in the domain of convergence, by the following elementary inequality:

$$|x'| + |x_n| \le \sqrt{2} \cdot \sqrt{|x'|^2 + x_n^2}.$$

∎

5 Generalizations

The Main Theorem generalizes, from harmonic functions in $\mathbb{R}^n$, to real analytic solutions of other partial differential equations.

Corollary 5.1. *Suppose $u(\vec{x})$ is a solution, on a ball $B(\vec{0}, \rho)$, of the Poisson equation $\Delta u = Q(\vec{x})$, where $Q(\vec{x})$ is a polynomial. If the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ converge for $|x'| < r$, then the Taylor series of $u(\vec{x})$ converges for all $\vec{x} = (x', x_n)$ such that $|x'| + |x_n| < r$.*

Proof. As shown by [Y], such a solution u must be the sum of a polynomial and a harmonic function, to which Theorem 1.1 applies. ∎

Another generalization is to modify the Laplacian operator:

Proposition 5.2. *Suppose $u(\vec{x})$ is a smooth solution, on a ball $B(\vec{0}, \rho)$, of the equation $(c^2\Delta' \pm (\frac{d}{dx_n})^2)u = 0$, where $c > 0$. If the Taylor series of $u(x', 0)$ and $u_{x_n}(x', 0)$ centered at $x' = 0'$ converge for $|x'| < r$, then the Taylor series of $u(\vec{x})$ converges for all $\vec{x} = (x', x_n)$ such that $|x'| + c|x_n| < r$.*

Sketch of Proof. The arguments of the previous Section require only straightforward modifications, introducing the constant c and the $+$ or $-$ sign to get a new recurrence relation in the Proof of Theorem 4.1. By the end of the Proof of Theorem 4.3, it would be enough to notice that the new version of quantity (4.5),

$$M(\sqrt{2})^{n-2} \binom{b+n-2}{n-2} \left(\frac{c \cdot y_n}{R}\right)^{2b},$$

is bounded for $c \cdot y_n < R$. ∎

Taylor series expansions of solutions of the $n = 2$ wave equation, $(c^2(\frac{d}{dx_1})^2 - (\frac{d}{dx_2})^2)u = 0$, were considered by [B] when $c = 1$, and more generally by [W$_1$], [W$_2$]. A formula resembling Corollary 4.2 is stated by [Chen] for series solutions of the $n \geq 2$ wave equation, without comment on the size of the domain of convergence. For $n \geq 2$, the shape of the domain from Proposition 5.2, $\{\vec{x} : |x'| + c|x_n| < r\}$, is similar (literally) to the "domain of determinacy" of the wave equation ([Rauch] §1.8). In fact, the following Example shows that the claim of Proposition 5.2 for the $n = 2$, $c = 1$ wave equation is sharp, in the sense that not only does the series for the given solution of the wave equation converge on $\{\vec{x} : |x'| + c|x_n| < r\}$, but that the function does not even extend continuously to any larger connected open set.

Example 5.3. The following function is a solution of the equation $u_{xx} - u_{tt} = 0$:

$$u(x,t) = \frac{1 - x^2 - t^2}{(1 - x^2 - t^2)^2 - 4x^2t^2}.$$

Note that $u(x,0) = \frac{1}{1-x^2}$ has a convergent Taylor series for $-1 < x < 1$, and $u_t(x,0) = 0$, so Proposition 5.2 says that the two-variable Taylor series of u converges on the open square $\{|x| + |t| < 1\}$. The denominator of u is zero on the union of the four lines $x + t = \pm 1$, $x - t = \pm 1$, which contains the boundary of the square.

This behavior is unlike the harmonic functions from Section 2, where the domain of the harmonic functions extended beyond the domain of convergence of the Taylor series.

6 A less elementary approach

For purposes of comparison, we very briefly sketch an approach to the Main Theorem that uses methods of partial differential equations, which apply in far greater generality than is required for our problem.

The Main Theorem claims that the power series for a certain function u converges on a set of the form $\{|x'| + c|x_n| < r\} \subseteq \mathbb{R}^n$. If there were some extension of u to a complex analytic function on the domain $\{|z'| + c|z_n| < r\} \subseteq \mathbb{C}^n$, the convergence of the series would follow from Theorem 3.4 of [GF] §1.3.

In fact, since $u(x', 0)$ and $u_{x_n}(x', 0)$ are given by convergent series on the real ball $B'(0, r)$, they extend as complex analytic functions on the complex ball $S = \{(z', 0) : |z'| < r\} \subseteq \mathbb{C}^n$, by Proposition 3.5. By the Cauchy-Kovalevsky Theorem ([Hörmander],[Rauch]), there exists a unique complex analytic solution of the Laplace equation (or any of the generalizations from the previous Section) on a neighborhood of S in $\mathbb{C}^n$, which agrees with u and its first derivatives on S. The extension of u, from this neighborhood (or possibly an even smaller one with a suitable boundary) to the desired set in $\mathbb{C}^n$, is then a matter of analytic continuation, using a continuity method as presented by [Hörmander] §9.4.

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