BEREZIN'S OPERATOR CALCULUS AND HIGHER ORDER SCHWARZ-PICK LEMMA

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Abstract. We provide a new and simple proof to the result in our study of Berezin's operator calculus [17] that the mth order Bergman metric $(B_m\partial_v)(z)$ is a constant multiple of $\{(B_1\partial_v)(z)\}^m$ on the unit ball, where $(B_1\partial_v)(z)$ is the classical Bergman metric. Based on the reproducing-kernel theory, an approximation approach is developed to treat $(B_m\partial_v)(z)$ on the unit ball and $\mathbb{C}^n$ in a uniform way. Secondly, we discuss the interplay between our analysis in Berezin's operator calculus and the higher order Schwarz-Pick lemma in [10]. As a consequence, the mth order Carathéodory-Reiffen metric $(C_m\partial_v)(z)$ is shown to be a constant multiple of $\{(C_1\partial_v)(z)\}^m$ also on the unit ball, where $(C_1\partial_v)(z)$ is the classical infinitesimal Carathéodory-Reiffen metric.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{H}$ a Hilbert space of holomorphic functions on Ω. $\mathcal{H}$ is called a holomorphic Hilbert space if the evaluation functional at each point $z \in \Omega$ is continuous on $\mathcal{H}$ ([1]). In this case, there is a unique function $K_z \in H$ such that $f(z) = \langle f, K_z\rangle$ for every $f \in \mathcal{H}$. The function $K(w,z) = K_z(w)$ on $\Omega \times \Omega$ is called the reproducing kernel of $\mathcal{H}$ and holomorphic in the first variable and conjugate holomorphic in the second one. We will assume throughout the paper that $\|K_z\|^2 = K(z,z) > 0$ for all $z \in \Omega$, and the set $\mathcal{P}$ of holomorphic polynomials is included in $\mathcal{H}$.

For a fixed point $a \in \Omega$, let $V = \sum_{j=1}^n v_j(z)\frac{\partial}{\partial z_j} = \sum_{j=1}^n v_j(z)\partial_j$ be a nonzero holomorphic vector field on Ω with $v_j's$ holomorphic, and for $m \geq 1$ define

$$S_m(V,a) = \{f \in \mathcal{H} : \|f\| \leq 1,\ (V^j f)(a) = 0,\ j = 0, 1, \cdots, m-1\} \tag{1}$$

where $V^0 = I$ the identity operator. We denote by $S_0(V,a) = \{f \in \mathcal{H} : \|f\| \leq 1\}$ the closed unit ball of $\mathcal{H}$. It is clear that $S_m(V,a)$ is nonempty since it contains functions of the form $c(z-a)^\alpha = c\prod_{j=1}^n (z_j - a_j)^{\alpha_j}$ with $|\alpha| = \sum_{j=1}^n \alpha_j \geq m$, where $c = \|(z-a)^\alpha\|^{-1}$. Let

$$(R_m V)(a) = \sup_{f \in S_m(V,a)} |(V^m f)(a)|^2, \tag{2}$$

and write

$$\{(B_m V)(a)\}^2 = K(a,a)^{-1}(R_m V)(a). \tag{3}$$

It is standard that $(R_0 V)(a) = K(a,a)$ and $(B_0 V)(a) = 1$ ([14]).

Note that for $m \geq 1$, the iteration V^m of a holomorphic vector field $V = \sum_{j=1}^n v_j(z)\partial_j$ can be considered as a holomorphic differential operator of the form $V^m = \sum_{|\alpha|=0}^m f_\alpha(z)\partial^\alpha$ for some holomorphic functions f_α's on Ω. For fixed $a \in \Omega$ and each $m \geq 1$, the linear functional $V_a^m : f \to (V^m f)(a)$ defined on $\mathcal{H}$ is bounded by Cauchy Estimates and the fact that the norm of $f \in \mathcal{H}$ dominates its supremum norm over any compact set by the reproducing property $f(z) = \langle f, K_z\rangle$. So there exists a unique element $K_{V^m,a} \in \mathcal{H}$ such that

$$(V^m f)(a) = \langle f, K_{V^m,a}\rangle \tag{4}$$

and

$$K_{V^m,a}(z) = \langle K_{V^m,a}, K_z\rangle = \overline{\langle K_z, K_{V^m,a}\rangle} = \overline{(V^m K_z)(a)}.$$

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For fixed vector field V, we write $e_{a,j}(z) = K_{V^j,a}(z)$ for $j = 0, 1, \cdots, m$. In turn, (1) and (2) can be rewritten as

$$S_m(V, a) = \{f \in S_0(V, a) : \langle f, e_{a,j}\rangle = 0, j = 0, 1, \cdots, m-1\}$$

and

$$(R_m V)(a) = \sup_{f\in S_m(V,a)} |\langle f, e_{a,m}\rangle|^2. \tag{5}$$

Let $\mathcal{H}_{a,m}$ be the closed subspace spanned by $\{e_{a,j}\}_{j=0}^{m-1}$ and $P_{\mathcal{H}_{a,m}^\perp}$ be the orthogonal projection onto $\mathcal{H}_{a,m}^\perp$, the orthogonal complement space of $\mathcal{H}_{a,m}$ in $\mathcal{H}$. Then $(R_m V)(a)$ is exactly the square of the operator norm of the linear functional V_a^m restricted to $\mathscr{H}_{a,m}^\perp$ by (2), and

$$(R_m V)(a) = \|P_{\mathcal{H}_{a,m}^\perp} e_{a,m}\|^2 \tag{6}$$

by (5) in the point of view of Hilbert space geometry. We remark that it is convenient to view B_m in (3) as a mapping that associates a holomorphic vector field with a function on Ω.

Now suppose that $\mathcal{H}_1$ on Ω_1 and $\mathcal{H}_2$ on Ω_2 are two holomorphic Hilbert spaces normed by $\|f\|_{\mathcal{H}_1}^2 = \int_{\Omega_1} |f(z)|^2 d\mu_1(z)$ and $\|g\|_{\mathcal{H}_2}^2 = \int_{\Omega_2} |g(w)|^2 d\mu_2(w)$, where $d\mu_1$ and $d\mu_2$ are positive Borel measures on Ω_1 and Ω_2 respectively. Let ψ be a biholomorphic mapping of Ω_1 onto Ω_2 and assume that there is a nonvanishing holomorphic function j_ψ on Ω_1 such that

$$\mu_2(\psi(N)) = \int_N |j_\psi(z)|^2 d\mu_1(z) \tag{7}$$

for each Borel subset N of Ω_1. Then the change of variable formula (7) implies that the operator

$$(U_\psi f)(z) = (f \circ \psi)(z) j_\psi(z) \tag{8}$$

is an isometry of $\mathcal{H}_2$ onto $\mathcal{H}_1$, which in turn gives rise to

$$K^{(\Omega_1)}(z, w) = K^{(\Omega_2)}(\psi(z), \psi(w)) j_\psi(z) \overline{j_\psi(w)}. \tag{9}$$

Consequently, using the unitary operator U_ψ in (8) and the identity (9), as in the proof of Proposition 3.1 in [17], it is easy to show that for such ψ, B_m enjoys the transformation property as follows:

$$(B_m V)^{(\Omega_1)}(z) = (B_m \psi_*(V))^{(\Omega_2)}(\psi(z)), \tag{10}$$

where $\psi_*(V)$ is the push-forward of the vector field V under ψ.

In particular, we specialize the vector field V to be the constant vector field $\partial_v = \sum_{j=1}^n v_j \partial_j$ with $v = (v_1, \cdots, v_n) \in \mathbb{C}^n \backslash \{0\}$. In this case, we write $e_j(\cdot) = e_{a,j}(\cdot) = K_{\partial_v^j,a}(\cdot) = (\bar\partial_v^j K)(\cdot, a) = (\bar\partial_v^j K_a)(\cdot)$ if no confusion arises. Then the set $\{e_j\}_{j=0}^m$ is linearly independent in $\mathcal{H}$ since the set of holomorphic polynomials $\mathcal{P} \subset \mathcal{H}$ by our assumption. An application of the Gram-Schmidt orthogonalization process gives an orthonormal basis $\{\varphi_i\}_{i=0}^m$ for $\mathcal{H}_{a,m+1}$, where ([4])

$$\begin{aligned}
\varphi_0 &= J_0^{-\frac{1}{2}} e_0, \\
\varphi_i &= \{J_{i-1} J_i\}^{-\frac{1}{2}} \begin{vmatrix} \langle e_0, e_0\rangle & \cdots & \langle e_0, e_{i-1}\rangle & e_0 \\ \vdots & \ddots & \vdots & \vdots \\ \langle e_i, e_0\rangle & \cdots & \langle e_i, e_{i-1}\rangle & e_i \end{vmatrix}
\end{aligned}$$

for $i = 1, \ldots, m$, and

$$J_i = J_i(\partial_v, a) = \begin{vmatrix} \langle e_0, e_0\rangle & \cdots & \langle e_0, e_i\rangle \\ \vdots & \ddots & \vdots \\ \langle e_i, e_0\rangle & \cdots & \langle e_i, e_i\rangle \end{vmatrix}$$

is the determinant of the Gram-matrix of the system $\{e_j\}_{j=0}^i$ for $i = 0, 1, \ldots, m$.

Note that $J_i > 0$ due to the linear independence of $\{e_j\}_{j=0}^{i}$ (see [4]) for $i = 0, 1, \ldots, m$, and $(B_m\partial_v)(a)$ can be expressed in a closed form in terms of the kernel function, i.e.([4])

$$(R_m\partial_v)(a) = J_{m-1}^{-1}J_m, \quad \{(B_m\partial_v)(a)\}^2 = \{J_0J_{m-1}\}^{-1}J_m, \quad m \geq 1. \tag{11}$$

As remarked in [17], $(B_m\partial_v)(a)$ is homogeneous in v of degree m, and is NOT, strictly speaking, an infinitesimal metric as $m > 1$. Care must be taken in considering "transformation laws" (10) for $(B_m\partial_v)(a)$ for $m > 1$. We also remark that for fixed domain Ω, the quantity $(B_m\partial_v)(a)$ relies on the specific Hilbert space strcture of $\mathcal{H}$ on Ω. The notation $(B_m\partial_v)(a)$ will be abused without indicating the holomorphic Hilbert space $\mathcal{H}$ which is to be clear from the context.

The prototypes of the spaces $\mathcal{H}$ are the Bergman spaces $A^2(\Omega, dv)$ of all holomorphic functions in $L^2(\Omega, dv)$ on a bounded domain $\Omega \subset \mathbb{C}^n$ with the normalized volume measure $dv(z) = V(\Omega)^{-1}dV(z)$ where dV is the Lebesgue measure and $V(\Omega)$ is the volume of Ω with respect to dV, or the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$ of all entire functions in $L^2(\mathbb{C}^n, d\mu)$ for the Gaussian measure

$$d\mu(z) = (2\pi)^{-n}e^{-\frac{|z|^2}{2}}dV(z).$$

For $\mathcal{H} = A^2(\Omega, dv)$, the reproducing kernel $K(z, w)$ is then just the original kernel function of Bergman [4], and $(B_m\partial_v)(a)$ is exactly the mth order Bergman metric as introduced by Burbea in [5]. In particular, $(B_1\partial_v)(a)$ is the classical Bergman metric. For $\mathcal{H} = H^2(\mathbb{C}^n, d\mu)$, the kernel function $K(z, w) = e^{\frac{\langle z,w\rangle}{2}}$ and $(B_1\partial_v)(a)$ coincides (up to a constant factor) with the Euclidean metric. As a Hermitian metric, the classical Bergman metric can also be defined as

$$B_1(a, v) = (B_1\partial_v)(a) = \Big\{\sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j\partial\bar{z}_k}\log K(a, a)v_j\bar{v}_k\Big\}^{\frac{1}{2}} \tag{12}$$

for $v \in T_z(\Omega) \cong \mathbb{C}^n$ the tangent space at $z \in \Omega$ ([13]).

The motivation for us to study the mapping B_m originates from the recent intensive research on Berezin's operator calculus ([6], [7], [8], [11], [12], [16]). The Berezin transform, introduced by F. A. Berezin in his quantization program ([2], [3]), provides a general symbol calculus for linear operators on any holomorphic Hilbert space. More specifically, for $\mathcal{H}$ a holomorphic Hilbert space on Ω and $X \in Op(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, the Berezin transform (or symbol) of X is given by

$$\widetilde{X}(z) = \langle Xk_z, k_z\rangle = tr(XA(z)), \tag{13}$$

where $k_z(\cdot) = K(\cdot, z)K(z, z)^{-\frac{1}{2}}$ is the normalized kernel function at z, and

$$A(z) = k_z \otimes k_z = \langle\cdot, k_z\rangle k_z \tag{14}$$

is the projection onto the span of k_z. It is well-known that $\widetilde{X}$ is real analytic with $\|\widetilde{X}\|_\infty \leq \|X\|$, and X is uniquely determined by $\widetilde{X}$.

The study on the interaction between the mth order Bergman metric and Berezin's operator calculus was initiated in [17] most recently, where the relationship between $(B_m\partial_v)(a)$ and $(B_1\partial_v)(a)$ has also been investigated. In particular, it was shown there that for $\mathcal{H} = A^2(\mathbb{B}^n, dv)$ on the open unit ball,

$$(B_m\partial_v)(a) = \left\{\frac{m!(m+n)!}{n!(n+1)^m}\right\}^{\frac{1}{2}}\{(B_1\partial_v)(a)\}^m. \tag{15}$$

In the same paper, it was also proved separately that an identity similar to (15) holds with different constant factor on $\mathbb{C}^n$ for $\mathcal{H} = H^2(\mathbb{C}^n, d\mu)$ (see Theorem 3.14 and Theorem 3.18 of [17]).

In view of the simple intuition that $\mathbb{C}^n$ could be considered as a ball with infinite radius, and the fact that $(B_m\partial_v)(a)$ is "intrinsic" interpreted as in (10), it would be of interest to find an approach that could deal with both cases uniformly in this Bergman-type spaces setting. It turns out that this goal can be achieved by approximating the scaled Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu_p)$ by a class of holomorphic Hilbert spaces in an appropriate way, where $d\mu_p(z) = \frac{p^n}{\pi^n}e^{-p|z|^2}dV(z)$. This uniform treatment will be developed in Section 2 and Section 3, while a different proof of (15) will be provided in the setting of $\mathcal{H} = A^2(\mathbb{B}^n, dv_\alpha)$ the weighted Bergman space in Section 2.

Motivated by the definition of mth order Bergman metric, we introduce another mapping D_m from the space of holomorphic vector fields to the set of functions on Ω in Section 5, and provide an estimates for $(D_m\partial_v)(z)$ on the unit ball using our analysis of Berezin's operator calculus. It turns out that this estimate coincides with the other one which is a direct consequence of the higher order Schwarz-Pick lemma in [10] except a different constant factor. In our argument, we need an iteration formula in [17] which will be refined by determining all coefficients completely in Section 4.

In Section 6, using the refined formula in Section 4 and the higher order Schwarz-Pick lemma in [10], we are able to show that on the unit ball $\mathbb{B}^n$,

$$(C_m\partial_v)(z) = m!\{(C_1\partial_v)(z)\}^m,$$

where $(C_m\partial_v)(z)$ is the mth order Carathéodory-Reiffen metric ([5]), and $(C_1\partial_v)(z)$ is the classical infinitesimal Carathéodory-Reiffen metric.

We conclude by pointing out that by using the refined formula in Section 4 and some result in Section 6, our estimate for $(D_m\partial_v)(z)$ in Section 5 can be improved to be the same as the other one directly from the higher order Schwarz-Pick lemma. This perfect coincidence discloses some intimate relation between our analysis in Berezin's operator calculus and the higher order Schwarz-Pick lemma.

2. A new proof

In this section, we extend (15) to the setting of weighted Bergman space $A^2(\mathbb{B}^n, dv_\alpha)$ by a different proof, which provides the foundation for our method of approximation in Section 3.

For $\alpha > -1$, let $dv_\alpha(z) = c_\alpha\left(1-|z|^2\right)^\alpha dv(z)$ be the probability measure on $\mathbb{B}^n$ where $c_\alpha = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$. For $\mathcal{H} = A^2(dv_\alpha) = A^2(\mathbb{B}^n, dv_\alpha)$ the weighted Bergman space on the unit ball $\mathbb{B}^n$, the reproducing kernel is given by ([19])

$$K^\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}} = K(z,w)^{\frac{n+1+\alpha}{n+1}}, \tag{16}$$

where $K(z,w)$ is the kernel function of the standard Bergman space $A^2(\mathbb{B}^n, dv)$ as $\alpha = 0$. It is easy to check that the Bergman metric (12) induced by $K^\alpha(z,w)$ on $\mathbb{B}^n$ is

$$\{(B_1\partial_v)(a)\}^2 = \frac{(n+1+\alpha)\{(1-|a|^2)|v|^2 + |\langle a,v\rangle|^2\}}{(1-|a|^2)^2} \tag{17}$$

by (16) and Proposition 1.4.22 in [15].

It is standard that for each a in $\mathbb{B}^n$, there is a $\psi_a \in \mathrm{Aut}(\mathbb{B}^n)$, the group of all biholomorphic self-mapping on $\mathbb{B}^n$, with the properties $\psi_a(a) = 0$ and $\psi_a \circ \psi_a = I$, the identity map. More precisely,

$$\psi_a(z) = \frac{a - P_a z - \sqrt{1-|a|^2}Q_a z}{1-\langle z,a\rangle} \tag{18}$$

where $P_0 = 0$, $P_a = \frac{1}{|a|^2} a \otimes a$ for $a \neq 0$, and $Q_a = I - P_a$ ([18]).

Lemma 2.1. *For $a, z \in \mathbb{B}^n$, $v \in \mathbb{C}^n$ and $\psi_a \in \mathrm{Aut}(\mathbb{B}^n)$,*

$$\langle z,v\rangle(1-\langle\psi_a(z),a\rangle) = \langle a,v\rangle - \langle\psi_a(z),\lambda\rangle,$$

where $\lambda = \sqrt{1-|a|^2}v + \frac{\langle v,a\rangle}{1+\sqrt{1-|a|^2}}a$ with $|\lambda|^2 = (1-|a|^2)|v|^2 + |\langle a,v\rangle|^2$.

Proof. Without loss of generality, suppose that $a \neq 0$. By the involutive property of ψ_a, it suffices to show that

$$\langle\psi_a(z),v\rangle(1-\langle z,a\rangle) = \langle a,v\rangle - \langle z,\lambda\rangle.$$

By (18),

$$\begin{aligned}\langle \psi_a(z), v\rangle(1-\langle z,a\rangle) &= \langle a - P_a z - \sqrt{1-|a|^2}Q_a z, v\rangle \\ &= \langle a, v\rangle - \frac{1}{|a|^2}\langle z,a\rangle\langle a,v\rangle - \sqrt{1-|a|^2}\langle z,v\rangle + \frac{\sqrt{1-|a|^2}}{|a|^2}\langle z,a\rangle\langle a,v\rangle \\ &= \langle a,v\rangle - \langle z, \sqrt{1-|a|^2}v\rangle - \frac{1}{1+\sqrt{1-|a|^2}}\langle z,a\rangle\langle a,v\rangle \\ &= \langle a,v\rangle - \langle z, \sqrt{1-|a|^2}v + \frac{\langle v,a\rangle}{1+\sqrt{1-|a|^2}}a\rangle.\end{aligned}$$

Let $\lambda = \sqrt{1-|a|^2}v + \frac{\langle v,a\rangle}{1+\sqrt{1-|a|^2}}a$, and a direct calculation shows that $|\lambda|^2 = (1-|a|^2)|v|^2 + |\langle a,v\rangle|^2$, which finishes the proof. □

In the setting of weighted Bergman space $A^2(dv_\alpha)$, the key point of our new method is to express the basis $\{e_j = \bar{\partial}_v^j K_a^\alpha(\cdot)\}_{j=0}^m$ of $\mathcal{H}_{a,m+1}$ in terms of a set of orthogonal elements.

Proposition 2.2. *For $j \geq 0$,*

$$\bar{\partial}_v^j K_a^\alpha(w) = \frac{\Gamma(j+n+1+\alpha)}{\Gamma(n+1+\alpha)}(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+j)}\sum_{\beta=0}^{j}\binom{j}{\beta}(-1)^\beta\langle a,v\rangle^{j-\beta}\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w),$$

where $\lambda = \sqrt{1-|a|^2}v + \frac{\langle v,a\rangle}{1+\sqrt{1-|a|^2}}a$ with $|\lambda|^2 = (1-|a|^2)|v|^2 + |\langle a,v\rangle|^2$, and the set $\{\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w)\}_{\beta=0}^j$ consists of orthogonal elements in $A^2(dv_\alpha)$.

Proof. For $v \in \mathbb{C}^n\backslash\{0\}$, we write $\bar{\partial}_v^j K^\alpha(w,u)|_{u=a} = \bar{\partial}_v^j K_a^\alpha(w)$. By (16), it is easy to see that

$$\bar{\partial}_v^j K_a^\alpha(w) = \frac{\Gamma(j+n+1+\alpha)}{\Gamma(n+1+\alpha)}\langle w,v\rangle^j(1-\langle w,a\rangle)^{-(j+n+1+\alpha)}.$$

Since the normalized kernel function k_a^α at a is

$$k_a^\alpha(w) = \frac{K^\alpha(w,a)}{K^\alpha(a,a)^{\frac{1}{2}}} = \frac{(1-|a|^2)^{\frac{n+1+\alpha}{2}}}{(1-\langle w,a\rangle)^{n+1+\alpha}}$$

and ([18])

$$1 - \langle\psi_a(w), a\rangle = \frac{1-|a|^2}{1-\langle w,a\rangle},$$

it follows that

$$\begin{aligned}\langle w,v\rangle^j(1-\langle w,a\rangle)^{-(j+n+1+\alpha)} &= (1-|a|^2)^{-(\frac{n+1+\alpha}{2}+j)}k_a^\alpha(w)\left\{\langle w,v\rangle(1-\langle\psi_a(w),a\rangle)\right\}^j \\ &= (1-|a|^2)^{-(\frac{n+1+\alpha}{2}+j)}k_a^\alpha(w)\left\{\langle a,v\rangle - \langle\psi_a(w),\lambda\rangle\right\}^j,\end{aligned}$$

where the last equality follows from Lemma 2.1. Therefore,

$$\begin{aligned}\bar{\partial}_v^j K_a^\alpha(w) &= \frac{\Gamma(j+n+1+\alpha)}{\Gamma(n+1+\alpha)}(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+j)}k_a^\alpha(w)[\langle a,v\rangle - \langle\psi_a(w),\lambda\rangle]^j \\ &= \frac{\Gamma(j+n+1+\alpha)}{\Gamma(n+1+\alpha)}(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+j)}\sum_{\beta=0}^{j}\binom{j}{\beta}(-1)^\beta\langle a,v\rangle^{j-\beta}\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w).\end{aligned}$$

Next we need to show the orthogonality of elements in the set $\{\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w)\}_{\beta=0}^j$. For this purpose, we consider the operator U_a associated with ψ_a on $A^2(dv_\alpha)$, defined by

$$(U_a f)(w) = (f\circ\psi_a)(w)k_a^\alpha(w).$$

Then it is easy to see that U_a is unitary on $A^2(dv_\alpha)$ by Proposition 1.13 in [19]. Since $\left\{\langle w,\lambda\rangle^\beta\right\}_{\beta=0}^j$ is an orthogonal set in $A^2(dv_\alpha)$ by (1.21) in [19], and

$$U_a\{\langle w,\lambda\rangle^\beta\}=\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w),$$

our assertion follows easily and the proof is completed. $\square$

The next result gives the estimate for the greatest possible rate of growth of higher order directional derivatives of Bergman space functions on the unit ball.

Corollary 2.3. *For $f\in A^2(dv_\alpha)$ and $m\ge 0$,*

$$\begin{aligned}\max_{\|f\|\le 1}|(\partial_v^m f)(a)|^2 &= \frac{[\Gamma(m+n+1+\alpha)]^2}{\Gamma(n+1+\alpha)(n+1+\alpha)^m}K^\alpha(a,a)\{(B_1\partial_v)(a)\}^{2m}\\ &\quad\times\sum_{\beta=0}^m\frac{(m!)^2}{\beta![(m-\beta)!]^2\Gamma(\beta+n+1+\alpha)}\left\{\frac{|\langle a,v\rangle|^2}{|\lambda|^2}\right\}^{m-\beta}.\end{aligned}$$

In particular, for $\alpha=0$,

$$\begin{aligned}\max_{\|f\|\le 1}|(\partial_v^m f)(a)|^2 &= \frac{(m+n)!m!}{n!(n+1)^m}K(a,a)\{(B_1\partial_v)(a)\}^{2m}\sum_{\beta=0}^m\binom{m}{\beta}\binom{m+n}{\beta+n}\left\{\frac{|\langle a,v\rangle|^2}{|\lambda|^2}\right\}^{m-\beta}\\ &\le \frac{[(m+n)!]^2m!}{(n!)^2(n+1)^m}K(a,a)\{(B_1\partial_v)(a)\}^{2m}\left\{1+\frac{|\langle a,v\rangle|^2}{|\lambda|^2}\right\}^m.\end{aligned}\tag{19}$$

Proof. By (4) with $V=\partial_v$ and Proposition 2.2, we know that

$$\begin{aligned}&\max_{\|f\|\le 1}|(\partial_v^m f)(a)|^2=\|\bar\partial_v^m K_a^\alpha\|^2\\ &=\left\|\frac{\Gamma(m+n+1+\alpha)}{\Gamma(n+1+\alpha)}(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+m)}\sum_{\beta=0}^m\binom{m}{\beta}(-1)^\beta\langle a,v\rangle^{m-\beta}\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w)\right\|^2\\ &=\left|\frac{\Gamma(m+n+1+\alpha)}{\Gamma(n+1+\alpha)}(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+m)}\right|^2\sum_{\beta=0}^m\left|\binom{m}{\beta}\langle a,v\rangle^{m-\beta}\right|^2\left\|\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w)\right\|^2.\end{aligned}$$

The rest of the proof follows from (17) and the fact that

$$\left\|\langle\psi_a(w),\lambda\rangle^\beta k_a^\alpha(w)\right\|^2=\|\langle w,\lambda\rangle^\beta\|^2=\frac{\Gamma(n+\alpha+1)\beta!}{\Gamma(\beta+n+1+\alpha)}|\lambda|^{2\beta}$$

by Lemma 1.11 of [19]. $\square$

Now we are ready to state the main result of this section.

Theorem 2.4. *On $A^2(dv_\alpha)$,*

$$(B_m\partial_v)(a)=\left\{\frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)(n+1+\alpha)^m}\right\}^{\frac12}\{(B_1\partial_v)(a)\}^m.\tag{20}$$

Proof. By Proposition 2.2, we know

$$(P_{\mathcal{H}_{a,m}^\perp}e_m)(w)=[P_{\mathcal{H}_{a,m}^\perp}(\bar\partial_v^m K_a^\alpha)](w)=\frac{\Gamma(m+n+1+\alpha)}{\Gamma(n+1+\alpha)}(-1)^m(1-|a|^2)^{-(\frac{n+1+\alpha}{2}+m)}\langle\psi_a(w),\lambda\rangle^m k_a^\alpha(w).$$

The same proof as in Corollary 2.3 shows that

$$\|P_{\mathcal{H}_{a,m}^\perp}e_m\|^2=\frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)}K^\alpha(a,a)\left\{\frac{|\lambda|^2}{(1-|a|^2)^2}\right\}^m.$$

Then by (3) and (6),

$$\begin{aligned}\{(B_m\partial_v)(a)\}^2 &= K^\alpha(a,a)^{-1}\|P_{\mathcal{H}_{a,m}^\perp}e_m\|^2\\ &= \frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)}\left\{\frac{|\lambda|^2}{(1-|a|^2)^2}\right\}^m\\ &= \frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)(n+1+\alpha)^m}\{(B_1\partial_v)(a)\}^{2m},\end{aligned}$$

as desired. □

Remark 2.5. *When $\alpha=0$ in Theorem 2.4, (15) will be recovered simply, while it was proved differently by using the transformation formula (10) and establishing the iteration formula (24) of certain vector fields at the origin (see Section 4).*

3. A uniform approach

The objective of this section is to develop an approach which will deal with $(B_m\partial_v)(a)$ on balls of any radius and $\mathbb{C}^n$ uniformly.

For $0<r<\infty$ and $\alpha>-1$, we denote $\mathbb{B}_r=\{z\in\mathbb{C}^n : |z|<r\}$ and $dv_{\alpha,r}(z)=c_{\alpha,r}(1-\frac{|z|^2}{r^2})^\alpha dv(z)$ with $c_{\alpha,r}=\frac{\Gamma(n+\alpha+1)}{n!r^{2n}\Gamma(\alpha+1)}$, then $v_{\alpha,r}(\mathbb{B}_r)=1$, $\mathbb{B}_1=\mathbb{B}^n$ the open unit ball and $dv_{\alpha,1}(z)=dv_\alpha(z)$ accordingly. It is clear that $\mathbb{B}_1$ and $\mathbb{B}_r$ are biholomorphically equivalent, and any biholomorphic mapping ψ of $\mathbb{B}_1$ onto $\mathbb{B}_r$ is of the form $r\sigma$ where $\sigma\in\mathrm{Aut}(\mathbb{B}_1)$. In particular, for $w=\psi(z)=\frac{z}{r}$: $\mathbb{B}_r\to\mathbb{B}_1$, it is easy to see that $\psi_*(\partial_v)=\partial_{\frac{v}{r}}$ and

$$v_\alpha(\psi(N))=\int_N dv_{\alpha,r}(z)$$

for each Borel subset $N\subset\mathbb{B}_r$. Consequently, the kernel function of the weighted Bergman spaces $A^2(\mathbb{B}_r,dv_{\alpha,r})$ is $K_r^\alpha(z,w)=(1-\frac{\langle z,w\rangle}{r^2})^{-(n+1+\alpha)}$ by (9). It also follows from the transformation formula (10) and Theorem 2.4 that

$$\begin{aligned}(B_m\partial_v)^{(\mathbb{B}_r)}(a) &= (B_m\partial_{\frac{v}{r}})^{(\mathbb{B}_1)}(\frac{a}{r})=\left\{\frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)(n+1+\alpha)^m}\right\}^{\frac12}\{(B_1\partial_{\frac{v}{r}})^{(\mathbb{B}_1)}(\frac{a}{r})\}^m\\ &= \left\{\frac{\Gamma(m+n+1+\alpha)m!}{\Gamma(n+1+\alpha)}\right\}^{\frac12}\left\{\frac{(r^2-|a|^2)|v|^2+|\langle a,v\rangle|^2}{(r^2-|a|^2)^2}\right\}^{\frac{m}{2}}.\end{aligned}\tag{21}$$

Lemma 3.1. *For $p>0$, the functions $f_r(z)=f(z,r)=(1-\frac{|z|^2}{r^2})^{-pr^2}$ converge to $e^{p|z|^2}$ uniformly on the compacta of $\mathbb{C}^n$ as $r\to+\infty$.*

Proof. Using the limit definition of the natural base e, it is easy to see that

$$\lim_{r\to+\infty}f_r(z)=e^{p|z|^2}$$

for each z. We only need to show that the pointwise convergence is actually uniform on each compact subset M of $\mathbb{C}^n$. For such an M, we can choose $R>0$ such that $M\subset\mathbb{B}_R$. Note that

$$\frac{\partial}{\partial r}f(z,r)=-2pr\left(1-\frac{|z|^2}{r^2}\right)^{-pr^2}\left\{\ln(1-\frac{|z|^2}{r^2})+\frac{\frac{|z|^2}{r^2}}{1-\frac{|z|^2}{r^2}}\right\}.$$

For $r>R$ and $z\in M$, we claim that $\ln(1-\frac{|z|^2}{r^2})+\frac{\frac{|z|^2}{r^2}}{1-\frac{|z|^2}{r^2}}\geq 0$. Let $x=\frac{|z|^2}{r^2}$, then $0\leq x<1$ and we define $g(x)=\ln(1-x)+\frac{x}{1-x}$. So $g(0)=0$ and $g'(x)=\frac{x}{(1-x)^2}\geq 0$. Thus $g(x)\geq 0$ for $0\leq x<1$, which implies that $\frac{\partial}{\partial r}f(z,r)\leq 0$, and the functions $f_r(z)$ is decreasing in r as $r>R$ for $z\in M$. So the functions $f_r(z)$ are convergent to $e^{p|z|^2}$ uniformly on M as $r\to+\infty$ by Dini's Theorem. □

Theorem 3.2. *For fixed $p > 0$ and $pr_0^2 > n$ for some $r_0 > 0$, let $\alpha = pr^2 - n - 1$ for $r \geq r_0$. Then*

$$\lim_{r\to+\infty} K_r^\alpha(z,z) = e^{p|z|^2}$$

uniformly on the compacta of $\mathbb{C}^n$. Consequently, for $\mathcal{H} = H^2(\mathbb{C}^n, d\mu_p)$,

$$(B_m\partial_v)^{(\mathbb{C}^n)}(a) = \lim_{r\to+\infty}(B_m\partial_v)^{(\mathbb{B}_r)}(a) = \{m!p^m|v|^{2m}\}^{\frac{1}{2}}.$$

Proof. By our assumption, $\alpha = pr^2 - n - 1 > -1$ and $K_r^\alpha(z,z) = (1 - \frac{|z|^2}{r^2})^{-pr^2}$. Then by Lemma 3.1, we know

$$\lim_{r\to+\infty} K_r^\alpha(z,z) = e^{p|z|^2} \tag{22}$$

uniformly on the compacta of $\mathbb{C}^n$.

On the other hand, by (11) we know that $(B_m\partial_v)^{(\mathbb{B}_r)}(a)$ is a closed expression of

$$\{\langle e_j, e_i\rangle = (\partial_v^i\bar{\partial}_v^j K_r^\alpha)(a,a)\}_{i,j=0}^m.$$

So the interchange of taking limit and differentiation is assured by (22) so that

$$\lim_{r\to\infty}(B_m\partial_v)^{(\mathbb{B}_r)}(a) = (B_m\partial_v)^{(\mathbb{C}^n)}(a)$$

for some holomorphic Hilbert space $\mathcal{H}$ on $\mathbb{C}^n$ with kernel function $K^{(p)}(z,z) = e^{p|z|^2}$ on the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$. Since the function $K^{(p)}(z,w) = e^{p\langle z,w\rangle}$, as the kernel function of $H^2(\mathbb{C}^n, d\mu_p)$, is uniquely determined by $K^{(p)}(z,z) = e^{p|z|^2}$ by the standard uniqueness theorem, and there is a bijective correspondence between Hilbert function spaces on $\mathbb{C}^n$ and reproducing kernels on $\mathbb{C}^n$ (p.19, [1]), we know that $\mathcal{H}$ must be represented by $H^2(\mathbb{C}^n, d\mu_p)$. Finally, with (21), it is easy to see that

$$\begin{aligned}
(B_m\partial_v)^{(\mathbb{C}^n)}(a) &= \lim_{r\to\infty}(B_m\partial_v)^{(\mathbb{B}_r)}(a)\\
&= \lim_{r\to\infty}\left\{\frac{\Gamma(m+pr^2)m!}{\Gamma(pr^2)}\right\}^{\frac{1}{2}}\left\{\frac{(r^2-|a|^2)|v|^2+|\langle a,v\rangle|^2}{(r^2-|a|^2)^2}\right\}^{\frac{m}{2}}\\
&= \{m!p^m|v|^{2m}\}^{\frac{1}{2}},
\end{aligned}$$

where the last equality follows from Stirling's formula. □

4. Revisiting to an iteration formula

For $\Omega = \mathbb{B}^n$ and $w = \psi_a(z) \in \mathrm{Aut}(\mathbb{B}^n)$ of the form (18), we define $g(w) = (\psi_a)_*(v) = \psi_a'(z)v = (g_1(w), \cdots, g_n(w))$, then $g(0) = \psi_a'(a)v$ and (Lemma 3.7 [17])

$$|g(0)| = \frac{(B_1\partial_v)(a)}{\sqrt{n+1}} \tag{23}$$

where $(B_1\partial_v)(a)$ is the classical Bergman metric associated to the (unweighted) Bergman space $A^2(\mathbb{B}^n, dv)$. Moreover, let ∂_g^m stand for the operator $\{\sum_{j=1}^n g_j(w)\partial_{w_j}\}^m$ and $\partial_{g(0)}^m = \partial_{\psi_a'(a)v}^m$ for the operator $\{\sum_{j=1}^n g_j(0)\partial_{w_j}\}^m$. Let $\mathscr{X}$ be a Banach space and $C^\infty(\mathbb{B}^n, \mathscr{X})$ be the set of smooth mappings from $\mathbb{B}^n$ into $\mathscr{X}$ possessing *strong derivatives* of all orders (see e.g. [12]). An iteration formula for the action of ∂_g^m on $C^\infty(\mathbb{B}^n, \mathscr{X})$ at $w = 0$ has been established in [17] as follows: for $f \in C^\infty(\mathbb{B}^n, \mathscr{X})$ and $m \geq 1$,

$$(\partial_g^m f)(w)|_{w=0} = \sum_{j=1}^m C_j^{(m)} h(a,v)^{m-j}(\partial_{g(0)}^j f)(w)|_{w=0}, \tag{24}$$

where $h(a,v) = (1-|a|^2)^{-1}\langle v,a\rangle$, and $C_j^{(m)}$'s are constants depending on m and j with $C_m^{(m)} = 1$.

Moreover, a careful examination of the proof of (24) in [17] indicates that $C_j^{(m)}$'s are determined by the recurrence relation

$$C_j^{(m)} = C_{j-1}^{(m-1)} + 2jC_j^{(m-1)} + j(j+1)C_{j+1}^{(m-1)}, \quad 1 \le j \le m, \tag{25}$$

where $C_m^{(m)} = 1$ for $m \ge 1$. We denote $C_0^{(0)} = 1$ and $C_0^{(m)} = 0$ for $m > 0$, $C_j^{(m)} = 0$ for $j > m$ or $j < 0$. Actually, we can determine these coefficients explicitly.

Proposition 4.1. *For $m \ge 1$ and $1 \le j \le m$,*

$$C_j^{(m)} = \frac{(m-1)!m!}{(j-1)!j!(m-j)!}. \tag{26}$$

Proof. We proceed by induction on m and j. It is clear that for $m = 1$ and $j = 1$, $C_1^{(1)} = 1$. So (26) holds. Now we assume that (26) is true for some $m = k \ge 1$ and for all $1 \le j \le k$. Then for $m = k + 1$ and $j = 1$, by (25) and our hypotheses for $m = k$, we have

$$C_1^{(k+1)} = C_0^{(k)} + 2C_1^{(k)} + 2C_2^{(k)} = 2C_1^{(k)} + 2C_2^{(k)} = 2k! + (k-1)k! = (k+1)!,$$

which shows that (26) is true for $m = k + 1$ and $j = 1$.

Now for $m = k + 1$ and $2 \le j \le k + 1$, by (25) and our hypotheses for $m = k$ again, we have

$$\begin{aligned}
C_j^{(k+1)} &= C_{j-1}^{(k)} + 2jC_j^{(k)} + j(j+1)C_{j+1}^{(k)} \\
&= \frac{(k-1)!k!}{(j-2)!(j-1)!(k+1-j)!} + 2j\frac{(k-1)!k!}{(j-1)!j!(k-j)!} + j(j+1)\frac{(k-1)!k!}{j!(j+1)!(k-1-j)!} \\
&= \frac{(k-1)!k!}{(j-1)!j!(k+1-j)!)}\{(j-1)j + 2j(k+1-j) + (k-j)(k+1-j)\} \\
&= \frac{(k-1)!k!}{(j-1)!j!(k+1-j)!)}\{k(k+1)\} \\
&= \frac{k!(k+1)!}{(j-1)!j!(k+1-j)!)},
\end{aligned}$$

which shows that (26) is also true for $m = k + 1$ and $2 \le j \le k + 1$. Therefore, we can conclude that (26) holds for all $m \ge 1$ and $1 \le j \le m$. □

5. Connections to higher order Schwarz-Pick lemma

Most recently, a higher order Schwarz-Pick lemma formulated in terms of the classical Bergman metric has been proved in [10] for the Fréchet derivative of $f = (f_1, \cdots, f_k) \in H(\mathbb{B}^n, \mathbb{B}^k)$ the class of holomorphic mappings from $\mathbb{B}^n$ into $\mathbb{B}^k$ (even in the setting of complex Hilbert balls [9]). For $v \in \mathbb{C}^n \backslash \{0\}$, we observe that the Fréchet derivative $D_m(f, a, v)$ of f at point $a \in \mathbb{B}^n$ of order m defined in [10] can be expressed in our notations as

$$D_m(f, a, v) = (\partial_v^m f)(a) = ((\partial_v^m f_1)(a), \cdots, (\partial_v^m f_k)(a)) \in \mathbb{C}^k,$$

and the higher order Schwarz-Pick lemma in [10] can be rewritten as

$$(B_1\partial_{(\partial_v^m f)(a)})(f(a)) \le m! \left\{ \frac{1}{\sqrt{n+1}} \left(1 + \frac{|\langle a, v\rangle|}{|\lambda|} \right) \right\}^{m-1} \{(B_1\partial_v)(a)\}^m. \tag{27}$$

Note that the Bergman metric used in (27) differs from the one used in [10] by a factor $\sqrt{n+1}$. One immediate consequence of (27) combining with (17) ($\alpha = 0$) is that for $a \in \mathbb{B}^n$ and $v \in \mathbb{C}^n$,

$$\sup_{f \in H(\mathbb{B}^n, \mathbb{B}^k)} |(\partial_v^m f)(a)| \le \frac{m!}{(n+1)^{\frac{m}{2}}} \left\{ 1 + \frac{|\langle a, v\rangle|}{|\lambda|} \right\}^{m-1} \{(B_1\partial_v)(a)\}^m. \tag{28}$$

We notice that (28) has the same flavor as (19) in Corollary 2.3, and believe that there might have certain connection between our study of higher order "intrinsic" metrics in Berezin's operator calculus and the higher order Schwarz-Pick lemma (27). Actually, our belief will be enhanced by the fact that we are able to recover (28) with less sharp constant factor based on our analysis in Berezin's operator calculus.

First of all, motivated by (28) and the definition of the mth order Bergman metric, it would be interesting to consider another mapping $D_m^{(k)}$ defined on the space of holomorphic vector fields by

$$(D_m^{(k)}V)(a) = \sup_{f\in H(\Omega,\mathbb{B}^k)} |(V^m f)(a)| = \sup_{f\in H(\Omega,\mathbb{B}^k)} \left\{\sum_{j=1}^{k} |(V^m f_j)(a)|^2\right\}^{\frac{1}{2}},$$

where Ω is a bounded domain, V is a holomorphic vector field on Ω and $H(\Omega,\mathbb{B}^k)$ is the class of holomorphic mappings from Ω into $\mathbb{B}^k$. It is also easy to prove that

$$(D_m^{(k)}V)^{(\Omega_1)}(z) = (D_m^{(k)}\psi_*(V))^{(\Omega_2)}(\psi(z)) \tag{29}$$

for $\psi : \Omega_1 \to \Omega_2$ a biholomorphic mapping. We write $(D_m^{(1)}V)(a) = (D_m V)(a)$. It turns out that the definition of $(D_m^{(k)}V)(a)$ is independent of k.

Proposition 5.1. *For any $k \geq 1$,*

$$(D_m^{(k)}V)(a) = (D_m V)(a).$$

Proof. For $f \in H(\Omega,\mathbb{B}^1)$, we define $\hat{f} = (f, 0, \cdots, 0) \in H(\Omega,\mathbb{B}^k)$. Then

$$|(V^m \hat{f})(a)| = |(V^m f)(a)|$$

for any holomorphic vector field V. It follows that $(D_m V)(a) \leq (D_m^{(k)}V)(a)$. For the opposite inequality, we observe that for $f \in H(\Omega,\mathbb{B}^k)$,

$$\begin{aligned}
|(V^m f)(a)| &= \sup\{|\langle (V^m f)(a), \xi\rangle| : \xi \in \mathbb{C}^k, |\xi| = 1\} \\
&= \sup\{|\sum_{j=1}^{k} \bar{\xi}_j (V^m f_j)(a)| : \xi \in \mathbb{C}^k, |\xi| = 1\} \\
&= \sup\{|(V^m \sum_{j=1}^{k} \bar{\xi}_j f_j)(a)| : \xi \in \mathbb{C}^k, |\xi| = 1\} \\
&\leq \sup\{|(V^m h)(a)| : h \in H(\Omega,\mathbb{B}^1)\} \\
&= (D_m V)(a).
\end{aligned}$$

The proof is completed. ∎

Recall that for the rank one selfadjoint operator $A(z)$ in (14), the mapping L_m defined on the space of holomorphic vector fields by

$$(L_m V)(a) = \|(V^m A)(a)\|_{tr}$$

was introduced in [17].

Proposition 5.2. *On any bounded domain Ω,*

$$(D_m V)(a) \leq (L_m V)(a).$$

Proof. From the definition (13) of the Berezin transform, we know that for $X \in Op(A^2(\Omega))$,

$$|(V^m \widetilde{X})(a)| = |tr[X(V^m A)(a)]| \leq \|X\| \|(V^m A)(a)\|_{tr}. \tag{30}$$

If we take $X = M_f$ the multiplication operator on the Bergman space $A^2(\Omega)$ with the bounded symbol $f \in H(\Omega, \mathbb{B}^1)$ in (30), it is easy to check that $\widetilde{M_f} = f$ and $\|M_f\| = \|f\|_\infty \le 1$. Our assertion follows easily. $\square$

In our terminology, (28) is the same as the statement that on the unit ball $\mathbb{B}^n$,

$$(D_m^{(k)}\partial_v)(a) = (D_m\partial_v)(a) \le \frac{m!}{(n+1)^{\frac{m}{2}}}\left\{1 + \frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-1}\{(B_1\partial_v)(a)\}^m, \tag{31}$$

which can be recovered with less sharp constant factor as follows:

Proposition 5.3. *For $X \in Op(A^2(\mathbb{B}^n))$,*

$$|(\partial_v^m \widetilde{X})(a)| \le \|X\|\binom{m+n}{n}^{\frac{1}{2}}\frac{m!}{(n+1)^{\frac{m}{2}}}\left\{1 + \frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-1}\{(B_1\partial_v)(a)\}^m. \tag{32}$$

Consequently,

$$(D_m\partial_v)(a) \le \binom{m+n}{n}^{\frac{1}{2}}\frac{m!}{(n+1)^{\frac{m}{2}}}\left\{1 + \frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-1}\{(B_1\partial_v)(a)\}^m. \tag{33}$$

Proof. It was obtained by the formula (24) in section 4.1 of [17] that on the unit ball,

$$\begin{aligned}
(L_m\partial_v)(a) &= \left\{\sum_{j=1}^m |C_j^{(m)}|^2 \frac{j!(j+n)!}{n!}|h(a,v)|^{2(m-j)}|g(0)|^{2j}\right\}^{\frac{1}{2}} \\
&= \frac{(m-1)!m!\{(B_1\partial_v)(a)\}^m}{\sqrt{n!(n+1)^m}}\left\{\sum_{j=1}^m \frac{(j+n)!}{[(j-1)!]^2 j![(m-j)!]^2}\left\{\frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{2(m-j)}\right\}^{\frac{1}{2}},
\end{aligned}$$

where (23) and (26) are used in the last equality. Thus,

$$\begin{aligned}
\{(L_m\partial_v)(a)\} &\le \frac{m!\{(B_1\partial_v)(a)\}^m}{\sqrt{(n+1)^m}}\sum_{j=1}^m \binom{n+j}{n}^{\frac{1}{2}}\binom{m-1}{j-1}\left\{\frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-j} \\
&\le \binom{m+n}{n}^{\frac{1}{2}}\frac{m!}{(n+1)^{\frac{m}{2}}}\left\{1 + \frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-1}\{(B_1\partial_v)(a)\}^m.
\end{aligned}$$

Now (32) follows easily from (30) by taking $V = \partial_v$, and (33) is from Proposition 5.2. $\square$

6. mTH ORDER CARATHÉODORY-REIFFEN METRIC

In this section, we would like to derive an identity similar to (20) on the unit ball $\mathbb{B}^n$ between the mth order Carathéodory-Reiffen metric and the classical infinitesimal Carathéodory-Reiffen metric.

For a bounded domain Ω, we replace the Hilbert space $\mathcal{H}$ by $H(\Omega, \mathbb{D})$ the class of holomorphic functions f on Ω with values in the unit disk $\mathbb{D}$, and the Hilbert space norm by the supremum norm $\|\cdot\|_\infty$ in (1), and denote

$$T_m(V, a) = \{f \in H(\Omega, \mathbb{D}) : (V^j f)(a) = 0,\ j = 0, 1, \cdots, m-1\}$$

and define a mapping on the space of holomorphic vector fields by

$$(C_mV)(a) = \sup_{f \in T_m(V,a)} |(V^m f)(a)|.$$

Then it is clear that $(C_mV)(a) \le (D_mV)(a)$ and $(C_1V)(a) = (D_1V)(a)$ on any bounded domain. It is also easy to show that C_m satisfies the transformation formula

$$(C_mV)^{(\Omega_1)}(z) = (C_m\psi_*(V))^{(\Omega_2)}(\psi(z)), \tag{34}$$

where $\psi : \Omega_1 \to \Omega_2$ is biholomorphic and $\psi_*(V)$ is the push-forward of the vector field V under ψ. If the vector field $V = \partial_v$ for $v \in \mathbb{C}^n \backslash \{0\}$, $(C_m\partial_v)(a)$ is exactly the mth order Carathéodory-Reiffen metric introduced by Burbea in [5], and $(C_1\partial_v)(a)$ is just the classical infinitesimal Carathéodory-Reiffen metric ([15]).

Lemma 6.1. *For $e_1 = (1, 0, \cdots, 0) \in \mathbb{C}^n$ and $m \geq 1$,*

$$(C_m\partial_{e_1})(0) = (D_m\partial_{e_1})(0) = m!$$

on the unit ball $\mathbb{B}^n$.

Proof. It is clear that $(C_m\partial_{e_1})(0) \leq (D_m\partial_{e_1})(0)$ by their definitions. For $f(z) = z_1^m$, it is easy to check that $f \in T_m(\partial_{e_1}, 0)$ and $(\partial_{e_1}^m f)(0) = m!$. So $m! \leq (C_m\partial_{e_1})(0)$, while $(D_m\partial_{e_1})(0) \leq m!$ by (31). $\square$

Theorem 6.2. *On the unit ball $\mathbb{B}^n$,*

$$(C_m\partial_v)(a) = m!\{(C_1\partial_v)(a)\}^m.$$

Proof. Based on (24), the same argument as in the proof of Proposition 3.13 of [17] yields that

$$(C_m\partial_g)(0) = (C_m\partial_{g(0)})(0). \tag{35}$$

By (34) and (35), we have

$$(C_m\partial_v)(a) = (C_m\partial_g)(0) = (C_m\partial_{g(0)})(0). \tag{36}$$

We choose a unitary transformation U such that $U(\frac{g(0)}{|g(0)|}) = e_1$ where $e_1 = (1, 0, \cdots, 0)$, then by (34) again,

$$(C_m\partial_{g(0)})(0) = |g(0)|^m (C_m\partial_{\frac{g(0)}{|g(0)|}})(0) = |g(0)|^m (C_m\partial_{U(\frac{g(0)}{|g(0)|})})(0) = |g(0)|^m (C_m\partial_{e_1})(0). \tag{37}$$

Since $|g(0)| = |\psi_a'(a)v| = (C_1\partial_v)(a)$, (36), (37) and Lemma 6.1 imply that

$$(C_m\partial_v)(a) = |g(0)|^m (C_m\partial_{e_1})(0) = m!\{(C_1\partial_v)(a)\}^m.$$

$\square$

Remark 6.3. *We remark that (31) could be recovered exactly by an alternative argument involving Lemma 6.1 in which a particular case of (31) has been used. That is, by (29) and (24),*

$$\begin{aligned}(D_m\partial_v)(a) = (D_m\partial_g)(0) &\leq \sum_{j=1}^m C_j^{(m)} |h(a,v)|^{m-j} (D_j\partial_{g(0)})(0) \\ &= \sum_{j=1}^m C_j^{(m)} |h(a,v)|^{m-j} |g(0)|^j (D_j\partial_{e_1})(0) \\ &= \frac{m!\{(B_1\partial_v)(a)\}^m}{\sqrt{(n+1)^m}} \sum_{j=1}^m \binom{m-1}{j-1} \left\{\frac{|\langle a, v\rangle|}{|\lambda|}\right\}^{m-j},\end{aligned}$$

which is exactly (31) after reindexing.

Remark 6.4. *Combining the results in [5], [17] and this note, we see that*

$$(C_m\partial_v)(a) \leq (B_m\partial_v)(a), (D_m\partial_v)(a) \leq (L_m\partial_v)(a)$$

on any bounded domain, and they are equivalent to each other and comparable to $\{(B_1\partial_v)(a)\}^m$ on the unit ball. We conjecture that they are also equivalent on any bounded symmetric domain in $\mathbb{C}^n$.

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