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The high order Schwarz-Pick lemma on complex Hilbert balls

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Abstract In this paper we prove a high order Schwarz-Pick lemma for holomorphic mappings between unit balls in complex Hilbert spaces. In addition, a Schwarz-Pick estimate for high order Fréchet derivatives of a holomorphic function f of a Hilbert ball into the right half-plane is obtained.

Keywords complex Hilbert ball, high order Schwarz-Pick lemma

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1 Introduction

For a complex Hilbert space X, $\mathbb{B} = \{z \in X : \|z\| < 1\}$ is called the unit ball in X. The unit disk in the complex plane $\mathbb{C}$ is denoted by $\mathbb{D}$, and the unit ball in the complex space $\mathbb{C}^n$ of dimension n is denoted by $\mathbb{B}_n$. For $z = (z_1, \dots, z_n)$ and $z' = (z'_1, \dots, z'_n) \in \mathbb{C}^n$, denote $\langle z, z'\rangle = z_1\overline{z}'_1 + \cdots + z_n\overline{z}'_n$ and $|z| = \langle z, z\rangle^{1/2}$.

Let X, Y be two complex Hilbert spaces, U be an open subset of X, and f be a continuous mapping of U into Y. We say that f is a holomorphic mapping if for any $z \in U$, there is a bounded linear operator $Df(z)$ of X into Y such that

$$\lim_{\beta\in X,\, \|\beta\|\to 0} \frac{\|f(z+\beta) - f(z) - Df(z)\cdot\beta\|}{\|\beta\|} = 0,$$

where $\|\cdot\|$ denotes the norm of the appropriate space, and $Df(z)\cdot\beta$ denotes the evaluation of $Df(z)$ on $\beta \in X$. $Df(z)$ is called the Fréchet derivative of f at z, and $Df(z)\cdot\beta$ is called the Fréchet derivative of f at z in the direction β. Furthermore, for a non-negative integer k, the k-th-order Fréchet derivative of a holomorphic mapping f at $z \in U$ and its evaluation on $(\beta_1, \dots, \beta_k)$, $\beta_j \in X$, $1 \leqslant j \leqslant k$, have been defined and denoted by $D^kf(z)$ and $D^kf(z)\cdot(\beta_1, \dots, \beta_k)$ respectively. The norm of $D^kf(z)$ is defined as

$$\|D^kf(z)\| = \sup\{\|D^kf(z)\cdot(\beta_1, \dots, \beta_k)\| \mid \|\beta_1\| = \cdots = \|\beta_k\| = 1\}.$$

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In addition, $D^kf(z)\cdot(\beta,\dots,\beta)$ is simply denoted by $D^kf(z)\cdot\beta^k$, where $\beta\in X$.

For a multi-index $v=(v_1,\dots,v_n)$ with non-negative integers $v_1,\dots,v_n$, denote $|v|=v_1+\cdots+v_n$ and $z^v=z_1^{v_1}\cdots z_n^{v_n}$ for $z=(z_1,\dots,z_n)\in\mathbb{C}^n$. Under these notations, if f is holomorphic, $X=\mathbb{C}^n, Y=\mathbb{C}^m$, we have

$$D^kf(z)\cdot\beta^k=\sum_{|v|=k}\frac{k!}{v!}\frac{\partial^k f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\beta^v,\quad z\in U,\quad \beta\in\mathbb{C}^n. \tag{1.1}$$

In particular, if $X=Y=\mathbb{C}$, then

$$D^kf(z)\cdot\beta^k=f^{(k)}(z)\beta^k,\quad z\in U\subset\mathbb{C},\quad \beta\in\mathbb{C}.$$

The Taylor formula for a holomorphic mapping f of $\mathbb{D}$ into a complex Hilbert space Y says that

$$f(z)=f(0)+\sum_{k=1}^{\infty}\frac{D^kf(0)\cdot z^k}{k!},\quad z\in\mathbb{D}.$$

In this paper, by $\mathbb{B}$ and $\tilde{\mathbb{B}}$ we denote the unit balls in complex Hilbert spaces X and Y respectively, by $\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$ we denote the class of all holomorphic mappings f from $\mathbb{B}$ into $\widetilde{\mathbb{B}}$, and by $\Phi_{\mathbb{B}}$, we denote the class of all holomorphic functions f defined on $\mathbb{B}$ such that $\mathrm{Re}\{f(z)\}>0$ for $z\in\mathbb{B}$.

For $f\in\Omega_{\mathbb{D},\mathbb{D}}$, the classical Schwarz-Pick lemma says that

$$\frac{|f'(z)|}{1-|f(z)|^2}\leqslant\frac{1}{1-|z|^2},\quad z\in\mathbb{D}.$$

This has been generalized to the derivatives of arbitrary order [4,5,7]. The best result, proved in [3], is

$$\frac{|f^{(k)}(z)|}{1-|f(z)|^2}\leqslant(1+|z|)^{k-1}\cdot\frac{k!}{(1-|z|^2)^k}\quad z\in\mathbb{D},\quad k\geqslant1. \tag{1.2}$$

Chen and Liu [2] generalized (1.2) by proving the following Schwarz-Pick estimate for partial derivatives of arbitrary order of a function $f\in\Omega_{\mathbb{B}_n,\mathbb{D}}$:

$$\left|\frac{\partial^{|v|}f(z)}{\partial {z_1}^{v_1}\cdots\partial {z_n}^{v_n}}\right|\leqslant n^{\frac{|v|}{2}}|v|!\binom{n+|v|-1}{n-1}^{n+2}\frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}(1+|z|)^{|v|-1} \tag{1.3}$$

holds for any $z\in\mathbb{B}_n$ and multi-index $v=(v_1,\dots,v_n)\neq0$.

Recently, the authors obtained a high order Schwarz-Pick lemma [3] for $f\in\Omega_{\mathbb{B}_n,\mathbb{B}_m}$, which is formulated by the Bergman metric. On the unit ball $\mathbb{B}_n$, the Bergman metric $H_z(\beta,\beta)$ may be defined by

$$H_z(\beta,\beta)=\frac{(1-|z|^2)|\beta|^2+|\langle\beta,z\rangle|^2}{(1-|z|^2)^2}\quad\text{for}\ \ z\in\mathbb{B}_n,\ \beta\in\mathbb{C}^n.$$

Commonly, there is a factor $(n+1)/2$ in the definition of the Bergman metric. In spite of ambiguity, we use the same notation for Bergman metrics in unit balls of different dimensions. With this notation, the main result in [3] is expressed as follows:

Theorem: *Let $f\in\Omega_{\mathbb{B}_n,\mathbb{B}_m}$. Then, for $k\geqslant1$, $z\in\mathbb{B}_n$ and $\beta\in\mathbb{C}^n\setminus\{0\}$, we have*

$$H_{f(z)}(D^kf(z)\cdot\beta^k,\ D^kf(z)\cdot\beta^k)\leqslant(k!)^2p(z,\beta)^{2(k-1)}H_z(\beta,\beta)^k, \tag{1.4}$$

where

$$p(z,\beta)=1+\frac{|\langle\beta,z\rangle|}{((1-|z|^2)|\beta|^2+|\langle\beta,z\rangle|^2)^{1/2}}\quad\textit{for}\ \ z\in\mathbb{B}^n,\ \beta\in\mathbb{C}^n\setminus\{0\}.$$

(1.4) coincides with (1.2) if $n=m=1$. As a consequence, the authors deduced from (1.4) a Schwarz-Pick estimate for partial derivatives of a function $f\in\Omega_{\mathbb{B}_n,\mathbb{D}}$:

$$\left|\frac{\partial^{|v|}f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right|\leqslant\sqrt{\frac{|v|^{|v|}}{v^v}}v!(1+|z|)^{|v|-1}\cdot\frac{1-|f(z)|^2}{(1-|z|^2)^{|v|}}. \tag{1.5}$$

Note that (1.5) is much better than (1.3).

The purpose of this paper is to generalize (1.4) to Fréchet derivatives of mappings in $\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$. For a Hilbert ball $\mathbb{B}$, we define

$$H_z(\beta,\beta)=\frac{(1-\|z\|^2)\|\beta\|^2+|\langle\beta,z\rangle|^2}{(1-\|z\|^2)^2}\quad\text{for}\quad z\in\mathbb{B},\ \beta\in X,$$

which may be called the Bergman metric on $\mathbb{B}$. Then, we prove that (1.4) is true also for mappings in $\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$. This is reformulated as follows.

Theorem 1.1. *If $f\in\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$, then*

$$H_{f(z)}(D^kf(z)\cdot\beta^k,D^kf(z)\cdot\beta^k)\leqslant(k!)^2p(z,\beta)^{2(k-1)}(H_z(\beta,\beta))^k \tag{1.6}$$

holds for $k\geqslant1$, $z\in\mathbb{B}$ and $\beta\in X\setminus\{0\}$, where

$$p(z,\beta)=1+\frac{|\langle\beta,z\rangle|}{((1-\|z\|^2)\|\beta\|^2+|\langle\beta,z\rangle|^2)^{1/2}}\quad\textit{for}\quad z\in\mathbb{B},\ \beta\in X\setminus\{0\}.$$

In addition, for $\varphi\in\Phi_{\mathbb{D}}$, Dai and Pan [4] have proved

$$|\varphi^{(k)}(z)|\leqslant\frac{2k!\mathrm{Re}\{\varphi(z)\}}{(1-|z|^2)^k}(1+|z|)^{k-1}. \tag{1.7}$$

Using this estimate and the same method as in the proof of Theorem 1.1, we obtain the following result, which generalizes (1.7) to the Fréchet derivatives of functions in $\Phi_{\mathbb{B}}$.

Theorem 1.2. *If $f\in\Phi_{\mathbb{B}}$, then*

$$|D^kf(z)\cdot\beta^k|\leqslant2k!Re\{f(z)\}p(z,\beta)^{k-1}H_z(\beta,\beta)^{k/2} \tag{1.8}$$

holds for $k\geqslant1$, $z\in\mathbb{B}$ and $\beta\in X\setminus\{0\}$, where $p(z,\beta)$ is defined as in Theorem 1.1.

From Theorems 1.1 and 1.2, we obtain the following estimates of $\|D^kf(z)\|$ for $\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$ and $\Phi_{\mathbb{B}}$.

Theorem 1.3. *If $f\in\Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$, then*

$$\|D^kf(z)\|\leqslant k^k\sqrt{1-\|f(z)\|^2}\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}$$

holds for $k\geqslant1$ and $z\in\mathbb{B}$.

Theorem 1.4. *If $f\in\Phi_{\mathbb{B}}$, then*

$$\|D^kf(z)\|\leqslant2k^k\mathrm{Re}\{f(z)\}\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}$$

holds for $k\geqslant1$ and $z\in\mathbb{B}$.

2 The Schwarz-Pick estimate for $\Omega_{\mathbb{D},\mathbb{B}}$

In this section, we prove Theorem 1.1 for the special case $f\in\Omega_{\mathbb{D},\mathbb{B}}$. On the basis of this, Theorem 1.1 will be proved in the next section.

For $a\in\mathbb{B}$, by φ_a we denote the holomorphic mapping of $\mathbb{B}$ onto itself, such that $\varphi_a(a)=0$ and $\varphi_a=\varphi_a^{-1}$. It is known [6] that

$$\varphi_a(z)=\frac{1}{1-\langle z,a\rangle}\left(a-\frac{\langle z,a\rangle a}{1+\sqrt{1-\|a\|^2}}-\sqrt{1-\|a\|^2}z\right).$$

If $a=0$, then $\varphi_a(z)=-z$; If $a\neq 0$, then $\varphi_a(z)$ is just

$$\varphi_a(z)=\frac{1}{1-\langle z,a\rangle}\left[a-\frac{\langle z,a\rangle}{\|a\|^2}a-\sqrt{1-\|a\|^2}\left(z-\frac{\langle z,a\rangle}{\|a\|^2}a\right)\right]. \tag{2.1}$$

Lemma 2.1. *If $f\in\Omega_{\mathbb{D},\mathbb{B}}$, then*

$$\sum_{k=0}^{\infty}\left\|\frac{D^kf(0)\cdot z^k}{k!}\right\|^2<1,\quad z\in\mathbb{D}.$$

Proof. Let $z\in\mathbb{D}$ be fixed. We have $f(ze^{i\theta})=\sum_{k=0}^{\infty}e^{ik\theta}\frac{D^kf(0)\cdot z^k}{k!}$ by the Taylor formula, where the first term under the summation is understood as $f(0)$. Then,

$$\begin{aligned}1&>\frac{1}{2\pi}\int_0^{2\pi}\left\|f(ze^{i\theta})\right\|^2d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left\langle f(ze^{i\theta}),f(ze^{i\theta})\right\rangle d\theta\\&=\frac{1}{2\pi}\sum_{k,j}\left\langle\frac{D^kf(0)\cdot z^k}{k!},\frac{D^jf(0)\cdot z^j}{j!}\right\rangle\int_0^{2\pi}e^{i(k-j)\theta}d\theta\\&=\sum_{k=0}^{\infty}\left\|\frac{1}{k!}D^kf(0)\cdot z^k\right\|^2.\end{aligned}$$

The lemma is proved. □

Lemma 2.2. *If $f\in\Omega_{\mathbb{D},\mathbb{B}}$, then*

$$\left|\left\langle\frac{D^kf(0)\cdot 1^k}{k!},f(0)\right\rangle\right|^2+(1-\|f(0)\|^2)\left\|\frac{D^kf(0)\cdot 1^k}{k!}\right\|^2\leqslant(1-\|f(0)\|^2)^2. \tag{2.2}$$

Proof. Denote $a_k=\frac{D^kf(0)\cdot 1^k}{k!}$, then $f(z)=\sum_{k=0}^{\infty}a_kz^k$. Note that $a_0=f(0)$.

If $a_0=0$, (2.2) is just Lemma 2.1. Now, assume that $a_0\neq 0$. Let a positive integer k be fixed, $\omega=e^{2\pi i/k}$ and $h(z)=\frac{1}{k}\sum_{l=1}^{k}f(\omega^lz)$. Then, $h(z)\in\Omega_{\mathbb{D},\mathbb{B}}$, $h(0)=a_0$, and

$$h(z)=a_0+\sum_{n=1}^{\infty}\frac{D^{nk}f(0)\cdot z^{nk}}{(nk)!}=a_0+\sum_{n=1}^{\infty}a_{nk}z^{nk}.$$

Let $\phi=\varphi_{a_0}\circ h$. Obviously, $\phi\in\Omega_{\mathbb{D},\mathbb{B}}$, and $\phi(0)=0$. By (2.1), we have

$$\begin{aligned}\phi(z)=&\frac{1}{1-\langle h(z),a_0\rangle}\left(-(a_0/\|a_0\|^2)\sum_{n=1}^{\infty}\langle a_{nk},a_0\rangle z^{nk}\right.\\&\left.-\sqrt{1-\|a_0\|^2}\sum_{n=1}^{\infty}a_{nk}z^{nk}+\sqrt{1-\|a_0\|^2}(a_0/\|a_0\|^2)\sum_{n=1}^{\infty}\langle a_{nk},a_0\rangle z^{nk}\right)\\=&-\frac{1}{1-\|a_0\|^2-\sum\limits_{n=1}^{\infty}\langle a_{nk},a_0\rangle z^{nk}}\left(\sum_{n=1}^{\infty}\frac{\langle a_{nk},a_0\rangle a_0z^{nk}}{1+\sqrt{1-\|a_0\|^2}}+\sqrt{1-\|a_0\|^2}\sum_{n=1}^{\infty}a_{nk}z^{nk}\right)\\=&-\frac{1}{1-\|a_0\|^2}\left(\frac{\langle a_k,a_0\rangle a_0}{1+\sqrt{1-\|a_0\|^2}}+\sqrt{1-\|a_0\|^2}a_k\right)z^k+\sum_{m=2}^{\infty}c_{mk}z^{mk}\\=&bz^k+\sum_{m=2}^{\infty}c_{mk}z^{mk}.\end{aligned}$$

Note that $b = \frac{D^k\phi(0)\cdot 1^k}{k!}$. Thus, using Lemma 2.1, we obtain

$$\begin{aligned}
1 \geqslant \|b\|^2 &= \frac{1}{(1-\|a_0\|^2)^2}\left\|\frac{\langle a_k, a_0\rangle a_0}{1+\sqrt{1-\|a_0\|^2}} + \sqrt{1-\|a_0\|^2}a_k\right\|^2 \\
&= \frac{1}{(1-\|a_0\|^2)^2}\left(\frac{|\langle a_k, a_0\rangle|^2\|a_0\|^2}{(1+\sqrt{1-\|a_0\|^2})^2} + (1-\|a_0\|^2)\|a_k\|^2 + \frac{2\sqrt{1-\|a_0\|^2}|\langle a_k, a_0\rangle|^2}{1+\sqrt{1-\|a_0\|^2}}\right) \\
&= \frac{1}{(1-\|a_0\|^2)^2}\left(|\langle a_k, a_0\rangle|^2 + (1-\|a_0\|^2)\|a_k\|^2\right).
\end{aligned}$$

This shows (2.2). □

Theorem 2.1. *If $f(z) \in \Omega_{\mathbb{D},\mathbb{B}}$, then*

$$|\langle D^k f(z)\cdot 1^k, f(z)\rangle|^2 + (1-\|f(z)\|^2)\|D^k f(z)\cdot 1^k\|^2 \leqslant \left[\frac{k!(1-\|f(z)\|^2)}{(1-|z|^2)^k}(1+|z|)^{k-1}\right]^2 \tag{2.3}$$

holds for $k \geqslant 1$ and $z \in \mathbb{D}$.

Proof. Let $\xi \in \mathbb{D}$ and a positive integer k be fixed. We consider $g = f \circ \varphi_\xi \in \Omega_{\mathbb{D},\mathbb{B}}$, where

$$\varphi_\xi(z) = \frac{\xi - z}{1-\bar{\xi}z}.$$

For g,

$$g(z) = \sum_{n=0}^{\infty}\frac{D^n g(0)\cdot z^n}{n!} = \sum_{n=0}^{\infty}\frac{D^n g(0)\cdot 1^n}{n!}z^n.$$

Denote $c_n = \frac{D^n g(0)\cdot 1^n}{n!}$, then $g(z) = \sum\limits_{n=0}^{\infty} c_n z^n$. By Lemma 2.2,

$$|\langle c_n, c_0\rangle|^2 + (1-\|c_0\|^2)\|c_n\|^2 \leqslant (1-\|c_0\|^2)^2 \tag{2.4}$$

holds for $n \geqslant 1$.

It is easy to verify that

$$\left.\frac{d^n(\varphi_\xi(z)^j)}{dz^n}\right|_{z=\xi} = \begin{cases} 0, & n < j; \\ \dfrac{(-1)^j(\bar{\xi})^{n-j}}{(1-|\xi|^2)^n}\dfrac{n!(n-1)!}{(n-j)!(j-1)!}, & n \geqslant j. \end{cases}$$

Let

$$A_j = \frac{(-1)^j\overline{\xi}^{k-j}}{(1-|\xi|^2)^k}\frac{k!(k-1)!}{(k-j)!(j-1)!}.$$

Since $f = g \circ \varphi_\xi$, we have

$$D^k f(\xi)\cdot 1^k = \sum_{j=1}^{k} c_j A_j,$$

and, using (2.4) and the Schwarz inequality,

$$\begin{aligned}
&|\langle D^k f(\xi)\cdot 1^k, f(\xi)\rangle|^2 + (1-\|f(\xi)\|^2)\|D^k f(\xi)\cdot 1^k\|^2 \\
&\quad = \left|\sum_{j=1}^{k} A_j\langle c_j, c_0\rangle\right|^2 + (1-\|c_0\|^2)\left\|\sum_{j=1}^{k} c_j A_j\right\|^2 \\
&\quad \leqslant \sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j||\langle c_j, c_0\rangle|^2 + (1-\|c_0\|^2)\sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j|\|c_j\|^2 \\
&\quad = \sum_{j=1}^{k}|A_j|\sum_{j=1}^{k}|A_j|\left(|\langle c_j, c_0\rangle|^2 + (1-\|c_0\|^2)\|c_j\|^2\right) \\
&\quad \leqslant (1-\|c_0\|^2)^2\left(\sum_{j=1}^{k}|A_j|\right)^2.
\end{aligned}$$

On the other hand,

$$\sum_{j=1}^{k}|A_j| = \frac{k!}{(1-|\xi|^2)^k}\sum_{j=1}^{k}\frac{(k-1)!|\xi|^{k-j}}{(k-j)!(j-1)!} = \frac{k!}{(1-|\xi|^2)^k}(1+|\xi|)^{k-1}.$$

(2.3) is proved. □

3 The proofs of the theorems

To prove Theorem 1.1, besides Theorem 2.1, the following lemmas are needed.

Lemma 3.1. *Let $\mathbb{B}$ be the unit ball in the Hilbert space X, for given $p, q \in \mathbb{B}$ with $q \neq p$, let $L(z) = p + z(q-p)$ for $z \in \mathbb{C}$. Then,*

$$L(\mathbb{D}_{c_{p,q},r_{p,q}}) \subset \mathbb{B}, \qquad L(\partial\mathbb{D}_{c_{p,q},r_{p,q}}) \subset \partial\mathbb{B},$$

where

$$D_{c_{p,q},r_{p,q}} = \{z \mid z \in \mathbb{C}, |z - c_{p,q}| < r_{p,q}\},$$

$$c_{p,q} = -\frac{\langle p, q-p\rangle}{\|q-p\|^2}, \quad r_{p,q} = \sqrt{\frac{1-\|p\|^2}{\|q-p\|^2} + \left|\frac{\langle p, q-p\rangle}{\|q-p\|^2}\right|^2}.$$

Proof. For $z \in \mathbb{C}$, if

$$\|L(z)\| = \|p + z(q-p)\|^2 < 1,$$

then

$$\|p\|^2 + 2\mathrm{Re}(\bar{z}\langle p, q-p\rangle) + |z|^2\|q-p\|^2 < 1,$$

and, consequently,

$$\left|z + \frac{\langle p, q-p\rangle}{\|q-p\|^2}\right|^2 < \frac{1-\|p\|^2}{\|q-p\|^2} + \left|\frac{\langle p, q-p\rangle}{\|q-p\|^2}\right|^2.$$

The converse is also true. This shows the lemma. □

Lemma 3.2 [1]**.** *Let G, E, F be Banach spaces, U be an open subset of G, A be a bounded linear operator of G to E, $f: A(U) \longrightarrow F$ be r times differentiable. Then,*

$$D^i(f \circ A)(v) \cdot (g_1, \dots, g_i) = D^i f(Av) \cdot (Ag_1, \dots, Ag_i)$$

exists for all $i \leqslant r$, where $v \in U$ and $g_1, \dots, g_i \in G$.

Lemma 3.3. *Let $g : \mathbb{D}_{z_0,\delta} \longrightarrow \mathbb{B}$ be a holomorphic mapping, where $\mathbb{D}_{z_0,\delta} = \{z \in \mathbb{C} : |z - z_0| < \delta\}$. Then*

$$|\langle D^k g(z)\cdot 1^k, g(z)\rangle|^2 + (1-\|g(z)\|^2)\|D^k g(z)\cdot 1^k\|^2 \leqslant \left[k!(1-\|g(z)\|^2)\frac{\delta(\delta+|z-z_0|)^{k-1}}{(\delta^2-|z-z_0|^2)^k}\right]^2.$$

Proof. Let $\varphi(z) = g(\delta z + z_0)$ for $z \in \mathbb{D}$. Using Theorem 2.1, we have

$$|\langle D^k\varphi(z)\cdot 1^k, \varphi(z)\rangle|^2 + (1-\|\varphi(z)\|^2)\|D^k\varphi(z)\cdot 1^k\|^2 \leqslant \left[\frac{k!(1-\|\varphi(z)\|^2)}{(1-|z|^2)^k}(1+|z|)^{k-1}\right]^2. \tag{3.1}$$

By Lemma 3.2, for $\xi \in \mathbb{D}_{z_0,\delta}$,

$$\delta^k D^k g(\xi)\cdot 1^k = D^k g(\xi)\cdot \delta^k = D^k\varphi\left(\frac{\xi - z_0}{\delta}\right)\cdot 1^k.$$

Letting $z = \frac{\xi - z_0}{\delta}$ in (3.1), we obtain

$$|\langle D^k g(\xi)\cdot 1^k, g(\xi)\rangle|^2 + (1-\|g(\xi)\|^2)\|D^k g(\xi)\cdot 1^k\|^2 \leqslant \left[k!(1-\|g(\xi)\|^2)\frac{\delta(\delta+|\xi-z_0|)^{k-1}}{(\delta^2-|\xi-z_0|^2)^k}\right]^2.$$

The lemma is proved. □

Proof of Theorem 1.1. For a given $p \in \mathbb{B}$, let $q \in \mathbb{B}$ with $q \neq p$. Let $g(z) = f(L(z))$ for $z \in \mathbb{D}_{c_{p,q}, r_{p,q}}$, where $\mathbb{D}_{c_{p,q}, r_{p,q}}$ is defined in Lemma 3.1 and $L(z) = p + z(q - p)$. Using Lemma 3.3 to $z = 0$, we have

$$|\langle D^k g(0) \cdot 1^k, g(0)\rangle|^2 + (1 - \|g(0)\|^2)\|D^k g(0) \cdot 1^k\|^2$$
$$\leqslant \left[k!(1 - \|g(0)\|^2)\frac{r_{p,q}(r_{p,q} + |c_{p,q}|)^{k-1}}{(r_{p,q}^2 - |c_{p,q}|^2)^k}\right]^2.$$

Note $g(0) = f(p)$ and, by Lemma 3.2,

$$D^k g(0) \cdot 1^k = D^k f(p) \cdot (q - p)^k,$$
$$\frac{r_{p,q}(r_{p,q} + |C_{p,q}|)^{k-1}}{(r_{p,q}^2 - |C_{p,q}|^2)^k} = \frac{\sqrt{A_{q-p}}\left(\sqrt{A_{q-p}} + |\langle p, q - p\rangle|\right)^{k-1}}{(1 - \|p\|^2)^k},$$

where

$$A_{q-p} = (1 - \|p\|^2)\|q - p\|^2 + |\langle p, q - p\rangle|^2.$$

So, we have

$$|\langle D^k f(p) \cdot \beta^k, f(p)\rangle|^2 + (1 - \|f(p)\|^2)\|D^k f(p) \cdot \beta^k\|^2$$
$$\leqslant \left[k!(1 - \|f(p)\|^2)\frac{\sqrt{A}(\sqrt{A} + |\langle p, \beta\rangle|)^{k-1}}{(1 - \|p\|^2)^k}\right]^2,$$

where $\beta = (q - p)/\|q - p\|$, and $A = 1 - \|p\|^2 + |\langle p, \beta\rangle|^2$. A simple calculation gives

$$\frac{\sqrt{A}(\sqrt{A} + |\langle p, \beta\rangle|)^{k-1}}{(1 - \|p\|^2)^k} = \left(1 + \frac{|\langle p, \beta\rangle|}{\sqrt{(1 - \|p\|^2)\|\beta\|^2 + |\langle p, \beta\rangle|^2}}\right)^{k-1}\left(\frac{\sqrt{(1 - \|p\|^2)\|\beta\|^2 + |\langle p, \beta\rangle|^2}}{1 - \|p\|^2}\right)^k$$
$$= \left(1 + \frac{|\langle p, \beta\rangle|}{[(1 - \|p\|^2)\|\beta\|^2 + |\langle p, \beta\rangle|^2]^{1/2}}\right)^{k-1}(H_p(\beta, \beta))^{k/2}.$$

Since β may be an arbitrary unit vector, this shows (1.6) for any $\beta \in \mathbb{B}$ and $\beta \in \partial\mathbb{B}$. (1.6) is homogeneous with respect to β, so (1.6) holds for $\beta \in X \setminus \{0\}$ also. Theorem 1.1 is proved.

The proof of Theorem 1.2 is almost the same as that of Theorem 1.1. The only change is that the use of Theorem 2.1 is replaced by that of (1.7).

To prove Theorem 1.3, we need the following lemma.

Lemma 3.4 [1]. *Let E, F be Banach spaces, A be a continuous symmetric k-multilinear map from $E \times \cdots \times E$ to F. The norm of A is defined as*

$$\|A\| = \sup\{\|A(e_1, \ldots, e_k)\| \mid \|e_1\| = \cdots = \|e_k\| = 1\}.$$

Then

$$\|A\| \leqslant (k^k/k!) \sup\{\|A(e, \ldots, e)\| \mid \|e\| = 1\}.$$

Proof of Theorem 1.3. It is easy to see that

$$H_{f(z)}(D^k f(z) \cdot \beta^k, D^k f(z) \cdot \beta^k) \geqslant \frac{\|D^k f(z) \cdot \beta^k\|^2}{1 - \|f(z)\|^2}.$$

So by Theorem 1.1,

$$\frac{\|D^k f(z) \cdot \beta^k\|^2}{1 - \|f(z)\|^2} \leqslant (k!)^2 p(z, \beta)^{2(k-1)}(H_z(\beta, \beta))^k \tag{3.2}$$

holds for $k \geqslant 1$, $z \in \mathbb{B}$ and $\beta \in X \setminus \{0\}$, where $p(z, \beta)$ is defined as in Theorem 1.1. Note that

$$H_z(\beta, \beta) \leqslant \frac{\|\beta\|^2}{(1 - \|z\|^2)^2}, \qquad \frac{|\langle z, \beta\rangle|}{[(1 - \|z\|^2)\|\beta\|^2 + |\langle z, \beta\rangle|^2]^{1/2}} \leqslant \|z\|. \tag{3.3}$$

Then by (3.2) we have

$$\|D^k f(z)\cdot\beta^k\| \leqslant k!\sqrt{1-\|f(z)\|^2}\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}\|\beta\|^k.$$

Thus

$$\sup\{\|D^k f(z)\cdot\beta^k\| \mid \|\beta\|=1\} \leqslant k!\sqrt{1-\|f(z)\|^2}\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}.$$

By Lemma 3.4, Theorem 1.3 is proved.

The proof of Theorem 1.4 is almost the same as that of Theorem 1.3. The only change is that the use of (3.2) is replaced by that of (1.8).

4 Corollaries

Note that (3.3). Thus, from Theorems 1.1 and 1.2, we have the following corollaries.

Corollary 4.1. *If $f\in\Omega_{\mathbb{B},\mathbb{D}}$, then*

$$|D^k f(z)\cdot\beta^k| \leqslant k!(1-|f(z)|^2)\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}\|\beta\|^k$$

holds for $k\geqslant 1$, $z\in\mathbb{B}$ and $\beta\in X$.

Corollary 4.2. *If $f\in\Omega_{\mathbb{B},\mathbb{B}_m}$, then*

$$|\langle D^k f(z)\cdot\beta^k, f(z)\rangle|^2+(1-|f(z)|^2)|D^k f(z)\cdot\beta^k|^2 \leqslant \left[k!(1-|f(z)|^2)\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}\|\beta\|^k\right]^2$$

holds for $k\geqslant 1$, $z\in\mathbb{B}$ and $\beta\in X$.

Corollary 4.3. *If $f\in\Phi_{\mathbb{B}}$, then*

$$|D^k f(z)\cdot\beta^k| \leqslant 2k!\mathrm{Re}\{f(z)\}\frac{(1+\|z\|)^{k-1}}{(1-\|z\|^2)^k}\|\beta\|^k$$

holds for $k\geqslant 1$, $z\in\mathbb{B}$ and $\beta\in X$.

Corollary 4.4. *Let $f\in\Phi_{\mathbb{B}_n}$, then*

$$\left|\frac{\partial^{|v|}f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\right| \leqslant v!\sqrt{\frac{|v|^{|v|}}{v^v}}\frac{2\mathrm{Re}\{f(z)\}}{(1-|z|^2)^{|v|}}(1+|z|)^{|v|-1} \tag{4.1}$$

holds for any multi-index $v\neq 0$.

Proof. Let $f\in\Phi_{\mathbb{B}_n}$, $z\in\mathbb{B}_n$ and a multi-index $v\neq 0$ be given. Denote $k=|v|$. By Corollary 4.3 and (1.1),

$$\left|\sum_{|v|=k}\frac{k!}{v!}\frac{\partial^k f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\beta^v\right| \leqslant 2k!\mathrm{Re}\{f(z)\}\frac{(1+|z|)^{k-1}}{(1-|z|^2)^k}|\beta|^k$$

holds for $\beta\in\mathbb{C}^n$. In particular,

$$\left|\sum_{|v|=k}\frac{k!}{v!}\frac{\partial^k f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\beta^v\right| < 2k!\mathrm{Re}\{f(z)\}\frac{(1+|z|)^{k-1}}{(1-|z|^2)^k}$$

holds for $\beta\in\mathbb{B}_n$. Let

$$A=2k!\mathrm{Re}\{f(z)\}\frac{(1+|z|)^{k-1}}{(1-|z|^2)^k},\quad g(\beta)=\frac{1}{A}\sum_{|v|=k}\frac{k!}{v!}\frac{\partial^k f(z)}{\partial z_1^{v_1}\cdots\partial z_n^{v_n}}\beta^v.$$

Since $g(\beta) \in \Omega_{\mathbb{B}_n,\mathbb{D}}$, using (1.5), we have

$$\left|\frac{\partial^{|v|} g(0)}{\partial \beta_1^{v_1} \cdots \partial \beta_n^{v_n}}\right| \leqslant v! \sqrt{\frac{|v|^{|v|}}{v^v}} (1 - |g(0)|^2).$$

Note that $g(0) = 0$, and

$$\frac{\partial^{|v|} g(0)}{\partial \beta_1^{v_1} \cdots \partial \beta_n^{v_n}} = \frac{k!}{A} \cdot \frac{\partial^k f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}}.$$

(4.1) follows and the corollary is proved. □

From Theorems 1.3 and 1.4, we have the following corollaries.

Corollary 4.5. *If* $f \in \Omega_{\mathbb{B},\widetilde{\mathbb{B}}}$, *then* $\|D^k f(0)\| \leqslant k^k \sqrt{1 - \|f(0)\|^2} \leqslant k^k$ *holds for* $k \geqslant 1$.

Corollary 4.6. *If* $f \in \Phi_{\mathbb{B}}$, *then* $\|D^k f(0)\| \leqslant 2k^k \mathrm{Re}\{f(z)\}$ *holds for* $k \geqslant 1$.

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