WHEN IS A FUNCTION NOT FLAT?

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Abstract. In this paper we prove a unique continuation property for functions of one variable satisfying certain differential inequality.

Key words: unique continuation, Carleman's method.

1. Introduction

The function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is well known for its property

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \underline{\text{but}} \qquad f \not\equiv 0.$$

Such a function is called *flat* at the origin. On the other hand, if f is a real analytic function with Taylor expansion on an open interval containing 0, then $f^{(k)}(0) = 0,\ \forall k \geq 0$ implies $f \equiv 0$. The unique continuation problem in PDE is to find conditions such that the solutions of PDE enjoy the same property. There are a large amount of literature in this area originated from the ideas by Carleman [1], called Carleman's method. In this paper we consider the simplest case of one variable.

Theorem 1. *Let $f(x) \in C^\infty([a,b]),\ 0 \in [a,b]$, and*

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad x \in [a,b] \tag{1}$$

for some constant C and $n \geq 2$. Then

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \text{implies} \qquad f \equiv 0 \quad on \quad [a, b].$$

From Theorem 1 we obtain the following corollary.

Corollary 2. *Let $f(x) \in C^\infty([a, b]),\ 0 \in [a, b]$, and (1) holds for some constant C and $n \geq 2$. Then*

$$f \not\equiv 0 \qquad \text{implies} \qquad \textit{the zero set} \quad \{f^{-1}(0)\} \subset [a, b] \quad \textit{is finite.}$$

An example.

We use an example in [2] to show that the order of singularity in (1) is best possible, i.e., there exists a function $f(x) \in C^\infty([-a, a]),\ a > 0$,

(2)
$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}} \quad for \quad x \in [-a, a] \qquad and \qquad f^{(k)}(0) = 0,\ \forall k \geq 0$$

for some constant C and $\varepsilon > 0$, but $f \not\equiv 0$ on $[-a, a]$.

For $m > 1,\ \varepsilon > 0$, the following equation is considered in [2]:

$$x^2 u''(x) + mxu'(x) - cx^{-\varepsilon} u(x) = 0, \qquad x \in (0, 1),$$

or equivalently,

$$u''(x) + \frac{m}{x} u'(x) - \frac{c}{x^{2+\varepsilon}} u(x) = 0, \qquad x \in (0, 1), \tag{3}$$

— a Bessel differential equation. The general Bessel differential equation takes the form

$$z^2 u''(z) + (1 - 2\alpha) z u'(z) + \left\{\beta^2 \gamma^2 z^{2\gamma} + (\alpha^2 - \nu^2 \gamma^2)\right\} u(z) = 0, \qquad z \in \mathbb{C}. \tag{4}$$

It is well known that for (non-integer) $\nu \notin \mathbb{Z}$, the solution for (4) is

$$u(z) = z^{\alpha}\,[\ C_1 J_{\nu}(\beta z^{\gamma}) + C_2 J_{-\nu}(\beta z^{\gamma})\ ]\,,$$

where $C_1,\ C_2$ are arbitrary complex numbers, and J_ν is the Bessel function of order ν, i.e., a solution of equation (4) with $\alpha = 0,\ \gamma = 1$. Notice that equation (3) with $c > 0$ is equation (4) with

$$\alpha = -\frac{m-1}{2}, \quad \beta = i\frac{2\sqrt{c}}{\varepsilon}, \quad \gamma = -\frac{\varepsilon}{2}, \quad \nu = \frac{m-1}{\varepsilon}.$$

By choosing $m > 1,\ \varepsilon \in (0,1)$ such that

$$C_1 = -\ C_2 = \frac{\pi}{2\ e^{i\nu\pi}\sin(\nu\pi)}, \qquad \nu = \frac{m-1}{\varepsilon} \notin \mathbb{Z},$$

the solution of equation (3) can be written as

$$u(x) = |x|^{-(m-1)/2} K_{(m-1)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right), \qquad x \in (0,1) \tag{5}$$

where

$$K_\nu(z) = \frac{\pi}{2}\ \frac{e^{-i\nu\pi}J_{-\nu}(iz) - J_\nu(iz)}{\sin(\nu\pi)}, \qquad \arg z \in (-\pi, \pi/2)$$

is the modified Bessel function of the third kind [4], with the asymptotic property

$$K_\nu(x) \approx \frac{\pi}{2}x^{-1/2}e^{-x} \qquad as \quad x \to +\infty.$$

Therefore in (5), the function

$$u(x) \approx \frac{\pi}{2}\left(\frac{2\sqrt{c}}{\varepsilon}\right)^{-1/2} x^{\,-\frac{m-1}{2}+\frac{\varepsilon}{4}} \exp\left\{-\ \frac{2\sqrt{c}}{\varepsilon}x^{-\varepsilon/2}\right\} \qquad as \quad x \to 0$$

is a nontrivial solution of (3) vanishing at $x = 0$ of infinite order. Hence

$$f(x) = u(|x|), \qquad x \in [-a, a], \quad a \in (0,1)$$

is well defined and $f \in \mathcal{C}^\infty([-a,a])$. Taking derivative of equation (3) $n-2$ times,

$$\frac{d^{n-2}}{dx^{n-2}} \left\{ u''(x) + \frac{m}{x} u'(x) - \frac{c}{x^{2+\varepsilon}} u(x) \right\} = 0, \qquad x \in (0,1),$$

we obtain

$$u^{(n)}(x) + a_{n-1}(x) u^{(n-1)}(x) + \cdots + a_0(x) u(x) = 0, \qquad x \in (0,1).$$

For given $m,\ c$ and ε, the coefficients

$$|a_j(x)| \le C_o \left(\frac{1}{|x|^{n-j+\varepsilon}} \right), \quad j = 0, 1, \cdots, n-1, \quad x \in (0,1)$$

for some constant $C_o > 0$. By the property of $u(x)$ near $x = 0$, we have

$$f^{(n)}(x) + a_{n-1}(x) f^{(n-1)}(x) + \cdots + a_0(x) f(x) = 0, \qquad x \in [-a,a]$$

with

$$f^{(k)}(0) = 0,\ \forall k \ge 0, \qquad |a_j(x)| \le C \left(\frac{1}{|x|^{n-j+\varepsilon}} \right), \quad j = 0, 1, \cdots, n-1.$$

for some constant $C >$. Thus

$$f^{(k)}(0) = 0,\ \forall k \ge 0, \qquad |f^{(n)}(x)| \le C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}}, \quad x \in [-a,a],$$

and $f \not\equiv 0$ from the non-triviality of u.

Theorem 1 leads to applications in ODE as stated in the following two propositions. Based on the above example, the order of the singularity of the coefficients in the assumption of the propositions is sharp.

Proposition 3. *Let $f(x) \in \mathcal{C}^\infty$ be a solution of*

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = 0, \qquad x \in [-a,a],\ a > 0$$

with

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

Then

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \Longrightarrow \qquad f \equiv 0 \quad on \quad [-a, a].$$

Proposition 4. *Let $f(x),\ g(x) \in \mathcal{C}^\infty$ be solutions of*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x), \qquad x \in [-a, a],\ a > 0$$

with

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

Then

$$f^{(k)}(0) = g^{(k)}(0), \quad \forall k \geq 0 \qquad \Longrightarrow \qquad f \equiv g \quad on \quad [-a, a].$$

2. Proof of Theorem 1 and its corollary

Several lemmas are needed for the proof of Theorem 1. The basic idea of the following lemma was considered in [3].

Lemma 5. *Let $v(x) \in C^\infty([0, b])$. Assume $v^{(k)}(0) = 0$ for $k \geq 0$. Then for $\alpha \geq 1$,*

$$\int_0^b \frac{[v(x)]^2}{x^{\alpha+2}} dx \leq \frac{4}{(\alpha+1)^2} \int_0^b \frac{[v'(x)]^2}{x^\alpha} dx \tag{6}$$

Proof. Write $v = v(x), v' = v'(x)$ and so on.

$$\frac{d}{dx}\left(x^{-(\alpha+1)}v^2\right) = -(\alpha+1)x^{-(\alpha+2)}v^2 + x^{-(\alpha+1)}2vv'$$

Since $v^{(k)}(0) = 0$ for $k \geq [\alpha/2] + 1$, we have $[v(x)]^2 \sim O(x^{2(k+1)}) = o(x^{\alpha+1})$, thus

$$\int_0^b \frac{d}{dx}\left(x^{-(\alpha+1)}v^2\right) dx = \frac{[v(b)]^2}{b^{\alpha+1}} - \lim_{x\to 0+} \frac{[v(x)]^2}{x^{\alpha+1}} = \frac{[v(b)]^2}{b^{\alpha+1}} \geq 0.$$

Therefore,

$$\begin{aligned}
(\alpha+1)\int_0^b \frac{v^2}{x^{\alpha+2}}dx &\leq \int_0^b \frac{2vv'}{x^{\alpha+1}}dx \\
&= \int_0^b 2\left\{\left(\frac{\alpha+1}{2}\right)^{1/2} \frac{v}{x^{(\alpha+2)/2}}\right\}\left\{\left(\frac{2}{\alpha+1}\right)^{1/2} \frac{v'}{x^{\alpha/2}}\right\} dx \\
&\leq \int_0^b \frac{\alpha+1}{2}\frac{v^2}{x^{\alpha+2}}dx + \int_0^b \frac{2}{\alpha+1}\frac{(v')^2}{x^{\alpha}}dx
\end{aligned}$$

by applying $2ab \leq a^2 + b^2$ to the last inequality. Consequently,

$$\frac{\alpha+1}{2}\int_0^b \frac{v^2}{x^{\alpha+2}}dx \leq \frac{2}{\alpha+1}\int_0^b \frac{(v')^2}{x^{\alpha}}dx,$$

which is equivalent to (6). $\square$

Lemma 6. *Let $u(x) \in C^\infty([0,b])$. Assume $u^{(k)}(0) = 0$ for $k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \leq \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^{\beta}}dx, \qquad for \quad k = 0, \cdots, n-1.$$

Proof. Applying Lemma 5 to $v = u^{(n-1-j)}, \alpha = \beta + 2j, \beta \geq 1, j = 0, 1, \cdots, n-1$, we obtain

$$\int_0^b \frac{\left[u^{(n-1-j)}(x)\right]^2}{x^{\beta+2j+2}}dx \leq \frac{4}{(\beta+2j+1)^2}\int_0^b \frac{[u^{(n-j)}(x)]^2}{x^{\beta+2j}}dx, \quad j = 0, 1, \cdots, n-1,$$

or equivalently,

$$\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \leq \frac{4}{(\beta+2(n-k-1)+1)^2}\int_0^b \frac{[u^{(k+1)}(x)]^2}{x^{\beta+2(n-k-1)}}dx, \quad k = 0, 1, \cdots, n-1.$$

Applying the above inequality repeatedly, we have

$$\begin{aligned}\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx &\leq \left\{\prod_{m=k}^{n-2}\frac{4}{(\beta+2(n-1-m)+1)^2}\right\}\frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^\beta}dx \\ &\leq \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^\beta}dx,\end{aligned}$$

because $\beta \geq 1,\ \beta+2(n-m-1)+1 \geq 2$ for $m = 0, \ldots, n-1$. □

Lemma 7. *Let $u(x) \in C^\infty([0,b])$. Assume $u^{(k)}(0) = 0$ for $k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_0^b \frac{1}{x^\beta}\sum_{k=0}^{n-1}\left(\frac{u^{(k)}(x)}{x^{n-k}}\right)^2 dx \leq \frac{4n}{(\beta+1)^2}\int_0^b \frac{1}{x^\beta}[u^{(n)}(x)]^2 dx.$$

Proof. Apply Lemma 6 to $k = 0, \cdots, n-1$ and sum up both sides. □

Lemma 8. *Let $f(x) \in C^\infty([a,b]), 0 \in [a,b]$. Assume $f^{(k)}(0) = 0$ for $k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(x)}{x^{n-k}}\right)^2 dx \leq \frac{4n}{(\beta+1)^2}\int_a^b \frac{1}{|x|^\beta}[f^{(n)}(x)]^2 dx.$$

Proof. The function $u(x) = f(-x)$ satisfies the assumptions for Lemma 7 on $[0,-a]$, so

$$\int_0^{-a} \frac{1}{x^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(-x)}{x^{n-k}}\right)^2 dx \leq \frac{4n}{(\beta+1)^2}\int_0^{-a} \frac{1}{x^\beta}[f^{(n)}(-x)]^2 dx.$$

Substitute the variable x by $-x$. Since the terms in the sum are of even powers and both sides have the same x^β term, the above inequality can be written as

$$\int_a^0 a\frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(x)}{x^{n-k}}\right)^2 dx \leq \frac{4n}{(\beta+1)^2}\int_a^0 \frac{1}{|x|^\beta}[f^{(n)}(x)]^2 dx.$$

From Lemma 7 the desired inequality is already true for $f(x)$ on $[0,b]$. Combining the results on $[a,0]$ and $[a,b]$, Lemma 8 follows. $\square$

The following is the proof of Theorem 1.

Proof. $f(x)$ satisfies the assumptions in Lemma 8 on $[a,b]$, so for any $\beta \geq 1$,

$$\frac{(\beta+1)^2}{4n}\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(x)}{x^{n-k}}\right)^2 dx \leq \int_a^b \frac{1}{|x|^\beta}[f^{(n)}(x)]^2 dx. \tag{7}$$

From (1),

$$|f^{(n)}(x)|^2 \leq C^2 \sum_{k=0}^{n-1}\left(\frac{|f^{(k)}(x)|}{|x|^{n-k}}\right)^2$$

thus

$$\int_a^b \frac{1}{|x|^\beta}|f^{(n)}(x)|^2 dx \leq C^2 \int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{|f^{(k)}(x)|}{|x|^{n-k}}\right)^2 dx. \tag{8}$$

Combining (7) and (8) we have

$$\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(x)}{x^{n-k}}\right)^2 dx \leq \frac{4nC^2}{(\beta+1)^2}\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{|f^{(k)}(x)|}{|x|^{n-k}}\right)^2 dx.$$

If $f \not\equiv 0$ on $[a,b]$, we would have $|f^{(k)}(x)| > 0$ on some sub-interval of $[a,b]$ thus the integrals > 0, which would imply

$$1 \leq \frac{4nC^2}{(\beta+1)^2}, \qquad \forall \beta \geq 1 \qquad \Longrightarrow \textit{Contradiction.}$$

Therefore $f(x) \equiv 0$ on $[a.b]$. This completes the proof of Theorem 1. $\square$

The proof of Corollary 2 follows immediately.

Proof. Under the assumption of Corollary 2, $f \not\equiv 0$ implies $f^{(k)}(0) = \beta \neq 0$ for some $k \geq 0$ based on the result of Theorem 1. If $\beta > 0$, then $f \in \mathcal{C}^\infty$ yields $f^{(k)}(x) > \beta/2 > 0,\ x \in [-\delta_0, \delta_0] \subset [a,b]$ for some $\delta_0 > 0$. Consequently the k-fold integral

$$f(x) = \int_0^x \cdots \int_0^x f^{(k)}(y)dy \cdots dy > x^k \beta/2 > 0, \qquad 0 < |x| < \delta_0.$$

The case $\beta < 0$ implies $f(x) < 0$ on an open interval containing 0.

For any $x' \in [a, b]$ such that $f(x') = 0$, the condition $f \not\equiv 0$ implies $f(x) \neq 0,\ 0 < |x - x'| < \delta_{x'},\ x \in [a, b]$ for some $\delta_{x'} > 0$. By the compactness of $[a, b]$, the zero set $\{f^{-1}(0)\}$ is at most finite in $[a, b]$. This completes the proof of Corollary 2. □

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